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THE HUTCHINSON–BARNSLEY THEORY FOR INFINITE ITERATED FUNCTION SYSTEMS

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We show that some results of the Hutchinson–Barnsley theory for finite iterated function systems can be carried over to the infinite case. Namely, if $\{F_i : i \in \mathbb{N}\}$ is a family of Matkowski's contractions on a complete metric space (X, d) such that $(F_i x_0)_{i\in\mathbb{N}}$ is bounded for some $x_0 \in X$, then there exists a non-empty bounded and separable set K which is invariant with respect to this family, that is, $K = \bigcup_{i\in\mathbb{N}} F_i(K)$. Moreover, given $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$, the limit $\lim_{n\to\infty} F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(x)$ exists and does not depend on x. We also study separately the case in which (X, d) is Menger convex or compact. Finally, we answer a question posed by Máté concerning a finite iterated function system $\{F_1, \ldots, F_N\}$ with the property that each of F_i has a contractive fixed point.

1. INTRODUCTION

Let $\{F_i : i \in I\}$ be a countable family of selfmaps of a complete metric space (X, d), where either $I = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$, or $I = \mathbb{N}$. In case in which I is finite, Hutchinson [11] proved that if all F_i are Banach contractions, then the mapping

$$\mathcal{F}(A) := \bigcup_{i\in I} F_i(A) \quad \text{for } A \subseteq X$$

has a unique fixed point K in the hyperspace of all nonempty compact subsets of X. Subsequently, this result was popularised by Barnsley [4], and therefore, in the literature, $\mathcal{F}$ is usually said to be the *Hutchinson–Barnsley operator*, whereas K is called a *fractal in the sense of Barnsley* associated with the *iterated function system* iterated function system $\{F_i : i \in I\}$. It is worth emphasising that two different proofs of the above result were given in [11]. The first one—which is the most familiar—is based on an application of the Banach Contraction Principle to operator $\mathcal{F}$ with a use of the Hausdorff metric in the above hyperspace. Recently, this approach was extended to iterated multifunction systems by Andres and Górniewicz [3] (see also [1] and [2] for further generalisations

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involving weakly contractive mappings). The second Hutchinson's proof is more elementary and it uses neither the Hausdorff metric, nor the hyperspace (see [**11**, Section 3.1]). (Yet another proof based on ordering techniques is given in [**14**].) Also the advantage of this proof is that it gives another characterisation of a fractal. To see that, let us define first

$$\Gamma(\sigma, n, x) := F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(x) \text{ for } \sigma \in I^{\mathbb{N}} \text{ and } x \in X.$$

Following Máté [**15**], we say that a family $\{F_i : i \in I\}$ has *property* (P) if the limit

$$\Gamma(\sigma) := \lim_{n\to\infty} \Gamma(\sigma, n, x) \tag{1}$$

exists for all $\sigma \in I^{\mathbb{N}}$ and $x \in X$, and does not depend on x. (In fact, Máté considered only the case in which I is finite.) Then Hutchinson's argument shows that a finite family of Banach contractions has property (P), and the fractal K associated with this family coincides with the set $\Gamma(I^{\mathbb{N}})$, that is,

$$K = \left\{ \lim_{n\to\infty} F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(x) : \sigma \in \{1, \ldots, N\}^{\mathbb{N}} \right\},$$

where $x \in X$ is fixed or may vary with σ. This result was partially extended by Máté [**15**] in the following way. Let φ be an upper semi-continuous and non-decreasing function from $\mathbb{R}_+$, the set of all non-negative reals, into $\mathbb{R}_+$ such that $\varphi(t) < t$ for all $t > 0$. Let $F_1, \ldots, F_N$ be *φ-contractions*, that is,

$$d(F_i x, F_i y) \leqslant \varphi\big(d(x, y)\big) \text{ for } x, y \in X \text{ and } i \in \{1, \ldots, N\}. \tag{2}$$

(The class of such φ-contractions was introduced by Browder [**6**].) If, given $\sigma \in \{1, \ldots, N\}^{\mathbb{N}}$ and $x \in X$, the sequence $\big(\Gamma(\sigma, n, x)\big)_{n\in\mathbb{N}}$ is bounded, then the family $\{F_1, \ldots, F_N\}$ has property (P).

However, in practice the latter assumption is inconvenient for verifying unless (X, d) is bounded. Our first purpose here is to show that, in fact, it can be dropped if φ is such that

$$\limsup_{t\to\infty} \big(t - \varphi(t)\big) = \infty. \tag{3}$$

Moreover, the result is still true if we consider an infinite iterated function system as stated in Theorem 1, in which we also give a list of equivalent conditions for the existence of a non-empty bounded set $K \subseteq X$ such that

$$K = \bigcup_{i\in\mathbb{N}} F_i(K). \tag{4}$$

Following Hutchinson [**11**], every non-empty set K (not necessarily bounded) satisfying (4) is said to be *invariant with respect to* $\{F_i : i \in \mathbb{N}\}$. Simple examples show that a family $\{F_i : i \in \mathbb{N}\}$ of φ-contractions need not possess a compact invariant set (see Example 2). Also note that the limit condition (3)—introduced in the metric fixed point theory by Matkowski [**18**]—is unnecessary if (X, d) is metrically convex or compact (see Theorems 3 and 4, respectively).

In this paper we also study the following problem (see Theorem 3) posed by Máté [**15**]: Let each F_i $(i = 1, \dots, N)$ have a *contractive fixed point* x_i, that is, $x_i = F_i x_i$ and given $x \in X$, $(F_i^n x)_{n \in \mathbb{N}}$ converges to x_i. Is it true that the family $\{F_1, \dots, F_N\}$ has property (P)? We answer this question in the negative; moreover, our Example 4 inspired us to find another sufficient condition for property (P) (see Theorem 5). That extends the following well-known result from the theory of contractive mappings: If $F\colon X \to X$ is such that, for some $p \in \mathbb{N}$, F^p has a contractive fixed point x_*, then x_* is also a contractive fixed point of F.

2. Invariant sets of infinite families of φ-contractions

Throughout this section we assume that $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$. Then it is easy to show that $\varphi(t) < t$ for $t > 0$. Matkowski [**17**, Theorem 1.2] (also see [**9**, page 15]) proved that each φ-contraction has a contractive fixed point. For a detailed discussion of several classes of mappings satisfying nonlinear contractive conditions, see [**12**]; in particular, Matkowski's theorem extends an earlier result of Browder [**6**]. The main result of this section is the following theorem dealing with infinite families of Matkowski's contractions.

Theorem 1. *Let (X, d) be a complete metric space and $F_i\colon X \to X$ $(i \in \mathbb{N})$ be φ-contractions with φ (independent of $i \in \mathbb{N}$) satisfying the limit condition* (3). *The following statements are equivalent:*

(i) *there exists an $x_0 \in X$ such that $(F_i x_0)_{i \in \mathbb{N}}$ is bounded;*
(ii) *given $x \in X$, $(F_i x)_{i \in \mathbb{N}}$ is bounded;*
(iii) *$(x_i)_{i \in \mathbb{N}}$ is bounded, where x_i is a fixed point of F_i;*
(iv) *there exists a non-empty bounded set $K \subseteq X$ such that*

$$K = \bigcup_{i \in \mathbb{N}} F_i(K).$$

Moreover, if (i) *holds, then the family $\{F_i : i \in \mathbb{N}\}$ has property (P) and given $x \in X$, sequences $\big(\Gamma(\sigma, n, x)\big)_{n \in \mathbb{N}}$ converge to $\Gamma(\sigma)$ uniformly with respect to $\sigma \in \mathbb{N}^{\mathbb{N}}$. Furthermore, the set $K_* := \Gamma(\mathbb{N}^{\mathbb{N}})$ satisfies the following conditions:*

(a) *K_* is bounded and $K_* = \bigcup_{i \in \mathbb{N}} F_i(K_*)$;*
(b) *if $K \subseteq X$ is bounded and $K = \bigcup_{i \in \mathbb{N}} F_i(K)$, then $K \subseteq K_*$, that is, K_* is the greatest (with respect to the inclusion $\subseteq$) fixed point of the Hutchinson–Barnsley operator $\mathcal{F}$ in the hyperspace of all bounded subsets of X.*

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The proof of Theorem 1 will be preceded by the series of auxiliary results.

Lemma 1. *Let (X,d) be a metric space and $F_i\colon X \to X$ $(i \in \mathbb{N})$ be Lipschitzian with the Lipschitz constant L_i such that $L := \sup_{i\in\mathbb{N}} L_i < \infty$. Assume that each F_i has a fixed point x_i (not necessarily unique). Consider* (i), (ii) *and* (iii) *of Theorem 1. Then* (i)$\Leftrightarrow$(ii)$\Leftarrow$(iii).

Proof: (i)$\Rightarrow$(ii): Given $x \in X$, we have

$$d(F_i x, F_i x_0) \leqslant L\, d(x, x_0) \quad \text{for all} \quad i \in \mathbb{N}.$$

Set $A := \{F_i x_0 : i \in \mathbb{N}\}$ and $r := L\, d(x,x_0)$. Then

$$\{F_i x : i \in \mathbb{N}\} \subseteq A^r := \bigcup_{a\in A} \overline{B}(a,r).$$

Since A is bounded, so is A^r and hence (ii) holds.

(ii) implies (i) *a fortiori.* (iii)$\Rightarrow$(ii): Given $x \in X$,

$$d(F_i x, x_i) = d(F_i x, F_i x_i) \leqslant L\, d(x, x_i).$$

By (iii), $r := L \sup\big\{d(x,x_i) : i \in \mathbb{N}\big\} < \infty$, so $\{F_i x : i \in \mathbb{N}\} \subseteq \{x_i : i \in \mathbb{N}\}^r$ which yields (ii). ∎

Lemma 2. *Let (X,d) be a metric space. Assume that $F_i\colon X \to X$ $(i \in \mathbb{N})$ are continuous and each of them has a contractive fixed point x_i. Consider* (iii) *and* (iv) *of Theorem 1. Then* (iv)$\Rightarrow$(iii).

Proof: (iv) implies that given $i \in \mathbb{N}$, $F_i(K) \subseteq K$. Hence and by continuity, we get

$$F_i(\overline{K}) \subseteq \overline{F_i(K)} \subseteq \overline{K}.$$

Let $x \in K$. By hypothesis, $F_i^n x \to x_i$ as $n \to \infty$. Since $F_i^n x \in K$, we infer $x_i \in \overline{K}$. Since, by (iv), also $\overline{K}$ is bounded, so is $\{x_i : i \in \mathbb{N}\}$. ∎

Let $s\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the *shift operator*, that is,

$$s(\sigma_1, \ldots, \sigma_n, \ldots) := (\sigma_2, \ldots, \sigma_{n+1}, \ldots) \quad \text{for} \quad \sigma \in \mathbb{N}^{\mathbb{N}}.$$

Proposition 1. *Let (X,d) be a metric space. Assume that $F_i\colon X \to X$ $(i \in \mathbb{N})$ are continuous and such that $\{F_i : i \in \mathbb{N}\}$ has property (P). Let $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ be such that $\Sigma \neq \emptyset$, $s(\Sigma) \subseteq \Sigma$ and $s^{-1}(\Sigma) \subseteq \Sigma$. Then the set $\Gamma(\Sigma)$ is invariant with respect to $\{F_i : i \in \mathbb{N}\}$.*

Proof: Set $K := \Gamma(\Sigma)$. Clearly, K is non-empty. If $x \in K$, then

$$x = \Gamma(\sigma) = \lim_{n\to\infty} F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(x) \text{ for some } \sigma \in \Sigma.$$

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By property (P), there exists the limit

$$y := \lim_{n\to\infty} F_{\sigma_2} \circ \cdots \circ F_{\sigma_n}(x) = \Gamma\big(s(\sigma)\big).$$

Then, by continuity of F_{σ_1}, we have $x = F_{\sigma_1}(y)$. Since $s(\Sigma) \subseteq \Sigma$, we infer $y \in K$ and thus $x \in F_{\sigma_1}(K)$. This yields the inclusion: $K \subseteq \bigcup_{i\in\mathbb{N}} F_i(K)$. Conversely, if $x \in F_i(K)$ for some $i \in \mathbb{N}$, then there is a $y \in K$ such that $x = F_i(y)$ and $y = \Gamma(\sigma)$ for some $\sigma \in \Sigma$. Then, by continuity of F_i, we get

$$x = \lim_{n\to\infty} F_i \circ F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(y) = \Gamma(\sigma'),$$

where $\sigma' := (i, \sigma_1, \ldots, \sigma_n, \ldots)$. Since $\sigma' \in s^{-1}(\Sigma) \subseteq \Sigma$, we infer $x \in K$. ∎

EXAMPLE 1. Fix a $\sigma^* \in \mathbb{N}^{\mathbb{N}}$ and set

$$\Sigma_{\sigma^*} := \{\sigma \in \mathbb{N}^{\mathbb{N}} : \exists_{i,j\in\mathbb{N}}\ \forall_{n\in\mathbb{N}}\ \sigma_{i+n} = \sigma^*_{j+n}\}.$$

Then Σ_{σ^*} is the least element of the family

$$\{\Sigma \subseteq \mathbb{N}^{\mathbb{N}} : \sigma^* \in \Sigma,\ s(\Sigma) \subseteq \Sigma \text{ and } s^{-1}(\Sigma) \subseteq \Sigma\}.$$

LEMMA 3. *Under the assumptions of Theorem 1, if* (i) *holds, then, given* $x \in X$,

$$\sup\Big\{d\big(x, F_{i_1} \circ \cdots \circ F_{i_n}(x)\big) : n \in \mathbb{N},\ i_1, \ldots, i_n \in \mathbb{N}\Big\} < \infty,$$

that is, the semigroup generated by mappings F_i is pointwise bounded on X.

PROOF: Let $x \in X$. Given $n \in \mathbb{N}$, set

$$a_n := \sup\Big\{d\big(x, F_{i_1} \circ \cdots \circ F_{i_n}(x)\big) : i_1, \ldots, i_n \in \mathbb{N}\Big\}.$$

Then $a_1 = \sup\big\{d(x, F_i x) : i \in \mathbb{N}\big\}$. Since all F_i are, in particular, non-expansive, Lemma 1 yields a_1 is finite. By (3), there is an $M > 0$ such that $M - \varphi(M) \geqslant a_1$. Using induction we show that $a_n \leqslant M$ for all $n \in \mathbb{N}$. Clearly, $a_1 \leqslant M$. Let $n \in \mathbb{N}$ be such that $a_n \leqslant M$. Then, given $i_1, \ldots, i_{n+1} \in \mathbb{N}$,

$$\begin{aligned} d\Big(x, F_{i_1} \circ \cdots \circ F_{i_{n+1}}(x)\Big) &\leqslant d(x, F_{i_1}x) + \varphi\big(d(x, F_{i_2} \circ \cdots \circ F_{i_{n+1}}(x)\big) \\ &\leqslant a_1 + \varphi(a_n) \leqslant a_1 + \varphi(M) \leqslant M, \end{aligned}$$

since F_{i_1} is a φ-contraction, φ is non-decreasing and $d\big(x, F_{i_2} \circ \cdots \circ F_{i_{n+1}}(x)\big) \leqslant a_n$. Since $i_1, \ldots, i_{n+1}$ were arbitrary, we infer $a_{n+1} \leqslant M$. Thus, by induction, $(a_n)_{n\in\mathbb{N}}$ is bounded which completes the proof. ∎

PROOF OF THEOREM 1: Assume that (i) holds. We show $\{F_i : i \in \mathbb{N}\}$ has property (P). Let $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$. By Lemma 3, the constant

$$M := \sup\Big\{d\big(x, F_{i_1} \circ \cdots \circ F_{i_n}(x)\big) : n \in \mathbb{N},\ i_1, \ldots, i_n \in \mathbb{N}\Big\} \tag{5}$$

is finite. Hence, given $j, k \in \mathbb{N}$, we have

$$(6) \qquad \begin{aligned} d\big(F_{\sigma_1} \circ \cdots \circ F_{\sigma_j}(x), F_{\sigma_1} \circ \cdots \circ F_{\sigma_{j+k}}(x)\big) &\leqslant \varphi^j\Big(d\big(x, F_{\sigma_{j+1}} \circ \cdots \circ F_{\sigma_{j+k}}(x)\big)\Big) \\ &\leqslant \varphi^j(M), \end{aligned}$$

since $F_{\sigma_1}, \dots, F_{\sigma_j}$ are φ-contractions and φ is non-decreasing. Since $\varphi^j(M) \to 0$ as $j \to \infty$, given $\varepsilon > 0$, there is an $l > 0$ such that $\varphi^j(M) < \varepsilon$ for all $j \geqslant l$. Hence and by (6), we infer

$$(7) \qquad d\big(\Gamma(\sigma, j, x), \Gamma(\sigma, j+k, x)\big) < \varepsilon \text{ for all } j \geqslant l \text{ and } k \in \mathbb{N}$$

which means $\big(\Gamma(\sigma, n, x)\big)_{n \in \mathbb{N}}$ is a Cauchy sequence, so it converges to some $y \in X$. We show the limit does not depend on x. Given $x' \in X$,

$$d\big(\Gamma(\sigma, n, x), \Gamma(\sigma, n, x')\big) \leqslant \varphi^n\big(d(x, x')\big) \to 0 \text{ as } n \to \infty.$$

Thus we may set $\Gamma(\sigma) := \lim_{n \to \infty} \Gamma(\sigma, n, x)$. Letting $k \to \infty$ in (7), we obtain

$$d\big(\Gamma(\sigma, j, x), \Gamma(\sigma)\big) \leqslant \varepsilon \text{ for all } j \geqslant l.$$

Note that l does not depend on σ which means $\big(\Gamma(\sigma, n, x)\big)_{n \in \mathbb{N}}$ converges to $\Gamma(\sigma)$ uniformly with respect to $\sigma \in \mathbb{N}^{\mathbb{N}}$. In particular, $\{F_i : i \in \mathbb{N}\}$ has property (P).

Now, since all F_i are continuous, Proposition 1 yields K_* is invariant with respect to $\{F_i : i \in \mathbb{N}\}$. Moreover, if $x \in X$ and M is defined by (5), then given $\sigma \in \mathbb{N}^{\mathbb{N}}$,

$$d\big(x, F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(x)\big) \leqslant M \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we get $d\big(x, \Gamma(\sigma)\big) \leqslant M$ which yields the boundedness of K_* since M does not depend on σ. Thus K_* satisfies (a); in particular, the above argument shows that (i)⇒(iv) holds. Hence and by Lemmas 1 and 2, we infer (i), (ii), (iii) and (iv) are equivalent.

Finally, we prove (b). Assume that K is a non-empty bounded subset of X and $K = \bigcup_{i \in \mathbb{N}} F_i(K)$. Let $x \in K$. Then there are $\sigma_1 \in \mathbb{N}$ and $x_1 \in K$ such that $x = F_{\sigma_1} x_1$. Similarly, $x_1 = F_{\sigma_2} x_2$ for some $\sigma_2 \in \mathbb{N}$ and $x_2 \in K$. Continuing in this fashion, we get sequences $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in K$ and $x = F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(x_n)$ for all $n \in \mathbb{N}$. Set $y_n := \Gamma(\sigma, n, x)$. Then

$$\begin{aligned} d(x, y_n) = d\big(F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(x_n), F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(x)\big) &\leqslant \varphi^n\big(d(x, x_n)\big) \\ &\leqslant \varphi^n(\operatorname{diam} K) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence $y_n \to x$, that is, $x = \Gamma(\sigma)$ which means $x \in K_*$. This yields $K \subseteq K_*$. □

Theorem 1 implies the following somewhat surprising property of the Hutchinson–Barnsley operator $\mathcal{F}$.

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Corollary 1. *Let $\mathcal{H}_b(X)$ denote the hyperspace of all non-empty bounded subsets of X. Under the assumptions of Theorem 1, consider the operator*

$$\mathcal{F}(A) := \bigcup_{i\in\mathbb{N}} F_i(A) \text{ for } A \subseteq X.$$

The following statements are equivalent:

(i) *$\mathcal{F}$ has a fixed point in $\mathcal{H}_b(X)$;*

(ii) *$\mathcal{F}$ maps $\mathcal{H}_b(X)$ into $\mathcal{H}_b(X)$.*

Proof: (i)⇒(ii): By Theorem 1 ((iv)⇒(iii)), $(x_i)_{i\in\mathbb{N}}$ is bounded. Set $B := \{x_i : i \in \mathbb{N}\}$. Let $A \in \mathcal{H}_b(X)$ and $x \in A$. Given $i \in \mathbb{N}$, we have

$$d(F_i x, x_i) = d(F_i x, F_i x_i) \leqslant \varphi\big(d(x, x_i)\big) \leqslant \varphi(r),$$

where $r := \operatorname{diam}(A \cup B)$. Clearly, r is finite and $F_i x \in B^{\varphi(r)}$ which yields $\mathcal{F}(A) \subseteq B^{\varphi(r)}$. Thus $\mathcal{F}(A)$ is bounded.

(ii)⇒(i) follows immediately from Theorem 1 since (ii) implies that $(F_i x)_{i\in\mathbb{N}}$ is bounded whenever $x \in X$. □

The following example shows that an infinite iterated function system need not have a compact invariant set even if (X, d) is compact. Moreover, a bounded invariant set need not be unique.

Example 2. Let $X := [0,1]$ and $F_i x := x/2 + 1/2^i$ for $i \in \mathbb{N}$ and $x \in X$. Suppose, on the contrary, there exists a non-empty compact set $K \subseteq X$ such that $K = \bigcup_{i\in\mathbb{N}} F_i(K)$. Then $F_i(K) \subseteq K$, so by the Contraction Principle, F_i has a fixed point in K which means $K \supseteq \{1/2^{i-1} : i \in \mathbb{N}\}$. Hence $0 \in K$ since K is closed. However,

$$\bigcup_{i\in\mathbb{N}} F_i(K) \subseteq \bigcup_{i\in\mathbb{N}} F_i(X) = \bigcup_{i\in\mathbb{N}} [1/2^i, 1/2^i + 1/2] = (0,1],$$

so $0 \notin \bigcup_{i\in\mathbb{N}} F_i(K)$ which yields a contradiction. On the other hand, by Theorem 1, the above family has bounded invariant sets as, for example, $(0,1]$ $(= \Gamma(\mathbb{N}^{\mathbb{N}}))$ or $(0,1] \cap \mathbb{Q}$.

Though $\Gamma(\mathbb{N}^{\mathbb{N}})$ need not be compact, it is always separable as shown below.

Lemma 4. *Endow $\mathbb{N}$ with the discrete topology, and the product $\mathbb{N}^{\mathbb{N}}$ with the Tychonoff topology τ_T. Under the assumptions of Theorem 1, if* (i) *holds, then Γ is continuous from $(\mathbb{N}^{\mathbb{N}}, \tau_T)$ into (X, d).*

Proof: Let $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $\varepsilon > 0$. By Theorem 1, $K_*(= \Gamma(\mathbb{N}^{\mathbb{N}}))$ is bounded and invariant. Let $x \in K_*$. There is a $p \in \mathbb{N}$ such that $\varphi^p(\operatorname{diam} K_*) < \varepsilon$. Set

$$U := \{\sigma_1\} \times \cdots \times \{\sigma_p\} \times \mathbb{N}^{\mathbb{N}}.$$

Then U is a neighbourhood of σ. If $\sigma' \in U$ and $n \in \mathbb{N}$, then

$$\begin{aligned} &d\big(\Gamma(\sigma, p+n, x), \Gamma(\sigma', p+n, x)\big) \\ &\qquad = d\big(F_{\sigma_1} \circ \cdots \circ F_{\sigma_p} \circ \cdots \circ F_{\sigma_{p+n}}(x), F_{\sigma_1} \circ \cdots \circ F_{\sigma_p} \circ F_{\sigma'_{p+1}} \circ \cdots \circ F_{\sigma'_{p+n}}(x)\big) \\ &\qquad \leqslant \varphi^p\Big(d\big(F_{\sigma_{p+1}} \circ \cdots \circ F_{\sigma_{p+n}}(x), F_{\sigma'_{p+1}} \circ \cdots \circ F_{\sigma'_{p+n}}(x)\big)\Big) \\ &\qquad \leqslant \varphi^p(\operatorname{diam} K_*) < \varepsilon, \end{aligned}$$

because $\sigma'_i = \sigma_i$ for $i = 1, \ldots, p$; moreover, since x is in K_*, so are $F_{\sigma_{p+1}} \circ \cdots \circ F_{\sigma_{p+n}}(x)$ and $F_{\sigma'_{p+1}} \circ \cdots \circ F_{\sigma'_{p+n}}(x)$. Letting $n \to \infty$, we obtain

$$d\big(\Gamma(\sigma), \Gamma(\sigma')\big) \leqslant \varepsilon$$

which yields the continuity of Γ. $\square$

COROLLARY 2. *Under the assumptions of Theorem 1, if* (i) *holds, then the set* $\Gamma(\mathbb{N}^{\mathbb{N}})$ *is bounded, separable and invariant with respect to* $\{F_i : i \in \mathbb{N}\}$.

PROOF: In virtue of Theorem 1, we only need to show $\Gamma(\mathbb{N}^{\mathbb{N}})$ is separable. This follows immediately from Lemma 4 and [**8**, 1.4.11 and 2.3.16] since Γ is continuous and $(\mathbb{N}^{\mathbb{N}}, \tau_T)$ is separable. $\square$

As a consequence of Theorem 1, we get the following extension of [**15**, Theorem 2] by omitting the assumption of boundedness of $\big(\Gamma(\sigma, n, x)\big)_{n \in \mathbb{N}}$.

THEOREM 2. *Let* (X, d) *be a complete metric space and* $N \in \mathbb{N}$. *Let* $F_1, \ldots, F_N$ *be* φ*-contractions on* X *with* φ *satisfying* (3). *Then* $\{F_1, \ldots, F_N\}$ *has property (P), and the set* $\Gamma(\{1, \ldots, N\}^{\mathbb{N}})$ *is compact and invariant with respect to this family.*

PROOF: Set $F_i := F_1$ for $i > N$. Then $(F_i)_{i \in \mathbb{N}}$ satisfies the assumptions of Theorem 1 including (i), so it has property (P). In particular, $\Gamma(\sigma)$ is well defined for all $\sigma \in \{1, \ldots, N\}^{\mathbb{N}}$ which means $\{F_1, \ldots, F_N\}$ has property (P). Moreover, if $\sigma \in \mathbb{N}^{\mathbb{N}}$ and we define

$$\sigma'_i := \sigma_i \text{ if } \sigma_i \in \{1, \ldots, N\}; \; \sigma'_i := 1 \text{ otherwise},$$

then it follows from the definition of $(F_i)_{i \in \mathbb{N}}$ that $\Gamma(\sigma) = \Gamma(\sigma')$. Consequently, $K_* = \Gamma(\mathbb{N}^{\mathbb{N}}) = \Gamma\big(\{1, \ldots, N\}^{\mathbb{N}}\big)$. Since, by Tychonoff's theorem, $\{1, \ldots, N\}^{\mathbb{N}}$ is a compact subset of $\mathbb{N}^{\mathbb{N}}$ and, by Lemma 4, Γ is continuous, we infer K_* is compact. Also, by Theorem 1, $K_* = \bigcup_{i \in \mathbb{N}} F_i(K_*)$ which yields $K_* = \bigcup_{i=1}^{N} F_i(K_*)$. $\square$

The following theorem illuminates connections between invariant sets of an infinite iterated function system $\{F_i : i \in \mathbb{N}\}$ and invariant sets of its finite subfamilies $\{F_1, \ldots, F_n\}$ $(n \in \mathbb{N})$.

PROPOSITION 2. *Let the assumptions of Theorem 1 be satisfied including condition* (i). *For* $n \in \mathbb{N}$, *let* $K_n\Big(= \Gamma\big(\{1, \ldots, n\}^{\mathbb{N}}\big)\Big)$ *be a compact invariant set with respect to* $\{F_1, \ldots, F_n\}$. *Then the set* $K := \bigcup_{n \in \mathbb{N}} K_n$ *is bounded and invariant with respect to* $\{F_i : i \in \mathbb{N}\}$.

PROOF: Set

$$\Sigma_n := \{1, \dots, n\}^{\mathbb{N}} \text{ for all } n \in \mathbb{N},$$

and $\Sigma := \bigcup_{n\in\mathbb{N}} \Sigma_n$. Then

$$\Gamma(\Sigma) = \bigcup_{n\in\mathbb{N}} \Gamma(\Sigma_n) = \bigcup_{n\in\mathbb{N}} K_n = K,$$

so $K \subseteq \Gamma(\mathbb{N}^{\mathbb{N}})$. Since, by Theorem 1, $\Gamma(\mathbb{N}^{\mathbb{N}})$ is bounded, so is K. In virtue of Proposition 1 it suffices to show that $s(\Sigma) \subseteq \Sigma$ and $s^{-1}(\Sigma) \subseteq \Sigma$. Clearly, $s(\Sigma_n) \subseteq \Sigma_n$ for $n \in \mathbb{N}$ which yields $s(\Sigma) \subseteq \Sigma$. Now assume $\sigma \in s^{-1}(\Sigma)$, that is, $s(\sigma) \in \Sigma_n$ for some $n \in \mathbb{N}$. If $\sigma_1 \leqslant n$, then $\sigma \in \Sigma_n$; otherwise, $\sigma \in \Sigma_{\sigma_1}$. Thus in both cases we have $\sigma \in \Sigma$. That means $s^{-1}(\Sigma) \subseteq \Sigma$. ∎

Now we discuss a question whether condition (i) of Theorem 1 is necessary for property (P). Also one may ask if the assumption that all F_i are φ-contractions with φ independent of i is essential. It turns out that for some families of mappings both conditions are also necessary for property (P) as shown in the following

EXAMPLE 3. Let $X := \mathbb{R}$ be endowed with the Euclidean metric. Let $\alpha_i \in \mathbb{R} \setminus \{0\}$ and $\beta_i \in \mathbb{R}$ $(i \in \mathbb{N})$. Set

$$F_i x := \alpha_i x + \beta_i \text{ for } x \in X \text{ and } i \in \mathbb{N}.$$

We show that $\{F_i : i \in \mathbb{N}\}$ has property (P) if and only if

$$\alpha := \sup\left\{|\alpha_i| : i \in \mathbb{N}\right\} < 1 \text{ and } \beta := \sup\left\{|\beta_i| : i \in \mathbb{N}\right\} < \infty,$$

that is, F_i are Banach α-contractions satisfying (i) of Theorem 1. The sufficiency part follows of course from Theorem 1. So assume $\{F_i : i \in \mathbb{N}\}$ has property (P). Suppose, on the contrary, $\alpha \geqslant 1$. Then, given $n \in \mathbb{N}$, $2^n/(2^n+1) < \alpha$, so there is a $\sigma_n \in \mathbb{N}$ such that

$$|\alpha_{\sigma_n}| \geqslant 2^n/(2^n+1).$$

It is easy to verify that

$$F_{\sigma_1} \circ \dots \circ F_{\sigma_n}(x) = \alpha_{\sigma_1} \cdot \ldots \cdot \alpha_{\sigma_n} x + \beta_{\sigma_1} + \sum_{j=2}^{n} \alpha_{\sigma_1} \cdot \ldots \cdot \alpha_{\sigma_{j-1}} \beta_{\sigma_j}. \tag{8}$$

Denote $a_n := \alpha_{\sigma_1} \cdot \ldots \cdot \alpha_{\sigma_n}$, $b_n := \beta_{\sigma_1} + \sum_{j=2}^{n} \alpha_{\sigma_1} \cdot \ldots \cdot \alpha_{\sigma_{j-1}} \beta_{\sigma_j}$. By property (P), $F_{\sigma_1} \circ \dots \circ F_{\sigma_n}(0) \to \Gamma(\sigma)$, that is, $b_n \to \Gamma(\sigma)$, and $F_{\sigma_1} \circ \dots \circ F_{\sigma_n}(1) \to \Gamma(\sigma)$, that is, $a_n + b_n \to \Gamma(\sigma)$. Hence $a_n \to 0$. On the other hand, $|a_n| \geqslant \prod_{k=1}^{n} 2^k/(1+2^k)$ which yields a contradiction since $\prod_{k=1}^{n} 2^k/(1+2^k) \not\to 0$. Thus $\alpha < 1$. Now suppose, on the contrary, $\beta = \infty$. By induction we define a sequence $(\sigma_n)_{n\in\mathbb{N}}$. Set $\sigma_1 := 1$. For $n \geqslant 2$, take σ_n such that

$$|\beta_{\sigma_n}| \geqslant 1/|\alpha_{\sigma_1} \cdot \ldots \cdot \alpha_{\sigma_{n-1}}|.$$

Then the series $\sum_{n=2}^{\infty} \alpha_{\sigma_1} \cdot \ldots \cdot \alpha_{\sigma_{n-1}} \beta_{\sigma_n}$ diverges. On the other hand, $F_{\sigma_1} \circ \cdots \circ F_{\sigma_n}(0) \to \Gamma(\sigma)$ which, by (8), implies the convergence of the above series; a contradiction. Thus β is finite.

REMARK 1. Note that the assumption that F_i of Example 3 are not constant functions is essential. Indeed, any family of constant functions does have property (P) though it need not satisfy (i) of Theorem 1. So, in general, condition (i) is not necessary for (P).

3. Invariant sets in Menger convex spaces and compact spaces

Recall that (X, d) is said to be *metrically convex* or *Menger convex* (see, for example, [5]) if given $x, y \in X$, $x \neq y$, there is a $z \in X$ such that $x \neq z \neq y$ and $d(x,y) = d(x,z) + d(z,y)$. In this case, with the help of Matkowski's [19] result, the assumptions of Theorem 1 may be weakened in the following way.

THEOREM 3. *Let (X, d) be a complete metrically convex space and $F_i \colon X \to X$ ($i \in \mathbb{N}$) be φ-contractions, where $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\varphi(t) < t$ for $t > 0$. Then the assertion of Theorem 1 holds.*

PROOF: By hypothesis, we have

$$d(F_i x, F_i y) \leqslant \varphi\big(d(x,y)\big) \text{ for } x, y \in X \text{ and } i \in \mathbb{N}.$$

Without loss of generality, we may assume $\varphi(0) = 0$. Then φ is continuous at 0, so by [19, Proposition 3], there is an increasing concave function $\gamma \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\gamma(t) < t$ for $t > 0$ and

$$d(F_i x, F_i y) \leqslant \gamma\big(d(x,y)\big) \text{ for } x, y \in X \text{ and } i \in \mathbb{N}.$$

Then γ is continuous and $\lim_{n\to\infty} \gamma^n(t) = 0$ (see, for example, [6]). Moreover, by [10, Theorem 7.2.5], γ is subadditive. Using [10, Theorem 7.6.1] we infer

$$\lim_{t\to\infty} \frac{\gamma(t)}{t} = \inf\left\{\frac{\gamma(t)}{t} : t > 0\right\} \leqslant \gamma(1) < 1$$

which yields $\lim_{t\to\infty}\big(t - \gamma(t)\big) = \infty$. Thus we may apply Theorem 1 substituting there γ for φ. ∎

In the same way we could restate Theorem 2 for finite iterated function system. Moreover, in case in which (X, d) is compact, we need not use a function φ at all letting F_i be Edelstein's [7] contractions, according to the following

THEOREM 4. *Let (X, d) be a compact metric space, $N \in \mathbb{N}$ and $F_i \colon X \to X$ $\big(i \in \{1, \ldots, N\}\big)$ be such that*

$$d(F_i x, F_i y) < d(x,y) \text{ for all } x, y \in X,\ x \neq y.$$

Then $\{F_1, \ldots, F_N\}$ has property (P), and there exists a compact and invariant set with respect to this family.

PROOF: By [**13**, Proposition 1], given $i \in \{1,\dots,N\}$, there is a subadditive and non-decreasing function $\varphi_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi_i(t) < t$ for all $t > 0$ and

$$d(F_i x, F_i y) \leqslant \varphi_i\big(d(x,y)\big) \text{ for } x, y \in X.$$

Set $\varphi(t) := \max\big\{\varphi_1(t),\dots,\varphi_N(t)\big\}$ for $t \in \mathbb{R}_+$. Clearly, the above properties of φ_i carry over to φ. Thus repeating the argument of the preceding proof, we infer $\lim_{t\to\infty}\big(t - \varphi(t)\big) = \infty$. Now the assertion follows directly from Theorem 2. ∎

REMARK 2. Though the assumptions of Theorem 3 on function φ are essentially weaker than those of Theorem 1, both theorems deal, in fact, with the same class of mappings as can be deduced from the proof of Theorem 3. The same comment concerns relations between Theorems 2 and 4.

4. AROUND MÁTÉ'S QUESTION

Example 4 given below settles in the negative a question posed by Máté [**15**]. Following [**15**], we say that a mapping $F\colon X \to X$ has *property* (Q) if, for every $x \in X$, there exists the limit $\lim_{n\to\infty} F^n x$ and does not depend on x.

EXAMPLE 4. Let $X := [0,1]$ be endowed with the Euclidean metric. Set

$$\begin{aligned} &Fx := x + 1/2 \text{ if } x \in [0,1/2); \qquad &&Fx := 1 \text{ if } x \in [1/2,1],\\ &Gx := 0 \text{ if } x \in [0,1/2); &&Gx := x - 1/2 \text{ if } x \in [1/2,1]. \end{aligned}$$

Clearly, F and G are continuous selfmaps of X. Since F^2 and G^2 are constant ($F^2 \equiv 1$, $G^2 \equiv 0$), we infer F and G have property (Q). We show, however, $\{F, G\}$ has no property (P). Moreover, sequences $\big(\Gamma(\sigma,n,x)\big)_{n\in\mathbb{N}}$ need not converge. To see it, set

$$\sigma_{2j-1} := 1 \text{ and } \sigma_{2j} := 2 \text{ for } j \in \mathbb{N}.$$

Then, given $x \in X$, $\Gamma(\sigma, 2n, x) = (F\circ G)^n(x) = F \circ G(x)$, because

$$F\circ G(x) = 1/2 \text{ if } x \in [0,1/2); \quad F\circ G(x) = x \text{ if } x \in [1/2,1],$$

so $F\circ G$ is an involution. On the other hand,

$$\Gamma(\sigma, 2n+1, x) = F\circ(G\circ F)^n(x) = F\circ G\circ F(x)$$

since $G\circ F(x) = x$ if $x \in [0,1/2)$; $G\circ F(x) = 1/2$ if $x \in [1/2,1]$. Hence we get

$$\Gamma(\sigma, 2n+1, x) = x + 1/2 \text{ if } x \in [0,1/2); \quad \Gamma(\sigma, 2n+1, x) = 1 \text{ if } x \in [1/2,1].$$

Thus we may conclude that $\big(\Gamma(\sigma,n,x)\big)_{n\in\mathbb{N}}$ is convergent if and only if $x = 0$ or $x = 1$. So (P) is not valid for the family $\{F, G\}$ though (Q) holds for each of these mappings.

Observe that the essence of Example 4 lies in the fact that F^2 and G^2 are Banach contractions whereas $F \circ G$ and $G \circ F$ are not contractive. Indeed, if both these compositions had been Banach contractions, then (P) would have been valid for $\{F, G\}$. That can be deduced from the following more general result (mentioned by Máté [**16**] only for families of affine selfmaps of $\mathbb{R}^n$): If $F_1, \ldots, F_N$ are selfmaps of a complete metric space (X, d) such that for some $p \in \mathbb{N}$ all mappings from the family

$$\mathfrak{F}_p := \{F_{i_1} \circ \cdots \circ F_{i_p} : i_1, \ldots, i_p \in \{1, \ldots, N\}\} \tag{9}$$

are Banach contractions, then $\{F_1, \ldots, F_N\}$ has property (P). Note that, by the Hutchinson [**11**] theorem, the above assumption implies (P) is valid for $\mathfrak{F}_p$. It turns out that the latter condition is also sufficient for property (P) of $\{F_1, \ldots, F_N\}$, according to the following

THEOREM 5. *Let (X, d) be a metric space (not necessarily complete), $N \in \mathbb{N}$ and $F_1, \ldots, F_N$ be selfmaps of X. If, for some $p \in \mathbb{N}$, $\mathfrak{F}_p$ defined by (9) has property (P), then so does $\{F_1, \ldots, F_N\}$. Moreover, $\Gamma(\{1, \ldots, N\}^{\mathbb{N}}) = \Gamma_p(\{1, \ldots, N^p\}^{\mathbb{N}})$, where Γ_p is defined below by* (10).

PROOF: Clearly, $\mathfrak{F}_p$ is finite. Denote its elements by G_i, $i \in \{1, \ldots, N^p\}$. Given $\sigma \in \{1, \ldots, N^p\}^{\mathbb{N}}$, $n \in \mathbb{N}$ and $x \in X$, set

$$\Gamma_p(\sigma, n, x) := G_{\sigma_1} \circ \cdots \circ G_{\sigma_n}(x). \tag{10}$$

Now let $\sigma \in \{1, \ldots, N\}^{\mathbb{N}}$ and $x \in X$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned}\Gamma(\sigma, pn, x) &= (F_{\sigma_1} \circ \cdots \circ F_{\sigma_p}) \circ \cdots \circ (F_{\sigma_{p(n-1)+1}} \circ \cdots \circ F_{\sigma_{pn}})(x) \\ &= G_{\sigma'_1} \circ \cdots \circ G_{\sigma'_n}(x),\end{aligned}$$

where $\sigma' \in \{1, \ldots, N^p\}^{\mathbb{N}}$ is such that $G_{\sigma'_i} = F_{\sigma_{p(i-1)+1}} \circ \cdots \circ F_{\sigma_{pi}}$ for $i \in \mathbb{N}$. Hence we get $\Gamma(\sigma, pn, x) = \Gamma_p(\sigma', n, x)$. Since (P) is valid for $\mathfrak{F}_p$, we infer

$$\lim_{n \to \infty} \Gamma(\sigma, pn, x) = \Gamma_p(\sigma').$$

Since this limit does not depend on x, we may conclude substituting successively $F_1 x, \ldots, F_N x$ for x that

$$\forall_{\varepsilon > 0} \exists_{k \in \mathbb{N}} \forall_{n \geqslant k} \forall_{j \in \{1, \ldots, N\}} \, d(\Gamma(\sigma, pn, F_j x), \Gamma_p(\sigma')) < \varepsilon. \tag{11}$$

Fix an $\varepsilon > 0$. By (11), there is a $k \in \mathbb{N}$ such that if $n \geqslant k$, then the inequality of (11) holds for all $j \in \{1, \ldots, N\}$; in particular, for $j := \sigma_{pn+1}$, we have

$$d(\Gamma(\sigma, pn, F_{\sigma_{pn+1}} x), \Gamma_p(\sigma')) < \varepsilon.$$

Since $\Gamma(\sigma, pn, F_{\sigma_{pn+1}} x) = \Gamma(\sigma, pn + 1, x)$, the above argument shows

$$\lim_{n \to \infty} \Gamma(\sigma, pn + 1, x) = \Gamma_p(\sigma').$$

Continuing in this fashion, we obtain

$$\lim_{n\to\infty} \Gamma(\sigma, pn + k, x) = \Gamma_p(\sigma') \text{ for all } k \in \{0, 1, \ldots, p-1\}$$

which yields $\lim_{n\to\infty} \Gamma(\sigma, n, x) = \Gamma_p(\sigma')$. That means $\{F_1, \ldots, F_N\}$ has property (P). Moreover, the above argument shows

$$\Gamma(\{1, \ldots, N\}^{\mathbb{N}}) \subseteq \Gamma_p(\{1, \ldots, N^p\}^{\mathbb{N}}).$$

Since the opposite inclusion is obvious, the proof is completed. ∎

REMARK 3. Since property (P) of singleton $\{F\}$ is equivalent to property (Q) of F, Theorem 5 yields the result stated in the last sentence of the introduction.

As an immediate consequence of Theorems 2 and 5, we obtain the following

COROLLARY 3. *Let (X, d) be a complete metric space and $N \in \mathbb{N}$. Let $F_1, \ldots, F_N$ be selfmaps of X (not necessarily continuous) such that, for some $p \in \mathbb{N}$, all compositions $F_{i_1} \circ \cdots \circ F_{i_p}$ $\big(i_1, \ldots, i_p \in \{1, \ldots, N\}\big)$ are φ-contractions with φ as in Theorem 2. Then $\{F_1, \ldots, F_N\}$ has property (P), and there exists a compact and invariant set with respect to this family.*

To illuminate the assumptions of Corollary 3, consider the following

EXAMPLE 5. Let $X := \mathbb{R}$ be endowed with the Euclidean metric. Let a be any irrational number, and D denote the Dirichlet function. Set

$$F_1 := D \text{ and } F_2 := D + a.$$

It is easily seen that the assumptions of Corollary 3 are satisfied with $p = 2$ since all compositions $F_i \circ F_j$ $\big(i, j \in \{1, 2\}\big)$ are constant. On the other hand, Theorem 2 is not applicable here since F_1 and F_2 are discontinuous. So Corollary 3 does extend Theorem 2.

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