A GENERALIZED FREDHOLM THEORY FOR CERTAIN MAPS IN THE REGULAR REPRESENTATIONS OF AN ALGEBRA

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Introduction. Given an algebra A, the elements of A induce linear operators on A by left and right multiplication. Various authors have studied Banach algebras A with the property that some or all of these multiplication maps are completely continuous operators on A; see **(1–5)**. In **(3)** I. Kaplansky defined an element u of a Banach algebra A to be completely continuous if the maps $a \rightarrow ua$ and $a \rightarrow au$, $a \in A$, are completely continuous linear operators. The set of all completely continuous elements of A forms an ideal. Assume that A is a semisimple Banach algebra, and let B be the intersection of all the primitive ideals of A which contain the socle of A. Using **(1**, Theorem 7.2**)**, it can be shown that the ideal of completely continuous elements of A is contained in B.

In general the elements of B are not completely continuous (in fact there are important algebras A where $A = B$, but zero is the only completely continuous element of A). However, the multiplication maps induced by elements $u \in B$ do have special properties similar to those of completely continuous operators. It is the purpose of this paper to develop a generalized Riesz–Fredholm theory for these maps. We shall make only the assumption that A is semisimple and, in some cases, that A is a normed algebra. Theorem 3.6 serves as a partial summary of our results.

1. Preliminaries. Throughout this paper we shall assume that A is a complex semisimple algebra. We assume that the reader is acquainted with such notions as quasi-regularity of an element of A, left and right regular representations of A on A, primitive ideals, etc. We use in general the definitions in C. Rickart's book **(6)**. For B an algebra, we denote by E_B the set of all minimal idempotents of B, and by S_B, the socle of B; see **(6**, pp. 45–47**)**. A non-empty subset M of E_B is orthogonal if $ef = 0$ for any two distinct elements e and f in M.

We shall be interested in the elements in $k(h(S_A))$, the ideal which is the intersection of all those primitive ideals of A which contain S_A. Let B be the algebra $k(h(S_A))$. It is not difficult to verify that P is a primitive ideal of B if and only if P is of the form $B \cap Q$ where Q is a primitive ideal of A. Now

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$S_B = S_A \cap B$, and this in combination with the previous statement implies that S_B is contained in no primitive ideal of B. This is a necessary and sufficient condition that a semisimple algebra B be a modular annihilator algebra by **(1**, Theorem 4.3 (4), p. 570**)**. For the definition and elementary properties of modular annihilator algebras see either **(1)** or **(10)**. Since $k(h(S_A))$ is a modular annihilator algebra, we have the following result which is used repeatedly.

(1.1) *If* $u \in k(h(S_A))$, *then* u *is left* (*right*) *quasi-singular, i.e.,* $A(1 - u) \neq A$ $((1 - u)A \neq A)$, *if and only if there exists* $x \in A, x \neq 0$, *such that* $(1 - u)x = 0$ $(x(1 - u) = 0)$.

In §3 it will be necessary for us to assume that A is a normed algebra. Assume for the present that A has a norm $||\cdot||$. Then $B = k(h(S_A))$ is also a normed algebra. Let I be the norm closure in B of S_A. B/I is then a normed radical algebra (recall that $S_A = S_B$ is included in no primitive ideal of B). If $v \in B/I$ and $|\cdot|$ is the induced norm on the quotient algebra, it follows that $|v^n|^{1/n} \to 0$ as $n \to \infty$; see **(6**, Theorem (1.6.3), p. 28**)**. We can draw the following conclusion concerning elements in B:

(1.2) *Assume* A *has norm* $||\cdot||$. *If* $u \in k(h(S_A))$, *then there exists a sequence* $\{s_n\} \subset S_A$ *such that* $||u^n - s_n||^{1/n} \to 0$ *as* $n \to \infty$.

We do not assume that A has an identity. If A does have an identity, we denote it by 1; and if λ is a scalar, we denote $\lambda \cdot 1$ simply by λ. If A does not have an identity, 1 and $\lambda \cdot 1$, denoted again by λ, are symbolic, but make sense when multiplied by an element of A. Our main concern is with operators defined on A by left or right multiplication by $(\lambda - u)$ where λ is a scalar and $u \in A$; the left multiplication operator on A determined by $(\lambda - u)$ is the operator which takes $x \in A$ into $(\lambda - u)x \in A$. If M is any subset of A, we let $R[M] = \{a \in A \mid Ma = 0\}$ and $L[M] = \{a \in A \mid aM = 0\}$. With this notation the null space of the left multiplication operator determined by $(\lambda - u)$ is the right ideal $R[A(\lambda - u)]$; the range is the right ideal $(\lambda - u)A$. The right multiplication operator on A determined by $(\lambda - u)$ has a similar definition and similar properties.

In the course of studying left and right multiplication operators on A, we make important use of the concepts of ascent and descent of a linear operator. For the definitions and elementary properties of these concepts, see **(8**, pp. 271–274**)**. We denote the ascent of the left (right) multiplication operator on A determined by $(\lambda - u)$ by $\alpha_l(\lambda - u)$ $(\alpha_r(\lambda - u))$ and the descent by

$$\delta_l(\lambda - u)\ (\delta_r(\lambda - u)).$$

Finally we denote the spectrum of an element $u \in A$ by $\sigma(u)$.

2. Ideals of finite order and elements of the socle. In the generalized Fredholm theory that we develop for elements in $k(h(S_A))$, the concept of a

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left or right ideal of finite order replaces that of finite-dimensional subspace. In this section we study the elementary properties of ideals of finite order, and using these results, derive basic information concerning the socle of A.

Definition. A right (left) ideal K of A has finite order if and only if K can be written as the sum of a finite number of minimal right (left) ideals of A. We define the order of K to be the smallest number of minimal right (left) ideals of A which have sum K. For convenience we say that the zero ideal has finite order 0.

If I is a two-sided ideal of A, the definition of the order of I is ambiguous. However, it is a corollary of Theorem 2.2 that the order of I considered as a right ideal is the same as the order of I considered as a left ideal. Thus we shall ignore the ambiguity.

THEOREM 2.1. *Assume that M is a left ideal of A of finite order n. If $f_1, f_2, \ldots, f_m$ are in E_A, $Af_1 + Af_2 + \ldots + Af_m \subset M$, and this sum is direct, then $m \leqslant n$. A similar statement holds for right ideals of finite order.*

Proof. Choose $e_1, e_2, \ldots, e_n \in E_A$ such that $M = Ae_1 + Ae_2 + \ldots + Ae_n$. Since $f_1 \in M$, there exist elements $x_k \in A$ such that $f_1 = x_1 e_1 + \ldots + x_n e_n$. Assume that $x_j e_j \neq 0$. Then

$$Ae_j = Ax_j e_j \subset \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} Ae_k \right) + Af_1.$$

Thus M must be the sum on the right-hand side of this inclusion. Now $f_2 \in M$, and therefore there exist elements $y_k \in A$ and $z \in A$ such that

$$f_2 = zf_1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} y_k e_k.$$

Since the sum $Af_1 + Af_2 + \ldots + Af_m$ is direct, $y_i e_i \neq 0$ for some $i \neq j$. Then, proceeding as before, we have that

$$M = Af_1 + Af_2 + \left(\sum_{\substack{k=1 \\ k \neq j,\, i}}^{n} Ae_k \right).$$

By continuing in this manner, we can at each successive step replace an ideal Ae_q by an ideal Af_p. If $m > n$, then at the end of this process we have $M = Af_1 + Af_2 + \ldots + Af_n$. But this contradicts the assumption that the sum $Af_1 + \ldots + Af_m$ is direct. Therefore $m \leqslant n$.

THEOREM 2.2. *Assume that K is a non-zero right ideal of finite order n. Then any maximal orthogonal set of minimal idempotents in K contains n elements, and if $\mathfrak{M} = \{e_1, e_2, \ldots, e_n\}$ is such a set, then $K = eA$, where $e = e_1 + e_2 + \ldots + e_n$.*

Proof. Let $\mathfrak{M}$ be a maximal orthogonal set of minimal idempotents in K. By Theorem 2.1, $\mathfrak{M}$ must be a finite set (note that if $\{f_1, \ldots, f_k\}$ is an orthogonal set of minimal idempotents, then the sum $f_1 A + \ldots + f_k A$ is direct), so we

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write $\mathfrak{M} = \{e_1, e_2, \ldots, e_p\}$. Now assume that g is a minimal idempotent in K such that $e_k g = 0$ for $1 \leqslant k \leqslant p$. By the maximality of $\mathfrak{M}$, $ge_k \neq 0$ for some k, $1 \leqslant k \leqslant p$. By renumbering the elements of $\mathfrak{M}$ we may assume that $ge_j \neq 0$ if $1 \leqslant j \leqslant m$ and $ge_j = 0$ if $j > m$. Let

$$f = g - \sum_{k=1}^{m} ge_k.$$

Since $fg = g \neq 0$, then $f \neq 0$. It is easy to verify that $e_k f = fe_k = 0$ for all k, $1 \leqslant k \leqslant p$. Also

$$f^2 = \left(g - \sum_{k=1}^{m} ge_k\right)f = gf = f,$$

and $fA = gfA = gA$; thus f is a minimal idempotent. This contradicts the definition of $\mathfrak{M}$ as a maximal orthogonal set of minimal idempotent in K. Thus there can be no minimal idempotents $g \in K$ such that $e_k g = 0$ for all $e_k \in \mathfrak{M}$.

Now take $v \in K$ and define

$$w = v - \sum_{k=1}^{p} e_k v.$$

Then $e_k w = 0$ for all k, $1 \leqslant k \leqslant p$. If $w \neq 0$, then since $wA \subset K \subset S_A$, there exists $g \in E_A$ such that $g \in wA$. But then $e_k g = 0$ for $1 \leqslant k \leqslant p$. Therefore w must be 0. Thus it follows that for any $v \in K$,

$$v = \sum_{k=1}^{p} e_k v.$$

Let $e = e_1 + e_2 + \ldots + e_p$. We have proved that $K = eA$.

It remains to be shown that $p = n$. First by Theorem 2.1, $p \leqslant n$. But p cannot be strictly less than n by the definition of the order of an ideal and the fact that $K = e_1 A + \ldots + e_p A$. This completes the proof of the theorem.

If K is any left or right ideal of finite order and $\mathfrak{M}$ is a maximal orthogonal set of minimal idempotents in K, we shall call $\mathfrak{M}$ an orthogonal basis for K. It is not difficult to verify that if K is a left ideal of finite order and J is a left ideal such that $J \subset K$, then J has finite order; furthermore, if J is properly contained in K, then the order of J is strictly less than the order of K, and any orthogonal basis for J can be extended to an orthogonal basis for K.

Now we turn to the investigation of the elements in S_A, although we state the next lemma more generally for elements in $k(h(S_A))$.

LEMMA 2.3. *Assume that* $u \in k(h(S_A))$. *Furthermore, assume that* $R[A(1-u)^m]$ *is of finite order and that* $\alpha_l(1-u) = m$. *Then*

(1) $\delta_r(1-u) = m$;

(2) $A(1-u) = A(1-e)$, *where* e *is an idempotent in* S_A *such that* $R[A(1-u)] = eA$.

Proof. By Theorem 2.2 there exists an idempotent $e_m \in S_A$ such that $R[A(1-u)^m] = e_m A$. Now consider the left ideal $M = A((1-u)^m - e_m)$

which is of the form $A(1-v)$ where $v \in k(h(S_A))$. We shall prove that $R[M] = 0$. Suppose that $Mx = 0$. Then $(1-u)^m x = e_m x$ and

$$(1-u)^{2m}x = (1-u)^m e_m x = 0.$$

But since $\alpha_l(1-u) = m$, then $R[A(1-u)^{2m}] = R[A(1-u)^m]$; it follows that $(1-u)^m x = 0$ and $e_m x = 0$. But $x \in R[A(1-u)^m]$, and hence $x = e_m x = 0$. Thus $R[M] = 0$ and, by (1.1), $M = A$.

Now suppose that $y \in A(1-u)^n$ for some $n \geqslant m$. Then $ye_m = 0$. But also $y = z((1-u)^m - e_m)$ for some $z \in A$. Then $ze_m = ye_m = 0$, and hence $y = z(1-u)^m$. Thus $y \in A(1-u)^m$, and it follows that

$$A(1-u)^n = A(1-u)^m.$$

This proves in fact that $\delta_r(1-u) = m$.

Since $M = A$, we have $A(1-u)^m + Ae_m = A$.

$$R[A(1-u)] \subset R[A(1-u)^m],$$

and is therefore of finite order. Let e be an idempotent in S_A such that $R[A(1-u)] = eA$. Let $B = k(h(S_A))$, and let $N = B(1-u) + Be$. N is a left ideal of B, and it is easy to verify that the right annihilator of N in B is 0. Since B is a modular annihilator algebra and N is a modular left ideal of B, it follows that $B = N$. Thus

$$Ae_m \subset B = B(1-u) + Be \subset A(1-u) + Ae.$$

But also $A(1-u)^m \subset A(1-u)$. Then

$$A = A(1-u)^m + Ae_m \subset A(1-u) + Ae.$$

Assume that $z \in A(1-e)$. z is of the form $z = w(1-u) + ye$ for some $w, y \in A$. But $ye = ze = 0$. Thus $z = w(1-u)$, and it follows that $A(1-u) = A(1-e)$.

Next we prove our main result concerning elements of the socle of A. All considerations are completely algebraic, as they have been up to this point in the paper.

THEOREM 2.4. *Assume that $s \in S_A$. Then*

(1) *$R[A(1-s)]$ and $L[(1-s)A]$ are of finite order*;

(2) *$\alpha_l(1-s) = \delta_l(1-s) = \alpha_r(1-s) = \delta_r(1-s)$ and all these quantities are finite*;

(3) *$A(1-s) = A(1-e)$ where e is an idempotent in S_A such that*

$$R[A(1-s)] = eA;$$

(4) *$\sigma(s)$ is finite.*

Proof. Let $K = R[A(1-s)]$. If $x \in K$, then $(1-s)x = 0$, and thus $x = sx \in sA$. But then $K \subset sA$, and since sA is of finite order, K must be of finite order. By a similar proof, we find that $L[(1-s)A]$ is of finite order.

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Next we prove that $\alpha_l(1-s)$ is finite. Suppose it is not; then setting $K_n = R[A(1-s)^n]$, we have that K_n is a proper subset of K_{n+1} for all $n \geqslant 0$. We may choose an orthogonal sequence $\{e_k\} \subset E_A$ with the property that $e_n \in K_n$ (choose first orthogonal bases $\mathfrak{M}_n$ for each K_n such that $\mathfrak{M}_{n+1}$ is an extension of $\mathfrak{M}_n$; next, choose e_k to be an element in $\mathfrak{M}_k$ not in $\mathfrak{M}_{k-1}$). But then $(1-s)^n e_n = 0$, and this implies that $e_n \in sA$. This contradicts the fact that sA is of finite order. Therefore $\alpha_l(1-s)$ must be finite. With a similar proof we find that $\alpha_r(1-s)$ is finite. By Lemma 2.3 (1) we have $\alpha_l(1-s) = \delta_r(1-s)$ and $\alpha_r(1-s) = \delta_l(1-s)$. Finally, $\alpha_l(1-s) = \delta_l(1-s)$ since when the ascent and descent of an everywhere defined linear operator are both finite, they are equal by (**8**, Theorem 5.41-E, p. 273). This completes the proof of (2).

Now having proved (2), (3) follows immediately by Lemma 2.3 (2).

Lastly, we prove (4). Assume that $\{\lambda_k\}$ is an infinite sequence of distinct non-zero elements in $\sigma(s)$. We may assume that there is a sequence $\{e_k\} \subset E_A$ such that $se_k = \lambda_k e_k$; see (1.1). It follows that $e_k \in sA$ for all k. Suppose that there are $x_k \in A$ such that

$$e_1 x_1 + e_2 x_2 + \ldots + e_n x_n = 0$$

and that $e_n x_n \neq 0$. Then

$$-e_n x_n = e_1 x_1 + \ldots + e_{n-1} x_{n-1}.$$

Therefore

$$\begin{aligned} 0 &= (\lambda_1 - u)(\lambda_2 - u) \ldots (\lambda_{n-1} - u) e_n x_n \\ &= (\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n) \ldots (\lambda_{n-1} - \lambda_n) e_n x_n. \end{aligned}$$

This contradiction implies that for any $n \geqslant 1$, the sum

$$e_1 A + e_2 A + \ldots + e_n A$$

is direct. This in turn contradicts the fact that sA has finite order.

3. The elements in $k(h(S_A))$. In this section we generalize the results of §2 concerning elements in S_A to the elements in $k(h(S_A))$. The first theorem is an easy extension of Theorem 2.4 (1).

THEOREM 3.1. *If $u \in A$ is quasi-regular modulo S_A, then $R[A(1-u)]$ and $L[(1-u)A]$ are of finite order. In particular, this conclusion holds whenever $u \in k(h(S_A))$.*

Proof. If u is left quasi-regular modulo S_A, then there exists $w \in A$ and $s \in S_A$ such that $(1-w)(1-u) = (1-s)$. Then $A(1-s) \subset A(1-u)$. It follows that $R[A(1-u)] \subset R[A(1-s)]$, and since $R[A(1-s)]$ is of finite order by Theorem 2.4 (1), then $R[A(1-u)]$ must have finite order. Similarly, if u is right quasi-regular modulo S_A, then $L[(1-u)A]$ must have finite order. Now $B = k(h(S_A))$ is a modular annihilator algebra, and thus B/S_A is a radical algebra. It follows in the case when $u \in k(h(S_A))$ that u is quasi-regular modulo S_A.

We shall usually find it necessary in this section to assume that A is a normed algebra. The proof of the next theorem depends in a crucial way upon this assumption. In the proof we use a version of a result proved by A. F. Ruston concerning a bounded linear operator T defined on a Banach space X which has the property that $\lim_{n\to\infty} ||T^n - C_n||^{1/n} = 0$ (where $||\cdot||$ is the operator norm) for some sequence $\{C_n\}$ of completely continuous operators on X. Ruston's proof of the result we use (**7**, Lemma 3.2, p. 323) does not require X to be a Banach space in the given norm. The conclusion of Ruston's Lemma 3.2 is that the ascent of $I - T$ must be finite where I is the identity operator on X.

THEOREM 3.2. *Let A be a normed algebra with norm $||\cdot||$. Assume that $u \in k(h(S_A))$. Then $\alpha_l(1 - u)$ and $\alpha_r(1 - u)$ are finite.*

Proof. We prove only that $\alpha_l(1 - u)$ is finite. Denote the right ideal $R[A(1 - u)] \cap (1 - u)^nA$ by K_n. Assume that $K_n \neq 0$ for all $n \geqslant 0$. Now by Theorem 3.1, $R[A(1 - u)]$ is of finite order. Also note that for all $k \geqslant 0$, $(1 - u)^{k+1}A \subset (1 - u)^kA$. It follows that there exists an integer m such that whenever $n \geqslant m$, then $K_n = K_m$. Since $K_m \neq 0$, there exists an $e \in E_A$ such that $e \in K_m$. Then $e \in K_n$ for all $n \geqslant 0$. It follows that for all integers $k \geqslant 0$,

$$e \in (R[A(1 - u)] \cap Ae) \cap (1 - u)^kAe.$$

Let $a \to T_a$ be the left regular representation of A on $Ae(T_a(xe) = axe$ for all $xe \in Ae)$. Ae is a normed linear space and T_a is a bounded operator on Ae. Now by assumption $u \in k(h(S_A))$. Therefore there exists a sequence $\{s_n\} \subset S_A$ such that $||u^n - s_n||^{1/n} \to 0$ as $n \to \infty$ by (1.2). Let $|T_a|$ denote the operator norm of T_a on the normed linear space Ae. Then we have immediately that $|T_u^n - T_{s_n}|^{1/n} \to 0$ as $n \to \infty$. But it can be shown that T_{s_n} is an operator of finite rank on Ae. By Ruston's result (see the discussion preceding the statement of this theorem), the ascent of $I - T_u$ on Ae must be finite. Lemma 3.4 (**9**, p. 22) implies that a linear operator W has finite ascent if and only if there exists an integer p such that the intersection of the null space of W with the range of W^p is 0. Letting W represent the operator $I - T_u$ on Ae, we have that $(R[A(1 - u)] \cap Ae) \cap (1 - u)^pAe$ must be 0 for some p. This is a contradiction, and we conclude that $K_m = 0$ for some m. But now let W represent the left multiplication operator on A determined by $(1 - u)$.

$$0 = K_m = R[A(1 - u)] \cap (1 - u)^mA,$$

and this last object is exactly the intersection of the null space of W with the range of W^m. Therefore $\alpha_l(1 - u)$ is finite.

THEOREM 3.3. *Assume that A is a normed algebra. If $u \in k(h(S_A))$, then*

(1) $\alpha_l(1 - u) = \delta_l(1 - u) = \alpha_r(1 - u) = \delta_r(1 - u)$ and all these quantities are finite;

(2) $A(1 - u) = A(1 - e)$, where e is an idempotent in S_A such that $R[A(1 - u)] = eA$.

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Proof. By Theorem 3.1, $R[A(1-u)^k]$ is of finite order for all $k \geqslant 1$. By Theorem 3.2, $\alpha_l(1-u)$ and $\alpha_r(1-u)$ are finite. Now (2) follows directly from Lemma 2.3 (2). Also by Lemma 2.3 (1), $\alpha_l(1-u) = \delta_r(1-u)$. Then since the ascent and descent of an operator are equal if they are finite by (**8**, Theorem 5.41-E, p. 273), it follows that

$$\delta_l(1-u) = \alpha_l(1-u) = \delta_r(1-u) = \alpha_r(1-u).$$

The next theorem concerns the spectrum of elements in $k(h(S_A))$. It has a direct application to modular annihilator algebras which we state as a corollary.

THEOREM 3.4. *Assume that A is normed with norm $\|\cdot\|$. If $u \in k(h(S_A))$, then $\sigma(u)$ is either finite or countable and has no non-zero limit points.*

Proof. Assume that $\lambda \neq 0$ is in $\sigma(u)$, and that $\{\lambda_n\}$ is a sequence of distinct non-zero elements in $\sigma(u)$ such that $\lambda_n \to \lambda$ as $n \to \infty$. We may assume by appealing to (1.1) that there exists a sequence $\{e_n\} \subset E_A$ with the property that $(\lambda_n - u)e_n = 0$ for $n \geqslant 1$. By Theorem 3.3 (1),

$$\alpha_r(1-u) = \delta_r(1-u) = m$$

for some integer m. Let $K = L[(\lambda - u)^m A]$. By (**8**, Theorem 5.41-F, p. 273) $A = A(\lambda - u)^m + K$. Now define M to be the left ideal

$$\{v \in A \mid \|ve_n/\|e_n\|\| \to 0 \text{ as } n \to \infty\}.$$

Now $(\lambda - u)^m e_k = (\lambda - \lambda_k)^m e_k$ for all $k \geqslant 1$, and therefore $A(\lambda - u)^m \subset M$. It also follows that $e_k \in (\lambda - u)^m A$ for all $k \geqslant 1$. Therefore $Ke_k = 0$ for all $k \geqslant 1$. Then $K \subset M$, and finally $A = K + A(\lambda - u)^m \subset M$. But

$$\|ue_k/\|e_k\|\| = |\lambda_k|$$

for all k, which implies that $u \notin M$, a contradiction.

COROLLARY. *If A is a semisimple normed modular annihilator algebra, then the spectrum of any element in A is either finite or countable, and has no non-zero limit points.*

THEOREM 3.5. *Assume that A is normed. Then if $u \in k(h(S_A))$, the order of $R[A(1-u)]$ is the same as the order of $L[(1-u)A]$. If $u \in S_A$, the same conclusion holds without the hypothesis that A have a norm.*

Proof. We prove the theorem for the case where $u \in k(h(S_A))$ and A is normed. By Theorem 3.1, we may assume that $R[A(1-u)]$ has finite order n and that $L[(1-u)A]$ has finite order m. The proof proceeds by induction on n. In the case when $n = 0$, $m = 0$ by Theorem 3.3 (1). Now assume that $n \geqslant 1$, and that the theorem holds for all k such that $0 \leqslant k < n$. First note that $m \geqslant 1$, again by Theorem 3.3 (1). Let $\mathfrak{M} = \{e_1, \ldots, e_n\}$ be a maximal orthogonal set of minimal idempotents in $R[A(1-u)]$, and let

$$\mathfrak{N} = \{f_1, \ldots, f_m\}$$

be a maximal orthogonal set of minimal idempotents in $L[(1-u)A]$.

Suppose that $f_k Ae_1 = 0$ for all k, $1 \leqslant k \leqslant m$. Let $P = L[Ae_1]$; then P is a primitive ideal of A. Let $B = A/P$, and let $\pi \colon A \to B$ be the natural projection of A onto the quotient algebra B. Note that B is a primitive normed algebra and that $\pi(u) \in k(h(S_B))$. Clearly $\pi(e_1) \neq 0$ and $(1 - \pi(u))\pi(e_1) = 0$. Then there exists $x \in A$ such that $\pi(x) \neq 0$ and $\pi(x)(1 - \pi(u)) = 0$ by Thorem 3.3 (1).Now by (**1**, Proposition 3.1 (1), p. 567), $P = L[Ae_1] = R[e_1 A]$. Then since $\pi(x - xu) = 0, e_1 A(x - xu) = 0$. Thus

$$(e_1 Ax) \subset A(f_1 + \ldots + f_m) \subset P;$$

hence $(e_1 Ax) \subset P$. Then $(e_1 A)(e_1 Ax) = 0$, and it follows that $e_1 Ax = 0$. Thus $\pi(x) = 0$, a contradiction.

Therefore there exists some j, $1 \leqslant j \leqslant m$, such that $f_j Ae_1 \neq 0$. We may assume that $j = 1$. Choose $y \in A$ such that $f_1 ye_1 \neq 0$, and let $w = u + f_1 ye_1$; note that $w \in k(h(S_A))$. Assume that $A(1 - w)v = 0$. Then

$$(1 - u)v = (f_1 ye_1)v.$$

Multiplying this equation on the left by f_1, we have that $(f_1 ye_1)v = 0$. It follows that $0 = A(f_1 ye_1 v) = Ae_1 v$, and hence that $e_1 v = 0$. But also $(1 - u)v = 0$. Thus

$$v = (e_1 + e_2 + \ldots + e_n)v = (e_2 + \ldots + e_n)v.$$

Therefore $R[A(1 - w)] = (e_2 + \ldots + e_n)A$. In a similar fashion we find that $L[(1 - w)A] = A(f_2 + \ldots + f_m)$. By the induction hypothesis, it follows that $n = m$.

The last theorem of this section is a summary of the main results given in this paper. We use the notations $\mathcal{N}(W)$, $\alpha(W)$, and $\delta(W)$ to stand for the null space, the ascent, and the descent of a linear operator W, respectively. We hope that the notation and the particular formulation of the results presented in this theorem will make explicit the concept of a generalized Fredholm theory for elements in $k(h(S_A))$.

THEOREM 3.6. *Assume that A is a semisimple normed algebra, and that $u \in k(h(S_A))$. Let $a \to T_a$ be the left regular representation of A on A, and let $a \to T'_a$ be the right regular representation of A on A. Assume that λ is a non-zero scalar. Then*:

(1) *The orders of $\mathcal{N}(\lambda I - T_u)$ and $\mathcal{N}(\lambda I - T'_u)$ are finite and equal.*

(2) *$\alpha(\lambda I - T_u) = \delta(\lambda I - T_u) = \alpha(\lambda I - T'_u) = \delta(\lambda I - T'_u)$ and all these quantities are finite.*

(3) *The equation $(\lambda I - T_u)x = y$ has a solution $x \in A$ if and only if $zy = 0$ for all $z \in \mathcal{N}(\lambda I - T'_u)$. The equation $(\lambda I - T'_u)x = y$ has a solution $x \in A$ if and only if $yz = 0$ for all $z \in \mathcal{N}(\lambda I - T_u)$.*

(4) *The equation $(\lambda I - T_u)x = y$ has a solution $x \in A$ for all given $y \in A$, except for at most a countable set of λ. If there is an infinite sequence of such exceptional values $\{\lambda_n\}$, then $\lambda_n \to 0$ as $n \to \infty$.*

(5) *If $u \in S_A$, then* (1)–(4) *hold without the assumption that A have a norm, and in fact in* (4) *only a finite number of exceptional values is possible.*

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