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GENERATORS OF IDEALS DEFINING CERTAIN SURFACES IN PROJECTIVE SPACE

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ABSTRACT. We consider the surface obtained from the projective plane by blowing up the points of intersection of two plane curves meeting transversely. We find minimal generating sets of the defining ideals of these surfaces embedded in projective space by the sections of a very ample divisor class. All of the results are proven over an algebraically closed field of arbitrary characteristic.

0. **Introduction.** Consider a set of distinct points $p_1, \ldots, p_n$ of the projective plane $\mathbf{P}^2_{\mathfrak{K}}$ over a fixed ground field $\mathfrak{K}$ which is algebraically closed of arbitrary characteristic. Let X be the surface obtained by blowing up the points p_i. The problem of finding minimal generating sets, or more generally minimal free resolutions, for ideals of such surfaces, embedded in projective space by a linear system of forms on $\mathbf{P}^2$ vanishing at the points, has been studied by several authors (see for example [GG], [Gi], [GL], [GGH], and [GGP]). In [GG], resolutions of ideals defining Room surfaces in projective space are determined; these surfaces are blowings up of general sets of points in $\mathbf{P}^2$ (sets of $\binom{d+1}{2}$ points which do not lie on a curve of degree $d-1$), and the embedding in projective space is given by the linear system of forms of degree $d+1$ on $\mathbf{P}^2$ vanishing at these points. In [GGH], attention is paid to blowing up $\mathbf{P}^2$ at special sets of points. In particular, if X is the blowing up of $\mathbf{P}^2$ at the points of intersection of two plane curves P and Q meeting transversely, Geramita, Gimigliano and Harbourne show that X supports very ample superabundant divisor classes if and only if both curves have degree at least 4. The very ample superabundant classes found are *uniform, i.e.*, of the form $F_{d,m}$, corresponding to forms on $\mathbf{P}^2$ of degree d vanishing at each point of $P \cap Q$ to order at least m. (For the definition, see Section 1.) Moreover, in [GGH], minimal generating sets for the ideal defining the image of X in projective space, embedded by the sections of $F_{1+t,1}$ are found when P and Q have the same degree t.

In this paper we extend the work on minimal sets of generators. Given *any* very ample uniform divisor class on the blowing up X of $\mathbf{P}^2$ at the points of intersection of two curves P and Q meeting transversely, regardless of the degrees of P and Q (and in particular, not assuming P and Q have the same degree), we find a minimal set of generators for the ideal I_X defining the surface X embedded in projective space by the sections of the very ample class. This work is motivated by the results obtained in [GGH].

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In Section 1 we recall some preliminaries and introduce some notation. In Section 2 we explain the set up in which we will be working. In Section 3 we prove the main theorem of the paper.

Before we end this introduction, we wish to note that the results of this paper comprise part of the author's Ph.D. thesis, written under the guidance of Professor Brian Harbourne, to whom we express our sincere gratitude.

1. **Preliminaries.** Let X be a surface obtained by blowing up n distinct points $p_1, \ldots, p_n$ of $\mathbf{P}^2_{\mathfrak{K}}$ over a fixed ground field $\mathfrak{K}$ which is algebraically closed of arbitrary characteristic. The divisor class group $\mathrm{Pic}(X)$ of X is a free abelian group of rank $n+1$. A basis for $\mathrm{Pic}(X)$ is given by $e_0, \ldots, e_n$, where e_0 is the pullback to X of the class of a line in $\mathbf{P}^2$, and e_i, $i > 0$, is the class of the exceptional divisor E_i corresponding to the blowing up of p_i [Ha]. The intersection product on $\mathrm{Pic}(X)$ is described by saying $e_0, \ldots, e_n$ are pairwise orthogonal and $-1 = -e_0 \cdot e_0 = e_1 \cdot e_1 = \cdots = e_n \cdot e_n$. Following [GGH] we shall say a divisor class of the form $de_0 - me_1 - \cdots - me_n$ is *uniform* and denote it by $F_{d,m}$.

We recall that a divisor class F on a surface X is *very ample* if the global sections $H^0(X, F)$ define an embedding of X into $\mathbf{P}^N$, where $N + 1 = h^0(X, F)$. Also recall that a projective variety $X \subset \mathbf{P}^N$ is said to be *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring is integrally closed. We shall say a divisor class F on X is *projectively normal* if F is very ample and if X, with respect to the embedding in projective space given by the sections of F, is projectively normal.

We will follow the convention $\binom{a}{b} = 0$, if $a < b$.

2. **Set Up.** Let P and Q be two plane curves of degrees $d_1 \leq d_2$, respectively, meeting transversely. Let g and f be the homogeneous polynomials in the coordinate ring $R = \mathfrak{K}[w_1, w_2, w_3]$ of $\mathbf{P}^2$ defining P and Q, respectively. Let I be the ideal (f, g) in R. We shall denote $d_2 - d_1$ by δ and $d_1 d_2$ by ρ. Let X be the surface obtained by blowing up the points $p_1, \ldots, p_\rho \in \mathbf{P}^2$ of intersection of P and Q. Let $\pi: X \longrightarrow \mathbf{P}^2$ be the corresponding birational map. As before, $F_{d,m}$ denotes the uniform class $de_0 - me_1 - \cdots - me_\rho$.

We recall the following result of [GGH] (see Proposition II and Theorem IV of [GGH]).

PROPOSITION 2.1. *The divisor class $F_{d,m}$ is very ample on X if and only if $d > md_2$ and $m > 0$. Moreover, any very ample class $F_{d,m}$ on X is projectively normal.*

For simplicity of notation we will denote $F_{r+md_2,m}$ by $D_{r,m}$. To find generators for the ideal of X embedded by any such very ample class $D_{r,m}$, we first study a related but possibly degenerate embedding of X, induced by a morphism ϕ which we now define.

For $r > 0$ and $m > 0$, let $V_{sijk} = f^{m-s} g^s w_1^i w_2^j w_3^k$, where $i+j+k = r+s\delta$ and $0 \leq s \leq m$. Note there are $\sum_{s=0}^{m} \binom{r+s\delta+2}{2}$ elements V_{sijk}. Thus setting $\phi(a,b,c) = (\cdots, V_{sijk}(a,b,c), \cdots)$ defines a morphism

$$\phi: \mathbf{P}^2 - \{p_1, \ldots, p_\rho\} \longrightarrow \mathbf{P}^N$$

away from the points $p_1, \ldots, p_\rho$, where

$$(\dagger) \qquad N+1 = \sum_{s=0}^{m} \binom{r+s\delta+2}{2}.$$

Let $S = \mathfrak{K}[\cdots, X_{sijk}, \cdots]$ be the homogenous coordinate ring of $\mathbf{P}^N$.

LEMMA 2.2. *Pullback by π establishes a $\mathfrak{K}$-vector space isomorphism from the homogeneous component $(I^m)_{r+md_2}$ of the ideal I^m to $H^0(X, D_{r,m})$.*

PROOF. We have that the homogeneous coordinate ring R is just $\oplus_{i\geq 0} H^0(\mathbf{P}^2, ie_0)$ and that $H^0\big(X, (r+md_2)e_0\big) \cong H^0\big(\mathbf{P}^2, (r+md_2)e_0\big)$. Via

$$H^0(X, D_{r,m}) \subset H^0\big(X, (r+md_2)e_0\big),$$

$H^0\big(X, D_{r,m}\big)$ corresponds in $H^0\big(\mathbf{P}^2, (r+md_2)e_0\big)$ to the linear system of forms on $\mathbf{P}^2$ of degree $r+md_2$ vanishing at each point $p_1, \ldots, p_\rho$ to order m or more. Thus if P_i denotes the homogeneous ideal defining p_i, we have an isomorphism from $H^0(X, D_{r,m})$ to the homogeneous component $(P_1^m \cap \cdots \cap P_\rho^m)_{r+md_2}$ of $P_1^m \cap \cdots \cap P_\rho^m$ of degree $r+md_2$. By a theorem of Macaulay (Lemma 5 of Appendix 6 of volume II of [ZS]), $P_1^m \cap \cdots \cap P_\rho^m = (P_1 \cap \cdots \cap P_\rho)^m$. Thus now we have an isomorphism from $H^0(X, D_{r,m})$ to $(P_1 \cap \cdots \cap P_\rho)^m_{r+md_2}$. But $(P_1 \cap \cdots \cap P_\rho)^m_{r+md_2} = (I^m)_{r+md_2}$, which establishes the result. ■

Since $nD_{r,m} = D_{nr,nm}$, by Lemma 2.2 we see that

$$H^0(X, nD_{r,m}) \cong (I^{mn})_{n(r+md_2)}$$

as $\mathfrak{K}$-vector spaces. We will denote this isomorphism by ξ_n.

Now, for the reader's convenience, we state a part of Proposition III.1 of [GGH].

PROPOSITION 2.3. *Let X, $D_{r,m}$ and δ be as described above and let $r > 0$ and $m > 0$. Then*

$$h^0(X, D_{r,m}) = \sum_{i=0}^{m} \binom{r+i\delta+2}{2} - \sum_{i=0}^{m-1} \binom{r+i\delta-d_1+2}{2}.$$

Let $\chi: X \longrightarrow \mathbf{P}^\lambda$ be the morphism given by the sections of $D_{r,m}$, where, by Proposition 2.3,

$$\lambda = \sum_{i=0}^{m} \binom{r+i\delta+2}{2} - \sum_{i=0}^{m-1} \binom{r+i\delta-d_1+2}{2} - 1.$$

We also have the linear inclusion $\iota: \mathbf{P}^\lambda \hookrightarrow \mathbf{P}^N$ corresponding to $\mathfrak{K}[\mathbf{P}^N] \longrightarrow \mathfrak{K}[\mathbf{P}^\lambda]$ defined by $X_{sijk} \longmapsto \pi^*(V_{sijk})$, under the identification of linear forms on $\mathbf{P}^\lambda$ with sections of $D_{r,m}$.

PROPOSITION 2.4. *The closure of the image of ϕ is isomorphic to X.*

PROOF. Let $U = \pi^{-1}(\mathbf{P}^2 - \{p_1, \ldots, p_\rho\})$. By Proposition 2.1 the divisor $D_{r,m}$ is very ample if $r > 0$ and $m > 0$. Thus the map $\iota \circ \chi: X \longrightarrow \mathbf{P}^N$ is an isomorphism to its image. By Lemma 2.2 , we see that $\phi\pi$ and ι_χ have the same restriction to U. So $X \cong \iota_\chi(X) \cong \mathrm{clos}(\mathrm{im}\,\phi)$. ∎

In particular, $\phi|_U$ extends to an embedding $\phi': X \longrightarrow \mathbf{P}^N$, and we have $\phi' = \iota_\chi$. The reader may find the following geometric perspective helpful.

The image $\phi'(X)$ spans the subspace $L = \iota(\mathbf{P}^\lambda)$ of $\mathbf{P}^N$; L is a proper subspace when $d_1 \leq r + (m-1)\delta$ (as we see by comparing (†) with Proposition 2.3).

Our ultimate goal is to give generators for the homogeneous ideal I_X defining $\chi(X) \subset \mathbf{P}^\lambda$, but it is easier to work with the ideal I'_X defining $\phi'(X) \subset \mathbf{P}^N$, and derive our results for I_X from studying I'_X, using the fact that I_X is a quotient of I'_X by $N - \lambda$ linear forms.

3. **The Generators.** Before we get to the details of our main argument, we would like to give a brief overview of our approach.

Consider the morphism $\phi': X \longrightarrow \mathbf{P}^N$. We have the corresponding rational map $\mathbf{P}^2 \xrightarrow{\phi} \mathbf{P}^N$ which on coordinate rings factors as $S \xrightarrow{\psi} T \xrightarrow{\alpha} \mathfrak{K}[\mathbf{P}^2]$, where $T = \mathfrak{K}[a:b, w_1:w_2:w_3]$ and ψ maps X_{sijk} to $a^{m-s}b^s w_1^i w_2^j w_3^k$ where $i+j+k = r + s\delta$, $0 \leq s \leq m$ and α maps a to f, b to g, and w_i to w_i for $i = 1, 2, 3$. $\ker(\alpha \circ \psi)$ is the ideal we are looking for.

Note that T is bigraded, with

$$T = \mathfrak{K}[a:b, w_1:w_2:w_3] = \bigoplus_{0 \leq u,v} T_{u,v},$$

where $T_{u,v}$ is the $\mathfrak{K}$-span of the monomials of degree (u,v), where $\deg(a) = (1,\delta)$, $\deg(b) = (1,0)$ and $\deg(w_i) = (0,1)$ for $i = 1, 2, 3$. Note that in the case $d_1 = d_2$, the grading on T agrees with the standard grading on $\mathbf{P}^1 \times \mathbf{P}^2$ and the above factorization corresponds to factoring $\mathbf{P}^2 \xrightarrow{\phi} \mathbf{P}^N$ through $\mathbf{P}^1 \times \mathbf{P}^2$. We will show that generators coming from $\ker\psi$ are quadrics. The others map to S-module generators of $I_X / \ker\psi \cong \psi(I_X) \subset T$. To get a handle on these other generators we observe that the bihomogeneous elements of $\ker(\alpha)$ generate $\langle bf - ag\rangle$, which is isomorphic to T, shifted in degrees.

For example, $\psi(I_X) = \mathrm{im}(\psi) \cap (bf - ag)T$ which in the case $d_1 = d_2 = t$ is generated by the single component $(bf - ag)T_{mp-1,pr-t}$, where p is the least integer such that $pr \geq t$. But $T_{mp-1,pr-t}$ has $\mathfrak{K}$-dimension $mp\binom{pr-t+2}{2}$, and so one gets $mp\binom{pr-t+2}{2}$ forms of degree p as generators [Ho]. If moreover $r = m = 1$, this gives t forms of degree t, which agrees with the result of [GGH].

If $d_1 < d_2$, $\psi(I_X)$ is generated by various components of T and so generators coming from this are spread over various degrees.

Now we give the details of our argument.

Consider the morphism $\phi': X \longrightarrow \mathbf{P}^N$; recall that the hyperplane sections are precisely the sections of $D_{r,m}$. We get an exact sequence of sheaves of ideals

$$0 \longrightarrow \tilde{I_X} \longrightarrow \mathcal{O}_{\mathbf{P}^N} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Tensoring with $\mathcal{O}_{\mathbf{P}^N}(n)$ we get

$$0 \longrightarrow \tilde{I_X}(n) \longrightarrow \mathcal{O}_{\mathbf{P}^N}(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow 0.$$

Now taking cohomology and using the fact that $D_{r,m}$ is projectively normal (see Proposition 2.1) we get an exact sequence

$$0 \longrightarrow H^0\big(\mathbf{P}^N, \tilde{I_X}(n)\big) \longrightarrow H^0\big(\mathbf{P}^N, O_{\mathbf{P}^N}(n)\big) \longrightarrow H^0(X, nD_{r,m}) \longrightarrow 0.$$

Thus for $n > 0$ we get an exact sequence

$$(*) \qquad 0 \longrightarrow (I'_X)_n \longrightarrow S_n \xrightarrow{\phi'^*_n} H^0(X, nD_{r,m}) \longrightarrow 0.$$

Next let V_s denote $f^{m-s}g^s$. If $\mu = X^{n_1}_{s_1i_1j_1k_1}\cdots X^{n_p}_{s_pi_pj_pk_p}$ is a monomial of degree $n_1 + n_2 + \cdots + n_p = n$ in S_n then $\phi'^*_n(\mu) = V^{n_1}_{s_1}(w_1^{i_1}w_2^{j_1}w_3^{k_1})^{n_1}\cdots V^{n_p}_{s_p}(w_1^{i_p}w_2^{j_p}w_3^{k_p})^{n_p}$. Since $d = n_1s_1 + \cdots + n_ps_p$ is the power of g occurring in $V^{n_1}_{s_1}\cdots V^{n_p}_{s_p}$, we can factor the map ϕ'^*_n through $\bigoplus_{i=0}^{mn} R_{nr+i\delta}$ by defining $\psi_n \colon S_n \longrightarrow \bigoplus_{i=0}^{mn} R_{nr+i\delta}$ via

$$\psi_n(\mu) = 0 \oplus \cdots \oplus 0 \oplus (w_1^{i_1}w_2^{j_1}w_3^{k_1})^{n_1}\cdots(w_1^{i_p}w_2^{j_p}w_3^{k_p})^{n_p} \oplus 0 \oplus \cdots \oplus 0,$$

where the nonzero component is in the d-th place, and by defining

$$\alpha_n \colon \bigoplus_{i=0}^{mn} R_{nr+i\delta} \longrightarrow (I^{mn})_{n(r+md_2)}$$

via

$$\alpha_n(0 \oplus \cdots \oplus 0 \oplus h_d \oplus 0 \oplus \cdots \oplus 0) = h_d f^{mn-d} g^d.$$

Now using the exact sequence $(*)$ and the maps ψ_n and α_n we construct for $n > 0$ the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (I'_X)_n & \longrightarrow & S_n & \xrightarrow{\phi'^*_n} & H^0(X, nD_{r,m}) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \xi_n} & & \\ 0 & \longrightarrow & [\bigoplus_{i=0}^{mn-1} R(-\eta_i)]_\nu & \xrightarrow{\beta_n} & [\bigoplus_{i=0}^{mn} R(-\tau_i)]_\nu & \xrightarrow{\alpha_n} & (I^{mn})_\nu & \longrightarrow & 0 \end{array}$$

DIAGRAM 3.1

where η_i denotes $(nmd_2 - i\delta + d_1)$, ν denotes $n(r + md_2)$, and τ_i denotes $(nmd_2 - i\delta)$. In the above diagram define β_n via

$$\beta_n(h_0 \oplus \cdots \oplus h_{mn-1}) = -gh_0 \oplus (fh_0 - gh_1) \oplus \cdots \oplus (fh_{mn-2} - gh_{mn-1}) \oplus fh_{mn-1}.$$

LEMMA 3.2. *Diagram 3.1 is commutative with exact rows.*

PROOF. The top row is exact as observed above $(*)$. To check that the bottom row is exact, first note that $\alpha_n\beta_n = 0$, so $\operatorname{im}(\beta_n) \subset \ker(\alpha_n)$. To see $\ker(\alpha_n) \subset \operatorname{im}(\beta_n)$, let $h_0 \oplus \cdots \oplus h_{mn} \in \ker(\alpha_n)$; *i.e.*, $\sum_{i=0}^{mn} h_i f^{mn-i} g^i = 0$. Now g divides $\sum_{i=1}^{mn} h_i f^{mn-i} g^i$. So g divides $h_0 f^{mn}$, which implies g divides h_0. Thus $h_0 = gk_0$ for some $k_0 \in R$. But $\beta_n(k_0 \oplus 0 \oplus \cdots \oplus 0) = -gk_0 \oplus fk_0 \oplus 0 \oplus \cdots \oplus 0 = -h_0 \oplus fk_0 \oplus 0 \oplus \cdots \oplus 0$. Thus modulo $\operatorname{im}(\beta_n)$, denoting $fk_0 + h_1$ by h'_1, we can replace $h_0 \oplus \cdots \oplus h_{mn}$ by $0 \oplus h'_1 \oplus h_2 \oplus \cdots \oplus h_{mn}$.

Arguing as above, there exists $k_1 \in R$ such that $h_1' = gk_1$ and so, modulo $\mathrm{im}(\beta_n)$, denoting $fk_1 + h_2$ by h_2', we can replace $0 \oplus h_1' \oplus h_2 \oplus \cdots \oplus h_{mn}$ by $0 \oplus 0 \oplus h_2' \oplus h_3 \oplus \cdots \oplus h_{mn}$. In this way we eventually obtain $0 \oplus \cdots \oplus 0 \oplus h_{mn}' \in \ker(\alpha_n)$, which is only possible if $h_{mn}' = 0$. *I.e.*, modulo $\mathrm{im}(\beta_n)$, $h_0 \oplus \cdots \oplus h_{mn} \equiv 0$, so $h_0 \oplus \cdots \oplus h_{mn} \in \mathrm{im}(\beta_n)$, as claimed.

Next, to see $\xi_n \phi_n'^{*} = \alpha_n \psi_n$, note that for appropriate a, b, c, and d (where the nonzero component of $0 \oplus \cdots \oplus {w_1}^a {w_2}^b {w_3}^c \oplus \cdots \oplus 0$ occurs in the d-th place) we have

$$\begin{aligned}
\alpha_n \psi_n\Big(\prod (X_{sijk})^{u_{sijk}}\Big) &= \alpha_n(0 \oplus \cdots \oplus {w_1}^a {w_2}^b {w_3}^c \oplus \cdots \oplus 0) \\
&= {w_1}^a {w_2}^b {w_3}^c f^{mn-d} g^d \\
&= \prod (f^{m-s} g^s {w_1}^a {w_2}^b {w_3}^c) \\
&= \xi_n \phi_n'^{*}\Big(\prod (X_{sijk})^{u_{sijk}}\Big),
\end{aligned}$$

where products are taken over $i+j+k = r+s\delta$ and $0 \leq s \leq m$. This completes the proof of the lemma. ∎

We now make a definition.

DEFINITION 3.3. *Let $\mu = X_{s_1 i_1 j_1 k_1} \cdots X_{s_p i_p j_p k_p} \in S$. We shall say that μ is lexicographically minimal if the factors $X_{s_l i_l j_l k_l}$ can be reordered so that*

(i) $s_\gamma > 0$ implies $s_l = m$ for all $l > \gamma$,

(ii) $i_\gamma > 0$ implies $j_l = k_l = 0$ for all $l < \gamma$,

(iii) $j_\gamma > 0$ implies $k_l = 0$ for all $l < \gamma$,

where $1 \leq l, \gamma \leq p$.

REMARKS 3.4. (1) To clarify the point of the above definition, consider an analogy between a monomial $X_{s_1 i_1 j_1 k_1} \ldots X_{s_p i_p j_p k_p}$ and a row of p boxes, the l-th box containing m cubes ($m - s_l$ being white, and s_l being black) and $r + s_l\delta$ balls (i_l being blue, j_l being green, and k_l being red). It is clear that by swapping cubes in adjacent boxes we can eventually force the white cubes to be as much as possible in the leftward boxes, and the black cubes as much as possible at the right, always maintaining m cubes in each box. In particular, we can force there to be at most one box with cubes of both colors, with all boxes to its left (if any) having only white cubes, and those to its right having only black cubes. Likewise, swapping balls in adjacent boxes can be done to eventually move the balls so that the blues are as far left as possible and the reds are as far right as possible. Having in this way the whites and blues at the left, and the blacks and reds at the right precisely means the corresponding monomial is lexicographically minimal.

(2) Note that the map ψ_n can be interpreted using the above analogy. The w_i correspond to the balls and s corresponds to the number of black cubes. The evaluation by ψ_n then precisely means lumping the contents of the boxes together.

(3) We will say that two monomials of the same degree u and u' are *equivalent* if u can be transformed into u' by operations corresponding to these swaps. Then it is easy to see that there is a unique lexicographically minimal monomial in each class, and that the classes are precisely the monomials with the same image under ψ.

(4) Let M_p denote the set of lexicographically minimal monomials in S_p. We now count the number of elements in M_p. In terms of the analogy given above in (1), there is

a unique lexicographically minimal arrangement of cubes and balls with given numbers of cubes and balls of each color. In the notation above, there are $r + s_l\delta$ balls in the l-th box, so there are $pr + s\delta$ balls altogether, where $s = s_1 + \cdots + s_p$ is the number of black cubes. Thus the number of arrangments with s black cubes is the number of ways to apportion $pr + s\delta$ balls among the colors blue, green and red, which is well-known to be $\binom{pr+s\delta+2}{2}$. Summing over s gives the number $\sum_{s=0}^{mp}\binom{pr+s\delta+2}{2}$ of elements in M_p.

Next to find generators of $\ker(\psi_n)$, we need a lemma.

LEMMA 3.5. *The set of lexicographically minimal monomials in S_n, M_n maps bijectively to a basis of* $[\bigoplus_{s=0}^{mn} R(-nmd_2 + s\delta)]_{n(r+md_2)}$.

PROOF. For simplicity of notation let us denote $[\bigoplus_{s=0}^{mn} R(-nmd_2 + s\delta)]_{n(r+md_2)}$ by T. Let $0\oplus\cdots\oplus w_1^i w_2^j w_3^k\oplus\cdots\oplus 0$ be a basis element of T, where the only nonzero component is at the σ-th position, and $i+j+k = nr+\sigma\delta$. Consider a monomial $\mu = X_{s_1 i_1 j_1 k_1}\cdots X_{s_p i_p j_p k_p}$ in S such that following conditions hold.

(i) $s_1 + s_2 + \cdots + s_p = mn - \sigma$ and if $s_\gamma > 0$ then $s_l = m$ for all $l > \gamma$. In other words, if $(l-1)m \leq \sigma < lm$ then $s_\eta = 0$ for all $\eta = 1, 2, \cdots, l-1$, $s_l = ml - \sigma$, $s_\eta = m$ for all $\eta = l+1, \cdots, n$ and if $\sigma = mn$ then $s_l = 0$ for all l.

(ii) $i_1+\cdots+i_p = i, j_1+\cdots+j_p = j, k_1+\cdots+k_p = k$ such that if $i_\gamma > 0$ then $j_l = k_l = 0$ for all $l < \gamma$, and if $j_\gamma > 0$ then $k_l = 0$ for all $l < \gamma$, $1 \leq l, \gamma \leq p$.

Then by construction μ is lexicographically minimal, and one checks that

$$\psi_n(\mu) = 0 \oplus \cdots \oplus w_1^j w_2^j w_3^k \oplus \cdots \oplus 0.$$

This proves ψ_n is surjective. Also, by Remark 3.4(4) the number of elements in M_n is $\sum_{s=0}^{mn}\binom{nr+s\delta+2}{2}$, which is the dimension of the image T. Thus M_n maps bijectively to a basis of T. ∎

LEMMA 3.6. *Let J be the ideal of S generated by* $\ker(\psi_2)$. *Then J is generated by q quadrics where*

$$q = \binom{2+N}{N} - \sum_{s=0}^{2m}\binom{2r+s\delta+2}{2},$$

where $N = \sum_{s=0}^{m}\binom{r+s\delta+2}{2} - 1$, *and* $\ker(\psi_n) \subset J$, *for all* $n \geq 0$.

PROOF. Clearly J is generated by q quadrics, where $q = \dim\ker(\psi_2)$. To see that

$$\dim\ker(\psi_2) = \binom{2+N}{N} - \sum_{s=0}^{2m}\binom{2r+s\delta+2}{2}$$

consider the short exact sequence

$$0 \longrightarrow \ker(\psi_2) \longrightarrow S_2 \xrightarrow{\psi_2} \bigoplus_{s=0}^{2m} R_{2r+s\delta} \longrightarrow 0.$$

Then $\dim\ker(\psi_2) = \dim(S_2) - \dim(\bigoplus_{s=0}^{2m} R_{2r+s\delta}) = \binom{2+N}{N} - \sum_{s=0}^{2m}\binom{2r+s\delta+2}{2}$.

Next note that there is a bijection between coordinates X_{sijk} on $\mathbf{P}^N$ and the elements $V_{sijk} = f^{m-s}g^s w_1{}^i w_2{}^j w_3{}^k$, where $i+j+k = r+s\delta$ and $0 \le s \le m$. Let V_s denote $f^{m-s}g^s$. If $\mu = X^{n_1}_{s_1 i_1 j_1 k_1} \cdots X^{n_p}_{s_p i_p j_p k_p}$ is a monomial of degree $n_1+n_2+\cdots+n_p = n$ on $\mathbf{P}^N$ then

$$\psi_n^*(\mu) = V_{s_1}^{n_1}(w_1{}^{i_1}w_2{}^{j_1}w_3{}^{k_1})^{n_1}\cdots V_{s_p}^{n_p}(w_1{}^{i_p}w_2{}^{j_p}w_3{}^{k_p})^{n_p},$$

where $\psi_n{}^*$ is a map from S_n to $\mathfrak{K}[w_1,w_2,w_3,f,g]$ such that f and g are treated as variables in $\mathfrak{K}[w_1,w_2,w_3,f,g]$ and $\nu = n_1(m-s_1)+\cdots+n_p(m-s_p)$ is the power of f occurring in $V_{s_1}^{n_1}\cdots V_{s_p}^{n_p}$. Thus

$$\psi_n(\mu) = 0\oplus\cdots\oplus(w_1{}^{i_1}w_2{}^{j_1}w_3{}^{k_1})^{n_1}\cdots(w_1{}^{i_p}w_2{}^{j_p}w_3{}^{k_p})^{n_p}\oplus\cdots\oplus 0$$

which is in $[\bigoplus_{i=0}^{mn} R(-nmd_2+i\delta)]_{n(r+md_2)}$, where the nonzero component is in the ν-th place.

By Lemma 3.5 we know that M_n maps bijectively to a basis of

$$\left[\bigoplus_{i=0}^{mn} R(-nmd_2+i\delta)\right]_{n(r+md_2)},$$

and so ψ_n is surjetive. Thus for every s, $0 \le s \le mn$, and for any monomial μ' in $[\bigoplus_{i=0}^{mn} R(-nmd_2+i\delta)]_{n(r+md_2)} = \bigoplus_{i=0}^{mn} R_{nr+i\delta}$, there is a unique monomial μ in M_n such that $\psi(\mu) = 0\oplus\cdots\oplus\mu'\oplus 0\oplus\cdots\oplus 0$, where the nonzero component occurs in the ν-th position. From uniqueness we see that $S_n = \ker(\psi_n)\oplus\langle M_n\rangle$, where $\langle M_n\rangle$ denotes the span of M_n.

Now, as noted in Remark 3.4 (3), we see that if u is any monomial in S_n, then there exist monomials $u_1,\cdots,u_l$ in S_n such that $u = u_1$ is equivalent to u_l and u_l is lexicographically minimal. Here $u_{\gamma+1}$ is obtained from u_γ by the operation corresponding to a single swap of objects in corresponding adjacent boxes for $\gamma = 1,2,\cdots,l-1$. We claim that $u_\gamma - u_{\gamma+1}$ is in J. The proof of the claim is essentially writing out an algebraic interpretation of a single swap mentioned in Remark 3.4 (1). Write u_γ as $X_{s_1 i_1 j_1 k_1}\cdots X_{s_p i_p j_p k_p}$ such that $s_1 \le s_2 \le \cdots \le s_p$. Now transform the adjacent factors $X_{s_\delta i_\delta j_\delta k_\delta}X_{s_{\delta+1} i_{\delta+1} j_{\delta+1} k_{\delta+1}}$ into $X_{s'_\delta i'_\delta j'_\delta k'_\delta}X_{s'_{\delta+1} i'_{\delta+1} j'_{\delta+1} k'_{\delta+1}}$ in the following way. If $s_\delta = 0$ then $s'_\delta = s_\delta$ and $s'_{\delta+1} = s_{\delta+1}$, and if $s_\delta > 0$, and $s_\delta + s_{\delta+1} \le m$ then $s'_\delta = 0$ and $s'_{\delta+1} = s_\delta + s_{\delta+1}$, and if $s_\delta > 0$, and $s_\delta + s_{\delta+1} > m$ then $s'_\delta = s_\delta + s_{\delta+1} - m$ and $s'_{\delta+1} = m$, and $i_\delta + i_{\delta+1} = i'_\delta + i'_{\delta+1}$, $j_\delta + j_{\delta+1} = j'_\delta + j'_{\delta+1}$, $k_\delta + k_{\delta+1} = k'_\delta + k'_{\delta+1}$, such that if $i'_{\delta+1} > 0$ then $j'_\delta = k'_\delta = 0$, and if $j'_{\delta+1} > 0$ then $k'_{\delta+1} = 0$. Let $u_{\gamma+1}$ be $X_{s_1 i_1 j_1 k_1}\cdots X_{s_{\delta-1} i_{\delta-1} j_{\delta-1} k_{\delta-1}} X_{s'_\delta i'_\delta j'_\delta k'_\delta} X_{s'_{\delta+1} i'_{\delta+1} j'_{\delta+1} k'_{\delta+1}}\cdots X_{s_p i_p j_p k_p}$. Then note that $u_\gamma - u_{\gamma+1} \in J$.

Now if for a monomial μ we denote by μ' the lexicographically minimal monomial with the same image under ϕ'^*, then the set $\{\mu - \mu' \mid \mu \text{ monomial in } S_n\}$ is a basis of $\ker(\psi_n)$. The result follows, since we have already shown $\mu - \mu' \in J$. ■

To state the main theorem, we need some notation. Let p_1 be the least integer such that $p_1 r + (mp_1 - 1)\delta \ge d_1$, and let $j_1 = mp_1 - 1$. For $k \ge 2$, inductively define j_k as the largest integer such that $p_{k-1}r + j_k\delta < d_1$, and p_k as the least integer such that $p_k r + j_k\delta \ge d_1$. The procedure stops at that $k = t$ for which p_t is the least integer with $p_t r \ge d_1$. Note that $p_1 < p_2 < \cdots < p_t$. Also let

$$\lambda = \sum_{i=0}^{m}\binom{r+i\delta+2}{2} - \sum_{i=0}^{m-1}\binom{r+i\delta-d_1+2}{2} - 1.$$

Now we can state the main theorem.

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THEOREM 3.7. *With λ, t, p_k and j_k as above, the ideal I'_X is generated by q' quadrics and, for each $k = 1, \cdots, t$ such that $p_k \neq 2$, by σ_k forms of degree p_k, where*

$$q' = \binom{2+\lambda}{\lambda} - \sum_{s=0}^{2m} \binom{2r+s\delta+2}{2} + \sum_{s=0}^{2m-1} \binom{2r+s\delta-d_1+2}{2},$$

and

$$\sigma_k = \sum_{i=0}^{j_k} \binom{p_k r + i\delta - d_1 + 2}{2}.$$

PROOF. We continue to use notation introduced for Lemma 3.6. We shall denote

$$\binom{2+N}{2} - \sum_{s=0}^{2m} \binom{2r+s\delta+2}{2}$$

by q, where $N = \sum_{s=0}^{m} \binom{r+s\delta+2}{2}$-1.

Consider Diagram 3.1. By Lemma 3.2 this diagram is commutative with exact rows, so $(I'_X)_n = \ker(\phi_n'^{*}) = \psi_n{}^{-1}\big(\mathrm{im}(\beta_n)\big)$.

Next note that $[\bigoplus_{i=0}^{mn-1} R(-nmd_2 + i\delta - d_1)]_{n(r+md_2)} = 0$ if $n < p_1$. So for $n < p_1$, $(I'_X)_n = \ker(\psi_n)$. Therefore by Lemma 3.6 $(I'_X)_n = J_n$ for $n < p_1$; *i.e.*, q quadrics account for everything when $n < p_1$.

For $n = p_1$,

$$(I'_X)_n / J_n \cong \left[\bigoplus_{i=0}^{mn-1} R(-nmd_2 + i\delta - d_1)\right]_{n(r+md_2)} = \bigoplus_{i=0}^{mp_1-1} R_{p_1 r + i\delta - d_1},$$

which has dimension

$$\sigma_1 = \sum_{i=0}^{mp_1-1} \binom{p_1 r + i\delta - d_1 + 2}{2}.$$

Thus we require σ_1 forms of degree p_1. Let J'_1 be the ideal generated by these forms. For $n > p_1$,

$$(I'_X)_n / (J_n + J'_{1n}) \cong \left[\bigoplus_{i=0}^{j_2} R(-nmd_2 + i\delta - d_1)\right]_{n(r+md_2)} = \bigoplus_{i=0}^{j_2} R_{nr+i\delta-d_1},$$

where j_2 is the largest integer such that $p_1 r + j_2\delta - d_1 < 0$. If $n < p_2$ then $\bigoplus_{i=0}^{j_2} R_{nr+i\delta-d_1} = 0$. Thus for $n < p_2$, q quadrics and σ_1 forms of degree p_1 account for everything. For $n = p_2$,

$$(I'_X)_n / (J_n + J'_{1n}) \cong \left[\bigoplus_{i=0}^{j_2} R(-nmd_2 + i\delta - d_1)\right]_{n(r+md_2)} = \bigoplus_{i=0}^{j_2} R_{p_2 r + i\delta - d_1},$$

which has dimension

$$\sigma_2 = \sum_{i=0}^{j_2} \binom{p_2 r + i\delta - d_1 + 2}{2}.$$

Thus we require σ_2 forms of degree p_2.

Inductively one checks that we require σ_k forms of degree p_k, for $k = 1, 2, \cdots, t$, where p_t is the least integer such that $p_t r \geq d_1$.

To show these suffice, note that for $n > p_t$, $\mathrm{im}(\beta_n)$ is spanned by things of the form $h_\gamma(0, \cdots, 0, -g, f, 0, \cdots, 0)$ where $h_\gamma \in R_{nr+\gamma\delta-d_1}$, and only nonzero components occur at the γ-th and $(\gamma + 1)$-th places.

If $j_1 < \gamma \leq mn - 1$, then write $h_\gamma = \sum_{j=1}^{n_\gamma} u_{\gamma j} v_{\gamma j}$, where $u_{\gamma j} \in R_{nr-p_1 r}$ and $v_{\gamma j} \in R_{p_1 r+\gamma\delta-d_1}$.

If $j_2 < \gamma \leq j_1$, then write $h_\gamma = \sum_{j=1}^{n_\gamma} u_{\gamma j} v_{\gamma j}$, where $u_{\gamma j} \in R_{nr-p_2 r}$ and $v_{\gamma j} \in R_{p_2 r+\gamma\delta-d_1}$. and so on.

Thus $\mathrm{im}(\beta_n)$ is spanned by things of the form

$$h_\gamma(0, \cdots, 0, -g, f, 0, \cdots, 0) = \sum_j u_{\gamma j}(0, \cdots, 0, -g v_{\gamma j}, f v_{\gamma j}, 0, \cdots, 0).$$

But the right hand side lies in the image under ψ_n of $S_{n-p_l}(I'_X)_{p_l}$ when $j_l < \gamma \leq j_{l-1}$.

Thus we know that I'_X is generated by the q quadrics and σ_k forms of degree p_k, $1 \leq k \leq t$.

We consider three cases.

First, if $p_1 > 2$ then we are done. Note that in this case $\lambda = N$ and

$$\sum_{s=0}^{2m-1} \binom{2r + s\delta - d_1 + 2}{2} = 0.$$

Hence $q' = q$. Thus I'_X is generated by q' quadrics and σ_k forms of degree p_k.

Secondly, if $p_1 = 2$ then $\lambda = N$. In this case I'_X is generated by the

$$q + \sum_{s=0}^{2m-1} \binom{2r + s\delta - d_1 + 2}{2}$$

quadrics and for each $k = 2, \cdots, t$ such that $p_k \neq 2$, by σ_k forms of degree p_k. Note that since $\lambda = N$,

$$q + \sum_{s=0}^{2m-1} \binom{2r + s\delta - d_1 + 2}{2} = q'.$$

Thus in this case we are done.

Finally, if $p_1 = 1$ then I'_X is minimally generated by $\sum_{s=0}^{m-1} \binom{r+s\delta-d_1+2}{2}$ linear forms,

$$\begin{aligned}\left[h^0\big(\mathbf{P}^N, \mathcal{O}(2)\big) - h^0(X, 2D_{r,m})\right] - \left[h^0\big(\mathbf{P}^N, \mathcal{O}(2)\big) - h^0\big(\mathbf{P}^\lambda, \mathcal{O}(2)\big)\right] \\ = h^0\big(\mathbf{P}^\lambda, \mathcal{O}(2)\big) - h^0(X, 2D_{r,m})\end{aligned}$$

quadrics, and for each $k = 2, \cdots, t$ such that $p_k \neq 2$, by σ_k forms of degree p_k.

Now by Proposition 2.3 we see that

$$h^0\big(\mathbf{P}^\lambda, \mathcal{O}(2)\big) - h^0(X, 2D_{r,m}) = q'.$$

Thus we get the result. ■

COROLLARY 3.8. *Let $\chi: X \hookrightarrow \mathbf{P}^\lambda$ be the embedding given by the global sections of $D_{r,m}$, where*

$$\lambda = \sum_{i=0}^{m} \binom{r+i\delta+2}{2} - \sum_{i=0}^{m-1} \binom{r+i\delta-d_1+2}{2} - 1.$$

Let I_X be the ideal defining X in $\mathbf{P}^\lambda$. Let t, p_k and j_k be as in Theorem 3.7. Then I_X is generated by q' quadrics, where

$$q' = \binom{2+\lambda}{\lambda} - \sum_{s=0}^{2m} \binom{2r+s\delta+2}{2} + \sum_{s=0}^{2m-1} \binom{2r+s\delta-d_1+2}{2},$$

and, for each $k = 1, 2, \cdots, t$, such that $p_k \geq 3$ in addition by σ_k forms of degree p_k, where

$$\sigma_k = \sum_{i=0}^{j_k} \binom{p_k r + i\delta - d_1 + 2}{2}.$$

PROOF. If $r + (m-1)\delta < d_1$ then by (†) and Proposition 2.3, the embedding χ coincides with the embedding ϕ'. So in this case the result follows from Theorem 3.7.

If $r + (m-j-1)\delta < d_1 \leq r + (m-j)\delta$ then the ideal I'_X defining X in $\mathbf{P}^N$ where $N = \sum_{s=0}^{m} \binom{r+s\delta+2}{2} - 1$ is generated by $\sigma_1 = \sum_{i=0}^{j} \binom{r+(m-i)\delta-d_1+2}{2}$ linear forms, q' quadrics and σ_k forms of degree $p_k, p_k \geq 3$. The σ_1 linear forms define a linear subspace $L \cong \mathbf{P}^\lambda \subset \mathbf{P}^N$. Thus the generators for the ideal I'_X modulo σ_1 linear forms give the generators for the ideal I_X of X in $\mathbf{P}^\lambda$. Thus I_X is generated by q' quadrics, and if $p_k \geq 3$ for $k = 1, 2, \cdots, t$, in addition by σ_k forms of degree p_k. ∎

REFERENCES

[GG] A. Geramita and A. Gimigliano, *Generators for the defining ideals of certain rational surfaces*, Duke Math J. **62**(1991), 61–83.

[GGH] A. Geramita, A. Gimigliano and B. Harbourne, *Projectively normal but superabundant embedding of rational surfaces in projective space*, J. Algebra **169**(1994), 791–804.

[GGP] A. Geramita, A. Gimigliano and Y. Pitteloud, *Graded Betti numbers of some embedded rational n-folds*, Math. Ann. **301**(1995), 363–380.

[Gi] A. Gimigliano, *On Veronesean surfaces*, Proc. Konin. Ned. Akda. van Wetenschappen, Ser. A **92**(1989), 71–85.

[GL] M. L. Green, and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Compositio Math. **67**(1988), 301–314.

[Ha] R. Hartshorne, Algebraic Geometry, Grad texts in Math, **52**, Springer-Verlag, New York, 1977.

[Ho] S. H. Holay, *Generators and resolutions of ideals defining certain surfaces in projective space*, Ph.D. Thesis, University of Nebraska-Lincoln, 1994.

[ZS] O. Zariski and P. Samuel, Commutative Algebra, Vol. I, II, Van Nostrand, Princeton, 1958, 1960.

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