Canad. Math. Bull. Vol. 63 (3), 2020 pp. 610–623
http://dx.doi.org/10.4153/S0008439519000699

Separable Quotients of Free Topological Groups

Arkady Leiderman and Mikhail Tkachenko

Abstract. We study the following problem: For which Tychonoff spaces X do the free topological group $F(X)$ and the free abelian topological group $A(X)$ admit a quotient homomorphism onto a separable and nontrivial (*i.e.*, not finitely generated) group? The existence of the required quotient homomorphisms is established for several important classes of spaces X, which include the class of pseudocompact spaces, the class of locally compact spaces, the class of σ-compact spaces, the class of connected locally connected spaces, and some others.

We also show that there exists an infinite separable precompact topological abelian group G such that every quotient of G is either the one-point group or contains a dense non-separable subgroup and, hence, does not have a countable network.

1 Introduction

The famous, still open, problem of Banach–Mazur asks whether every infinite-dimensional Banach space has an infinite-dimensional separable quotient Banach space. Similar problems have been studied for various classes of topological vector spaces and, in particular, for spaces of continuous functions with the pointwise convergence topology [2, 5]. This line of research has been continued in [7], where the following general problem has been investigated in the class of topological groups.

Problem 1.1 *Characterize topological groups that have an infinite separable topological quotient group.*

Let us mention several positive results obtained there for the topological groups more general than compact groups.

Theorem 1.2 ([7]) *Let G be an infinite σ-compact topological group. Then G has an infinite quotient group with a countable network (hence the quotient group is separable).*

Theorem 1.3 ([7]) *Let G be an infinite pseudocompact topological group. Then G has an infinite quotient group that is compact and metrizable.*

It turns out that Theorem 1.3 cannot be extended to precompact groups.

Received by the editors May 31, 2019; revised November 18, 2019.
Published online on Cambridge Core November 29, 2019.
Author M. T. gratefully acknowledges the financial support received from the Center for Advanced Studies in Mathematics of the Ben Gurion University of the Negev during his visit in May, 2019.
AMS subject classification: **22A05**, 54D65, 54C10.
Keywords: free topological group, quotient, separable.

Theorem 1.4 ([7]) *There exists an uncountable dense subgroup G of the compact abelian group* $\mathbb{T}^{\mathfrak{c}}$ *satisfying* $\dim G = 0$ *such that every quotient group of G is either the one-element group or non-separable.*

Free topological groups constitute a prominent class of topological groups — every topological group G is a quotient group of a free topological group, namely, $F(X)$, where X is a space homeomorphic to G. The group $F(X)$ contains X as a subspace that generates $F(X)$ algebraically and is characterized by the property that every continuous mapping of X to a topological group H extends to a continuous homomorphism of $F(X)$ to H (see [8] or [1, Section 7.1]). In the category of topological abelian groups, a similar object is known as the free abelian topological group, which is denoted by $A(X)$. Our definition of free topological groups follows Markov's approach [8].

In this article we consider the special case of Problem 1.1 for the free (abelian) topological groups.

Problem 1.5 *Is it true that for any infinite Tychonoff space X, the free topological group* $F(X)$ *and the free abelian topological group* $A(X)$ *have nontrivial separable quotient groups?*

Throughout the paper all topological spaces under consideration are assumed to be Tychonoff. It is known that $A(X)$ is a quotient of $F(X)$ (see [3] or [1, Theorem 7.1.11]), so for our purposes it suffices to find a separable quotient group for the free abelian group $A(X)$. To shorten arguments, we frequently denote the groups $F(X)$ and $A(X)$ by $G(X)$.

It is known that the groups $G(X)$ are not locally compact for any non-discrete space X [9, Corollary 2]. Also, since X is closed in $G(X)$, the group $G(X)$ is σ-compact if and only if X is σ-compact. Therefore, according to Theorem 1.2, $F(X)$ and $A(X)$ each has an infinite quotient group with a countable network for every nonempty σ-compact space X. The latter conclusion, however, is trivial, as explained below.

Quotient groups provided by Theorem 1.2 can be finitely generated. It turns out that for every nonempty space X, the group $A(X)$ (hence, also $F(X)$) has the discrete group of integers $\mathbb{Z}$ as a quotient group. Indeed, for every non-empty Tychonoff space X, we can take the constant mapping f from X to $\mathbb{Z}$ defined by $f(x) = 1$ for each $x \in X$. Then f admits an extension to a continuous homomorphism $f^*\colon A(X) \to \mathbb{Z}$, which is open by [1, Corollary 7.1.10] since the mapping f is evidently quotient. Hence, the discrete group $\mathbb{Z}$ is always a quotient of $A(X)$. Observe that due to this fact the groups $A(X)$ and $F(X)$ are not pseudocompact for any nonempty space X.

To exclude this trivial quotient of $G(X)$, by *nontrivial* groups we always mean the groups that are *not finitely generated*.

Improving upon Theorem 1.2, one of the main results of our paper, Theorem 2.10, says that for every infinite σ-compact space, the groups $G(X)$ have a nontrivial quotient group with a countable network. In Corollary 3.5, we extend this conclusion to the more general case of an infinite space X with a dense σ-compact subspace.

In Section 3, we show that if a dense subgroup H of a topological group G has an infinite separable quotient group, then so has G, and a similar conclusion remains valid for nontrivial separable quotients provided the group H is abelian (see Proposition 3.1). We apply the latter result in Example 3.6 to describe a technique for construction of some topological groups with separable quotients that are not covered by the results in Section 2.

Our main tool for finding separable quotients of free topological groups is continuous *R-quotient* mappings of Tychonoff spaces. Given a space X, we try to find an R-quotient mapping $f: X \to Y$ onto an infinite separable space Y. Once this is done, we extend f to a continuous surjective homomorphism $f^*: G(X) \to G(Y)$. By Theorem 2.1, the homomorphism f^* is open, while the group $G(Y)$ is separable, since it contains the countable dense subgroup generated by a countable dense subset of Y. We do not know, however, whether every infinite Tychonoff space admits an R-quotient mapping onto an infinite separable Tychonoff space or even onto an infinite subspace of the unit interval $[0,1]$ (see Problem 2.18). The existence of the required mapping $f: X \to Y$ is established for several important classes of spaces X that include the class of pseudocompact spaces (Corollary 2.5), the class of locally compact spaces (Lemma 2.8), the class of Lindelöf Σ-spaces (Lemma 2.11), the class of connected locally connected spaces (Lemma 2.16), and some others.

In fact, in all aforementioned cases, we show that there exists a quotient homomorphism from $G(X)$ onto a nontrivial $G(Y)$ such that Y, and hence $G(Y)$, has a countable network.

The following question arises naturally.

Problem 1.6 *Does there exist a separable space X such that free abelian topological group $A(X)$ does not admit a nontrivial quotient with a countable network?*

While Problem 1.6 remained unsolved, we prove that there exists a separable precompact topological abelian group G such that every quotient of G is either the one-point group or contains a dense non-separable subgroup and, hence, does not have a countable network (Theorem 4.1). In Sections 2, 3, and 4 we pose other open problems related to quotients of free (abelian) topological groups.

Several results of this article have been announced (without proofs) in the survey paper [6].

2 Classes of X for which $G(X)$ Admits a Nontrivial Separable Quotient Group

A continuous onto mapping $\varphi: X \to Y$ is said to be *R-quotient* [4] if for every real-valued function f on Y, the composition $f \circ \varphi$ is continuous if and only if f is continuous. Clearly, every quotient mapping is R-quotient, but the converse is false.

Let $\varphi: X \to Y$ be a continuous onto mapping, where the space Y is Tychonoff. Then Y admits the finest topology, say, σ such that the mapping $\varphi: X \to (Y, \sigma)$ is R-quotient. The topology σ of Y is initial with respect to the family of real-valued functions f on Y such that the composition $f \circ \varphi$ is continuous. It is easy to see that the space (Y, σ) is also Tychonoff and that σ is finer than the original topology of Y. We say that σ is the *R-quotient topology* on Y (with respect to φ). Notice that the mapping $\varphi: X \to (Y, \sigma)$ remains continuous.

https://doi.org/10.4153/S0008439519000699 Published online by Cambridge University Press

The next result proved by Okunev in [10, Proposition 1.8] explains the powerful role of R-quotient mappings when studying quotients of free topological groups.

Theorem 2.1 *Let X and Y be Tychonoff spaces and let $\varphi: X \to Y$ be a continuous surjective mapping. Denote by $\varphi^*: F(X) \to F(Y)$ the extension of φ to a continuous homomorphism of the free topological groups on X and Y, respectively. Then φ^* is open if and only if the mapping φ is R-quotient. The same conclusion remains valid for free abelian topological groups.*

Sometimes it is useful to represent a topological group G as a quotient group of the free topological group on G (see [1, Corollary 7.1.10]).

Lemma 2.2 *Let G be a Hausdorff topological group. Then G is a quotient group of the free topological group $F(G)$. If G is abelian, then G is a quotient group of the free abelian topological group $A(G)$.*

In the following fact we present a fundamental property of the free topological groups which generalizes [1, Corollary 7.1.10].

Proposition 2.3 *If a Tychonoff space X admits an R-quotient mapping onto a topological (abelian) group G, then $F(X)$ (resp., $A(X)$) admits an open continuous homomorphism onto G.*

Proof Let $\varphi: X \to G$ be an R-quotient mapping of X onto G. By Theorem 2.1, the extension of φ to a continuous homomorphism $\varphi^*: F(X) \to F(G)$ is open. It follows from Lemma 2.2 that there exists an open continuous homomorphism j of $F(G)$ onto G. Hence, the composition $j \circ \varphi^*$ is an open continuous homomorphism of $F(X)$ onto the group G. If G is abelian, a similar argument shows that G is a quotient group of $A(X)$. ∎

A Tychonoff space X is called *pseudocompact* if every continuous real-valued function on X is bounded.

Lemma 2.4 *Every continuous mapping of a pseudocompact space X onto a Tychonoff space Y of countable pseudocharacter is R-quotient.*

Proof Let $\varphi: X \to Y$ be a continuous mapping of X onto Y. Denote by τ_Y the topology of Y and let σ be the R-quotient topology on Y with respect to φ. Then the topology σ is Tychonoff and $\tau_Y \subset \sigma$. Clearly, the mapping $\varphi: X \to (Y, \sigma)$ is also continuous, so the space (Y, σ) is pseudocompact. In particular, Y is bounded in (Y, σ).

Consider the identity mapping $j: (Y, \sigma) \to (Y, \tau_Y)$. Since every point y in (Y, τ_Y) is the intersection of a countable family of closed neighborhoods of y, we can apply [12, Corollary 2.2] to conclude that j is a homeomorphism. Hence, $\sigma = \tau_Y$ and the mapping φ is R-quotient. ∎

Corollary 2.5 *Every continuous mapping of a pseudocompact space onto a first countable Tychonoff space is R-quotient.*

Lemma 2.6 *Let U be an open subset of a Tychonoff space X and let A be an infinite subset of U. There exists a continuous real-valued function f on X such that $f(A)$ is infinite and $f(X\setminus U) = \{0\}$.*

Proof Let $F = X\setminus U$. We can assume without loss of generality that $F \neq \varnothing$. Since the set $A \subset U$ is infinite, we can find a point $x_0 \in A$ and an open neighborhood V_0 of x_0 in X such that $\overline{V_0} \subset U$ and $A_1 = A\setminus\overline{V_0}$ is infinite. Again, we can find a point $x_1 \in A_1$ and an open neighborhood V_1 of x_1 in X such that $\overline{V_1} \subset U$, $\overline{V_1}$ is disjoint from $F \cup \overline{V_0}$ and the complement $A_1\setminus\overline{V_1}$ is infinite. Continuing in this way we obtain a sequence $\{x_n : n \in \omega\} \subset A$ and a pairwise disjoint sequence $\{V_n : n \in \omega\}$ of open sets in X, where $x_n \in V_n$ and $\overline{V_n} \subset U$ for each $n \in \omega$.

For every $n \in \omega$, take a non-negative continuous real-valued function f_n on X bounded by 2^{-n} such that $f_n(x_n) = 2^{-n}$ and $f_n(X\setminus V_n) = \{0\}$. Then the function $f = \sum_{n=0}^{\infty} f_n$ is continuous, $f(x_n) = 2^{-n}$, and $f(X\setminus U) = \{0\}$. Hence, $f(A)$ is infinite. ■

One can now apply the technique of R-quotient mappings described in the introduction and deduce the following result.

Theorem 2.7 *Let X be an infinite pseudocompact space. Then the groups $G(X)$ admit an open continuous homomorphism onto $A(K)$, where K is an infinite compact subspace of the closed unit interval $[0,1]$. In particular, $A(X)$ and $F(X)$ admit an open continuous homomorphism onto a nontrivial group with a countable network.*

Proof It suffices to consider the case $G(X) = A(X)$. Take a countably infinite subset A of X. By Lemma 2.6, there exists a continuous function f on X with values in $[0,1]$ such that $f(A)$ is infinite. Hence, $K = f(X)$ is infinite and pseudocompact as a continuous image of the pseudocompact space X. Therefore, K is a compact subset of $[0,1]$ and by Lemma 2.4, f is an R-quotient mapping of X onto K. According to Theorem 2.1, the mapping f extends to an open continuous homomorphism of $A(X)$ onto $A(K)$. Clearly, the group $A(K)$ is nontrivial and has a countable network. ■

Similarly to pseudocompact spaces, locally compact spaces admit "good" mappings onto compact subsets of the real line.

Lemma 2.8 *Every non-discrete locally compact space admits a continuous closed mapping onto an infinite compact subspace of the closed unit interval $[0,1]$.*

Proof Let X be a non-discrete locally compact space and let x_0 be a non-isolated point of X. Take open neighborhoods U and V of x_0 such that the closure of U is compact and $\overline{V} \subset U$. By Lemma 2.6, there exists a continuous real-valued function f on X such that $f(V)$ is infinite and $f(X\setminus U) = \{0\}$. Then $f(X) = f(\overline{U}) \cup \{0\}$ is an infinite compact subset of the real line. If F is a closed subset of X, then $f(F) = f(F \cap \overline{U}) \cup f(F \cap (X\setminus U))$ is a closed subset of $\mathbb{R}$, since $\overline{U}$ is compact and $f(F \cap (X\setminus U)) \subset \{0\}$. Hence, f is a closed mapping. ■

Theorem 2.9 *Let X be an infinite locally compact space.*

(i) *If X is discrete, then the group $G(X)$ admits an open continuous homomorphism onto the discrete nontrivial group $A(\mathbb{N})$.*

(ii) *If X is non-discrete, then the group $G(X)$ admits an open continuous homomorphism onto $A(K)$, where K is an infinite compact subspace of the closed unit interval $[0,1]$.*

In both cases, $A(X)$ and $F(X)$ admit an open continuous homomorphism onto a nontrivial group with a countable network.

Proof Again, it suffices to prove the conclusion for $A(X)$. If X is discrete, there exists a continuous mapping g of X onto the countable infinite discrete space $\mathbb{N}$. We extend g to a continuous homomorphism $g^*: A(X) \to A(\mathbb{N})$. So g^* is an open surjective homomorphism onto a nontrivial countable (hence, separable) discrete group $A(\mathbb{N})$.

Assume that X is non-discrete. By Lemma 2.8, there exists a closed continuous mapping $f: X \to K$ onto an infinite compact metrizable space K. Clearly, f is a R-quotient mapping. Hence, the continuous homomorphic extension of f to the homomorphism $f^*: A(X) \to A(K)$ is open by Theorem 2.1. Evidently the group $A(K)$ is nontrivial. Finally, K has a countable base, so $A(K)$ has a countable network. ∎

A similar argument is used to prove the following statement.

Theorem 2.10 *Let X be an infinite σ-compact space. Then the group $G(X)$ admits an open continuous homomorphism onto $A(Y)$, where the infinite space Y is a countable union of subspaces homeomorphic to compact subsets of the closed unit interval $[0,1]$. In particular, $G(X)$ admits an open continuous homomorphism onto a nontrivial group with a countable network.*

Proof Again it suffices to consider the case $G(X) = A(X)$. Let $X = \bigcup_{n\in\omega} C_n$, where each C_n is a compact subset of X. By Lemma 2.6, X admits a continuous mapping f to the closed unit interval $[0,1]$ such that the image $f(X)$ is infinite. Let τ be a topology on $f(X)$ such that the mapping $f: X \to (f(X), \tau)$ is R-quotient. Clearly, τ is finer than the topology of $f(X)$ inherited from $[0,1]$ and that the space $Y = (f(X), \tau)$ is Tychonoff. For every $n \in \omega$, consider the compact subspace $K_n = f(C_n)$ of Y. Then each K_n admits a continuous one-to-one mapping to $[0,1]$, so K_n is homeomorphic to a subspace of $[0,1]$. Since $Y = \bigcup_{n\in\omega} K_n$, the space Y has a countable network.

Extend f to a continuous onto homomorphism $f^*: A(X) \to A(Y)$. It follows from Theorem 2.1 that the homomorphism f^* is open. Also, the group $A(Y)$ is algebraically generated by Y and, hence, has a countable network. Since Y is infinite, the group $A(Y)$ is nontrivial. ∎

In fact, the last assertion can be generalized to the topological spaces X, which are Lindelöf Σ-spaces. Recall that the important class of Lindelöf Σ-spaces contains all σ-compact and all separable metrizable topological spaces and is closed with respect to countable products, closed subgroups and continuous images. Again, we start with a lemma about R-quotient mappings.

Lemma 2.11 *Every infinite Lindelöf Σ-space X admits an R-quotient mapping onto an infinite space with a countable network.*

Proof Fix a continuous mapping $f: X \to [0,1]$ such that the image $f(X)$ is infinite. Let τ be a topology on $f(X)$ such that the mapping $f: X \to (f(X), \tau)$ is R-quotient. We have that τ is finer than the topology of $f(X)$ inherited from $[0,1]$ and the space $Y = (f(X), \tau)$ is an infinite Lindelöf Σ-space. The space Y admits a continuous one-to-one mapping to $[0,1]$, hence, by [1, Proposition 5.3.15], we conclude that Y has a countable network. ∎

Since the free (abelian) topological group of a space with a countable network has a countable network, the following result is immediate from Lemma 2.11.

Theorem 2.12 *Let X be an infinite Lindelöf Σ-space. Then the groups $F(X)$ and $A(X)$ admit an open continuous homomorphism onto a nontrivial group with a countable network.*

Remark 2.13 One might ask whether the technique used in the proof of Theorem 2.12 can be applied for any infinite Lindelöf space X. Unfortunately, this is not the case. The reason is that there exists an infinite Lindelöf space Y that admits a continuous one-to-one mapping into $[0,1]$, but Y is not even separable. Indeed, there is a scattered separable σ-compact space X such that the function space $Y = C_p(X, D)$ has the following properties: Y^n is Lindelöf for every finite power n, but the countable power Y^ω is not Lindelöf [11]. Here D denotes the discrete space consisting of two points $\{0,1\}$. Then Y admits a continuous one-to-one mapping into the Cantor set $D^\omega \subset [0,1]$, since X is separable. However, Y cannot be separable, otherwise all compact subsets of X would be metrizable and X would have a countable network, which is not true, since Y^ω is not Lindelöf. So, we do not know if Theorem 2.12 remains true for all Lindelöf spaces X.

Lemma 2.14 *If a space X contains infinitely many clopen sets, then X contains an infinite disjoint family of clopen sets.*

Proof Let γ be an infinite family of nonempty clopen sets in X. Denote by γ^* the minimal by inclusion family of nonempty clopen sets in X which contains γ and is closed under finite intersections, finite unions and complements. Clearly, γ^* is countably infinite and each element of γ^* is a clopen set in X.

For every $U \in \gamma^*$, we put

$$\lambda_U = \{V \in \gamma^* : U \cap V = \varnothing\} \quad \text{and} \quad \delta_U = \{V \in \gamma^* : V \subset U\}.$$

Note that if $U, V \in \gamma^*$, then $V = (U \cap V) \cup (V \backslash U)$, where $U \cap V \in \delta_U \cup \{\varnothing\}$ and $V \backslash U \in \lambda_U \cup \{\varnothing\}$. Therefore, for each $U \in \gamma^*$, either γ_U or δ_U is infinite.

Take an arbitrary element $U \in \gamma^*$ with $U \neq X$. If λ_U is infinite, we put $U_0 = U$, otherwise let $U_0 = X \backslash U$. In either case, we have that $U_0 \in \gamma^*$ and the family λ_{U_0} is infinite. Let us assume that we have defined pairwise disjoint elements $U_0, \ldots, U_n$ of γ^* such that λ_W is infinite, where $W = \bigcup_{k=0}^n U_k$. Note that $W \in \gamma^*$. Again, we take an arbitrary element $U \in \lambda_W$ and argue as at the initial step of our construction to define an element $U_{n+1} \in \lambda_W$ such that the family $\lambda_W \cap \lambda_{U_{n+1}}$ is infinite. In fact, one defines U_{n+1} to be either U or $X \backslash (U \cup W)$ depending on the number of elements of λ_W disjoint from U. Then $U_0, \ldots, U_n, U_{n+1}$ are pairwise disjoint elements of γ^* and the family $\lambda_{W \cup U_{n+1}} = \lambda_W \cap \lambda_{U_{n+1}}$ is infinite.

Continuing this way, we obtain an infinite pairwise disjoint family $\{U_n : n \in \omega\}$ of nonempty clopen sets in X. ∎

Proposition 2.15 *Let a space X contain infinitely many clopen sets. Then the group $G(X)$ has a nontrivial countable quotient group.*

Proof By Lemma 2.14, one can find a family $\gamma = \{U_n : n \in \omega\}$ of pairwise disjoint nonempty clopen sets in X. Let $F = X \setminus \bigcup_{n\in\omega} U_n$. Diminishing the family γ, if necessary, we can assume that $F \neq \emptyset$. Fix a countable set $Y = \{y_n : n \in \omega\}$, where $y_n \neq y_m$ for distinct $m, n \in \omega$. We define a surjective mapping $\varphi\colon X \to Y$ by letting $\varphi(F) = \{y_0\}$ and $\varphi(U_n) = \{y_{n+1}\}$ for each $n \in \omega$. Denote by τ the topology on Y such that the mapping $\varphi\colon X \to (Y, \tau)$ turns to be R-quotient. It is easy to see that continuous real-valued functions separate points of Y. Indeed, let m and n be distinct non-negative integers, $m < n$. Also, let f be the real-valued function on Y defined by $f(y_n) = 1$ and $f(y_k) = 0$ if $k \neq n$. Then $f \circ \varphi$ is a continuous real-valued function on X, so f is continuous on (Y, τ), and we have that $1 = f(y_n) \neq f(y_m) = 0$. Therefore, a countable space $Y = (Y, \tau)$ is Tychonoff.

Now we extend φ to a continuous homomorphism φ^* of $A(X)$ onto $A(Y)$. Theorem 2.1 implies that the homomorphism φ^* is open. ∎

Proposition 2.15 applies only to the class of non-connected spaces. Now we deal with some class of connected spaces. First, we need an auxiliary fact on continuous real-valued functions defined on connected locally connected spaces.

Lemma 2.16 *Let $f\colon X \to \mathbb{R}$ be a continuous function on a connected locally connected space X. Then f is hereditarily quotient.*

Proof There is nothing to prove if f is constant, so we can assume that $|f(X)| > 1$. Let U be an open neighborhood of the fiber $f^{-1}(r)$, for some $r \in f(X)$. Let us show that the image $f(U)$ contains a neighborhood of r in $f(X)$.

Case 1. There exists a point $x_0 \in f^{-1}(r)$ such that for every neighborhood V of x_0 in X, the sets $f(V) \cap (-\infty, a)$ and $f(V) \cap (a, \infty)$ are nonempty. Since X is locally connected, there exists a connected neighborhood V_0 of x_0 with $V_0 \subset U$. It follows from our assumption about x_0 that V_0 contains elements x and y with $f(x) < a < f(y)$. Since the set $f(V_0)$ is connected, the image $f(U)$ contains the interval $(f(x), f(y))$, as required.

Case 2. Every point $x \in f^{-1}(a)$ has a neighborhood V_x such that either $f(V_x) \subset [a, \infty)$ or $f(V_x) \subset (-\infty, a]$. Clearly, we can choose an open connected neighborhood O_x of x satisfying $O_x \subset V_x \cap U$. Then $U^* = \bigcup\{O_x : x \in f^{-1}(a)\}$ is an open neighborhood of $f^{-1}(a)$ contained in U. Again, if both sets $f(U^*) \cap (-\infty, a)$ and $f(U^*) \cap (a, \infty)$ are nonempty, we take points $x, y \in f^{-1}(a)$ and elements $x' \in O_x$ and $y' \in O_y$ such that $f(x') < a$ and $f(y') > a$. Since the sets $f(O_x)$ and $f(O_y)$ are connected, we have that $[a, f(y')) \subset f(O_y)$ and $(f(x'), a] \subset f(O_x)$. Therefore, the image $f(U)$ contains the open neighborhood $(f(x'), f(y'))$ of a.

Finally, assume that either $f(U^*) \subset [a,\infty)$ or $f(U^*) \subset (-\infty,a]$. It suffices to consider the first of the two cases. Then $f^{-1}(a) \subset U \subset f^{-1}(a) \cup f^{-1}(a,\infty)$. Notice that the open sets $f^{-1}(-\infty,a)$ and $U^* \cup f^{-1}(a,\infty)$ are disjoint and cover X. By the connectedness of X, we conclude that $f(X) \subset [a,\infty)$. In particular, $f(O_x) \subset [a,\infty)$ for each $x \in f^{-1}(a)$. If f is constant on U^*, then $U^* = f^{-1}(a)$ is a clopen subset of X, which is possible only if $X = U^*$ or, equivalently, $f(X) = \{a\}$. The latter contradicts our assumption that $|f(X)| > 1$. Hence, f is not constant on at least one of the sets O_x, with $x \in f^{-1}(a)$. Since $f(O_x) \subset [a,\infty)$, we can find $y \in O_x$ with $f(y) > a$. Hence, $[a, f(y))$ is an open neighborhood of a in $f(X)$. This completes the proof. ∎

Combining Theorem 2.1, Lemma 2.6 and Lemma 2.16, we obtain the next result.

Theorem 2.17 *Let X be a connected and locally connected space with $|X| > 1$. Then the group $G(X)$ has a nontrivial quotient group with a countable network.*

We finish the section with a problem that reduces Problem 1.5 to a purely topological question.

Problem 2.18 *Is it true that every infinite Tychonoff space X admits an R-quotient mapping onto an infinite Tychonoff space with a countable network?*

In view of Proposition 2.3, the affirmative answer to Problem 2.18 would resolve both Problems 1.5 and 1.6.

3 Extending Classes of Groups with Separable Quotients

The following fact permits us to extend several results of Section 2 to wider classes of topological groups.

Proposition 3.1 *Let H be a dense subgroup of a topological group G.*

(i) *If H has an infinite separable quotient group, then so has G.*
(ii) *If H is abelian and has a nontrivial separable quotient group, then so has G.*

Proof Let N be a closed normal subgroup of H such that the quotient group H/N is infinite and separable. We denote by K the closure of N in G. It is clear that K is a closed normal subgroup of G. Hence, we can consider the quotient homomorphism $p\colon G \to G/K$. Since $N = H \cap K$ is dense in K, one can identify the group H/N, algebraically and topologically, with the subgroup $p(H)$ of G/K. It follows from the density of $p(H)$ in G/K that the group G/K is infinite and separable. This implies (i) of the proposition.

If H is abelian and the quotient group $H/N \cong p(H)$ is nontrivial and separable, the quotient group G/K is also nontrivial, since the groups G and G/K are abelian and every subgroup of a finitely generated abelian group is finitely generated. ∎

Proposition 3.3 complements [7, Proposition 4.2] in the case of *nontrivial commutative* Lindelöf Σ-groups (we recall that *nontrivial* means not finitely generated). First we recall an important property of Lindelöf Σ-groups established in [1, Section 5.3] and reformulated in [7, Lemma 4.1].

https://doi.org/10.4153/S0008439519000699 Published online by Cambridge University Press

Lemma 3.2 *Let G be a Lindelöf Σ-group. Then*

(i) *For every closed normal subgroup N of type G_δ in G, the quotient group G/N has a countable network.*
(ii) *The family $\{f^{-1}(V) : \pi$ is a continuous homomorphism of G to a topological group K with a countable network, V is open in $K\}$ constitutes a base for G.*

Let $\mathbb{P}$ be the set of prime numbers. In the proof of the following proposition, we use the so-called *torsion-free rank*, $r_0(G)$, and *p-rank*, $r_p(G)$ for $p \in \mathbb{P}$, of an abstract abelian group G defined as follows. The torsion-free rank of G is the cardinality of a maximal independent subset S of G that consists of elements of infinite order. Similarly, the p-rank of G is the cardinality of a maximal independent subset S consisting of elements of order p. In both cases, the definition of $r_0(G)$ and $r_p(G)$ does not depend on the choice of the set S (see [1, Section 9.9]).

Proposition 3.3 *Let G be a nontrivial commutative Lindelöf Σ-group. Then G has a nontrivial quotient group with a countable network.*

Proof The conclusion of the proposition is trivial if G is countable. We assume, therefore, that $|G| > \omega$. It follows from Lemma 3.2(ii) that the topology of G is initial with respect to the family of quotient homomorphisms onto topological groups with a countable network. Note that the identity of a topological group with a countable network is a G_δ-set (in fact, the singletons in every Hausdorff space with a countable network are G_δ-sets). Hence, Lemma 3.2(ii) implies that every neighborhood of the identity in G contains a closed normal subgroup of type G_δ in G. Therefore, for every countable subgroup A of G, there exists a closed normal subgroup N of G of type G_δ such that the restriction to A of the quotient homomorphism $\pi_N: G \to G/N$ is a monomorphism. Hence, the groups A and $\pi_N(A)$ are isomorphic.

It follows from [1, Proposition 9.9.20] that $|G| = r_0(G) + \sum_{p\in\mathbb{P}} r_p(G)$, where $\mathbb{P}$ is the set of prime numbers. Since G is uncountable, we have that either $r_0(G) > \omega$ or $r_p(G) > \omega$, for some prime p. Let S be a countable independent subset of G consisting of elements of infinite order in the first case, and a countable independent subset of elements of order p in the second case. Denote by A the subgroup of G generated by S. Since A is countable, there exists a closed invariant subgroup N of G such that the groups A and $\pi_N(A)$ are isomorphic. It follows from $r_0(G/N) \geq r_0(\pi_N(A)) = |S| = \omega$ in the first case and $r_p(G/N) \geq r_p(\pi_N(A)) = |S| = \omega$ in the second case that the rank of the group G/N is infinite. Hence, G/N cannot be finitely generated. Thus, the quotient group G/N is nontrivial and has a countable network, by Lemma 3.2(i). ∎

Problem 3.4 *Is Proposition 3.3 valid in the non-commutative case?*

Corollary 3.5 *Let an infinite Tychonoff space X contain a dense Lindelöf Σ-subspace (in particular, a dense σ-compact subspace). Then $G(X)$ admits a quotient group that is nontrivial and separable.*

Proof Let Y be a dense Lindelöf Σ-subspace of X. There exists a continuous real-valued function f on X such that the image $Z = f(X)$ is infinite. Then $f(Y)$ is dense

https://doi.org/10.4153/S0008439519000699 Published online by Cambridge University Press

in $f(X)$ and, hence, infinite. Let τ be the topology on $f(X)$ such that the mapping $f: X \to (Z, \tau)$ becomes R-quotient. Clearly, τ is finer than the topology of Z inherited from the real line, so the space (Z, τ) is Tychonoff. The image $Y' = f(Y)$ considered as a subspace of (Z, τ) is a Lindelöf Σ-space and admits a continuous one-to-one mapping onto a subspace of the real line, so the space Y' has a countable network by [1, Proposition 5.3.15]. In particular, the spaces Y' and $Z^* = (Z, \tau)$ are separable. Hence, the group $G(Z^*)$ is separable as well. Since the mapping $f: X \to Z^*$ is R-quotient, its extension to a continuous homomorphism $f^*: G(X) \to G(Z^*)$ is open by Theorem 2.1. Thus, the group $G(X)$ has a nontrivial separable quotient group, namely, $G(Z^*)$. ■

In the next example we show how Corollary 3.5 applies to widen the class of spaces X for which the groups $F(X)$ and $A(X)$ have separable quotients.

Example 3.6 Let $\mathbb{Q}$ be the space of rational numbers considered as a subspace of the real line $\mathbb{R}$. Also let $\sigma\mathbb{Q}^\tau$ be the σ-product of $\tau > \omega$ copies of $\mathbb{Q}$ considered as a subspace of $\mathbb{Q}^\tau$. Then $\sigma\mathbb{Q}^\tau$ is a dense σ-compact subspace of $\mathbb{Q}^\tau$ (see [1, Proposition 1.6.41]), which is in turn a dense subspace of $\mathbb{R}^\tau$. Take an arbitrary space X with $\sigma\mathbb{Q}^\tau \subset X \subset \mathbb{R}^\tau$. Then X can fail to be σ-compact, while the projections of X to countable subproducts $\mathbb{R}^A$ with $A \subset \tau$ need not be open or even quotient. However, Corollary 3.5 implies that the groups $A(X)$ and $F(X)$ admit an open continuous homomorphism onto a nontrivial separable topological group.

4 A Separable Group that has no Quotients with a Countable Network

The reader has surely noticed that all separable quotients of free (abelian) topological groups found in Sections 2 and 3 (besides perhaps Corollary 3.5) have a countable network or even are countable. This can tempt one to conjecture that every topological group with a nontrivial separable quotient has a nontrivial quotient with a countable network. In the next theorem we present an example of a separable precompact abelian topological group G such that every quotient of G either is the one-element group or has no countable network.

Theorem 4.1 *There exists an infinite separable precompact topological abelian group G such that every quotient of G either is the one-element group or contains a dense non-separable subgroup and, hence, does not have a countable network.*

Proof Let H be a dense uncountable subgroup of $\mathbb{T}^{\mathfrak{c}}$ constructed in [7, Theorem 3.5]. Then all countable subgroups of H are closed in H and h-embedded. We claim that there exists an element $b \in \mathbb{T}^{\mathfrak{c}}$ such that the cyclic subgroup $\langle b \rangle$ of $\mathbb{T}^{\mathfrak{c}}$ generated by b is dense in $\mathbb{T}^{\mathfrak{c}}$ and satisfies

$$\langle b_\alpha \rangle \cap \Big\langle \bigcup_{\nu \le \alpha} \pi_\nu(H) \cup \{b_\nu : \nu < \alpha\} \Big\rangle = \{1\} \quad \text{for each } \alpha \in \mathfrak{c}. \tag{4.1}$$

Here, π_α is the projection of $\mathbb{T}^{\mathfrak{c}}$ onto the αth factor $\mathbb{T}_{(\alpha)}$ and $b_\alpha = \pi_\alpha(b)$.

https://doi.org/10.4153/S0008439519000699 Published online by Cambridge University Press

The construction of H in [7] starts with choosing an independent subset E of $\mathbb{T}$ of elements of infinite order satisfying $|E| = \mathfrak{c}$ and partitioning it into $\mathfrak{c}$ pairwise disjoint parts E_α, $\alpha \in \mathfrak{c}$, where each part E_α is of cardinality $\mathfrak{c}$. Then one considers the free abelian group $A(X)$ on a set X of cardinality $\mathfrak{c}$ and defines a family $\{f_\alpha : \alpha \in \mathfrak{c}\}$ of homomorphisms of $A(X)$ to $\mathbb{T}$. The group H is then defined as the subgroup $f(A(X))$ of $\mathbb{T}^{\mathfrak{c}}$, where f is the diagonal product of the family $\{f_\alpha : \alpha \in \mathfrak{c}\}$. To be more precise, for every $\alpha \in \mathfrak{c}$, we also have a countable subset Y_α of X and its complement $Z_\alpha = X \setminus Y_\alpha$. The homomorphisms f_α satisfy several conditions of which we need to mention only one (we enumerate it as in [7]):

(ii) $f_\alpha \restriction Z_\alpha$ is a bijection of Z_α onto a subset of E_α.

We start defining b as follows. Since the group $\langle f_0(Y_0)\rangle$ is countable, there exists a countable set $F_0 \subset E$ such that $\langle E\rangle \cap \langle f_0(Y_0)\rangle \subset \langle F_0\rangle$. Then we choose $b_0 \in E_1 \setminus F_0$. It is easy to see that $\langle b_0\rangle \cap f_0(A(X)) = \{1\}$. Indeed, otherwise (ii) implies that for some integer $m \neq 0$,

$$b_0^m \in f_0(A(X)) = \langle f_0(Y_0) \cup f_0(Z_0)\rangle \subseteq \langle f_0(Y_0) \cup E_0\rangle = \langle f_0(Y_0)\rangle + \langle E_0\rangle.$$

Take $a \in \langle f_0(Y_0)\rangle$ and $c \in \langle E_0\rangle$ such that $b_0^m = a \cdot c$. Then $a = b_0^m \cdot c^{-1} \in \langle E_1\rangle\langle E_0\rangle \subset \langle E\rangle$, which in turn implies that $a \in \langle f_0(Y_0)\rangle \cap \langle E\rangle \subset \langle F_0\rangle$. Hence, $b_0^m = a \cdot c \in \langle F_0\rangle\langle E_0\rangle = \langle F_0 \cup E_0\rangle$. Since $(F_0 \cup E_0) \cap (E_1 \setminus F_0) = \varnothing$ and the set E is independent, the latter contradicts our choice $b_0 \in E_1 \setminus F_0$.

Assume that for some $\alpha \in \mathfrak{c}$, we have define independent elements $b_\nu \in \mathbb{T}$ with $\nu < \alpha$, where each $b_\nu \in E_{\nu+1}$ has infinite order. The group

$$\langle E\rangle \cap \Big\langle \{b_\nu : \nu < \alpha\} \cup \bigcup_{\nu \le \alpha} f_\nu(Y_\nu)\Big\rangle$$

has cardinality less than $\mathfrak{c}$, so we can find a set $F_\alpha \subset E$ with $|F_\alpha| < \mathfrak{c}$ such that

$$\langle E\rangle \cap \Big\langle \{b_\nu : \nu < \alpha\} \cup \bigcup_{\nu \le \alpha} f_\nu(Y_\nu)\Big\rangle \subset \langle F_\alpha\rangle.$$

Then we choose $b_\alpha \in E_{\alpha+1} \setminus F_\alpha$. Similarly to the first step of our construction, one can verify that

$$\langle b_\alpha\rangle \cap \Big\langle \bigcup_{\nu \le \alpha} f_\nu(A(X)) \cup \{b_\nu : \nu < \alpha\}\Big\rangle = \{1\} \tag{4.2}$$

holds at the step α. This completes the construction.

Since the set E is independent, so is $\{b_\alpha : \alpha \in \mathfrak{c}\}$. We define $b \in \mathbb{T}^{\mathfrak{c}}$ by letting $\pi_\alpha(b) = b_\alpha$ for each $\alpha \in \mathfrak{c}$. Since the set $\{b_\alpha : \alpha \in \mathfrak{c}\}$ is independent, the cyclic subgroup $\langle b\rangle$ is dense in $\mathbb{T}^{\mathfrak{c}}$. It is also clear that $f_\alpha(A(X)) = \pi_\alpha(H)$, so condition (4.1) follows from (4.2).

We define a group G with the required properties as the subgroup $H + \langle b\rangle$ of the compact group $\mathbb{T}^{\mathfrak{c}}$. Hence, G is an abelian precompact topological group. Clearly, G is dense in $\mathbb{T}^{\mathfrak{c}}$. It follows from (4.1) that the groups $\langle b_0\rangle$ and $\pi_0(H)$ have trivial intersection. Hence, the same is valid for $\langle b\rangle$ and H, so the group G is algebraically isomorphic to $H \oplus \langle b\rangle$.

Take a proper closed subgroup K of G and consider the quotient group G/K. Let $\pi_K : G \to G/K$ be the quotient homomorphism. Since $|G/K| > 1$, we can find a nontrivial continuous character χ on G/K. Then $\varphi = \chi \circ \pi_K$ is a nontrivial continuous character on G and, clearly, $K = \ker \pi_K \subset \ker \varphi$. We claim that $\ker \varphi \subseteq H$. Our proof

https://doi.org/10.4153/S0008439519000699 Published online by Cambridge University Press

of this fact is based on an (elementary) application of the Pontryagin–van Kampen duality.

Let $(\mathbb{T}^{\mathfrak{c}})^\wedge$ and $G^\wedge$ be the groups of continuous characters of $\mathbb{T}^{\mathfrak{c}}$ and G, respectively. It follows from Pontryagin's duality theorem for compact abelian groups that the group $(\mathbb{T}^{\mathfrak{c}})^\wedge$ is algebraically generated by the projections of $\mathbb{T}^{\mathfrak{c}}$ to the factors $\mathbb{T}_{(\alpha)}$, with $\alpha \in \mathfrak{c}$, considered as characters of $\mathbb{T}^{\mathfrak{c}}$ (see also [7, Lemma 3.1]). Since G is dense in $\mathbb{T}^{\mathfrak{c}}$, every continuous character of G extends to a continuous character of $\mathbb{T}^{\mathfrak{c}}$. Hence, the group $G^\wedge$ is algebraically generated by the elements $\pi_\alpha \restriction G$, $\alpha \in \mathfrak{c}$. Therefore, we can find pairwise distinct indices $\alpha_1, \dots, \alpha_k \in \mathfrak{c}$ and nonzero integers $n_1, \dots, n_k$ such that $\varphi(x) = \prod_{i=1}^k \pi_{\alpha_i}(x)^{n_i}$, for each $x \in G$. Take an arbitrary element $x \in G$ with $\varphi(x) = 1$. Then $x = hb^m$, for some $h \in H$ and $m \in \mathbb{Z}$. Let us show that $m = 0$ and, hence, $x \in H$. An easy calculation shows that

$$1 = \varphi(x) = \varphi(hb^m) = \prod_{i=1}^{k} h_{\alpha_i}^{n_i} \cdot \Big(\prod_{i=1}^{k} b_{\alpha_i}^{n_i} \Big)^m,$$

where $h_{\alpha_i} = \pi_{\alpha_i}(h)$, $i = 1, \dots, k$. We can assume that $\alpha_1 < \dots < \alpha_k$. Then

$$b_{\alpha_k}^{mn_k} = \Big(\prod_{i=1}^{k} h_{\alpha_i}^{n_i} \Big)^{-1} \cdot \Big(\prod_{i=1}^{k-1} b_{\alpha_i}^{n_i} \Big)^{-m},$$

which in turn implies that

$$b_{\alpha_k}^{mn_k} \in \Big\langle \bigcup_{\nu \le \alpha_k} \pi_\nu(H) \cup \{ b_\nu : \nu < \alpha_k \} \Big\rangle.$$

Since $n_k \neq 0$, property (4.1) implies that the latter is possible only if $m = 0$. Hence, $x = h \in H$, as claimed. We have thus proved that the kernel of φ is contained in H. Hence, $K \subset H \subset G$.

Finally, it follows from $K \subset H$ that $H = \pi_K^{-1}\pi_K(H)$. Hence, the restriction of π_K to H is an open continuous homomorphism of H onto the subgroup $\pi_K(H)$ of G/K and the groups $\pi_K(H)$ and H/K are topologically isomorphic. It is clear that $\pi_K(H)$ is dense in G/K and, hence, $|\pi_K(H)| > 1$. According to [7, Theorem 3.5], every quotient group of H is either the one-element group or non-separable. We conclude, therefore, that the subgroup $\pi_K(H) \cong H/K$ of G/K is not separable, so the groups $\pi_K(H)$ and G/K have no countable network. ∎

We conclude the article with the following unsolved question.

Problem 4.2 *Does there exist an uncountable Tychonoff space X such that every infinite separable quotient group of $A(X)$ is countable? What if X is the one-point compactification of an uncountable discrete space or the compact ordinal space $[0, \alpha]$ with $\alpha \ge \omega_1$?*

Acknowledgments The authors are grateful to the referee for an improvement in the formulation of Proposition 2.15 and several useful remarks.

References

[1] A. V. Arhangel'skii and M. G. Tkachenko, *Topological groups and related structures*. Atlantis Studies in Mathematics, 1, Atlantis Press, Paris; World Scientific, Hackensack, NJ, 2008. https://doi.org/10.2991/978-94-91216-35-0

[2] T. Banakh, J. Kąkol, and W. Śliwa, *Metrizable quotients of C_p-spaces*. Topol. Appl. 249(2018), 95–102. https://doi.org/10.1016/j.topol.2018.09.012
[3] M. I. Graev, *Free topological groups.* (Russian). Izvestiya Akad. Nauk SSSR Ser. Mat. 12(1948), 279–324. English translation in: Topology and Topological Algebra Translations Series 1, vol. 8 (1962), American Mathematical Society, pp. 305–364.
[4] S. M. Karnik and S. Willard, *Natural covers and r-quotient mappings*. Canad. Math. Bull. 25(1982), 456–461. https://doi.org/10.4153/CMB-1982-065-1
[5] J. Kąkol and W. Śliwa, *Efimov spaces and the separable quotient problem for spaces $C_p(X)$*. J. Math. Anal. Appl. 457(2018), 104–113. https://doi.org/10.1016/j.jmaa.2017.08.010
[6] A. G. Leiderman and S. A. Morris, *Separability of topological groups: a survey with open problems*. Axioms 8(2019), 1–18. https://doi.org/10.3390/axioms8010003
[7] A. G. Leiderman, S. A. Morris, and M. G. Tkachenko, *The separable quotient problem for topological groups*. Israel J. Math. 234(2019), no. 1, 331–369. https://doi.org/10.1007/s11856-019-1931-1
[8] A. A. Markov, *On free topological groups*. In: *Topology and topological algebra*, Translation Series 1, vol. 8, American Math. Society, Providence, RI, 1962, pp. 195–272. Russian original in: Izv. Akad. Nauk SSSR **9**(1945), 3–64.
[9] S. A. Morris and P. Nickolas, *Locally compact group topologies on algebraic free product of groups*. J. Algebra 38(1976), 393–397. https://doi.org/10.1016/0021-8693(76)90229-5
[10] O. G. Okunev, *A method for constructing examples of M-equivalent spaces*. Topol. Appl. 36(1990), 157–171; correction: Topol. Appl. **49**(1993), 191–192. https://doi.org/10.1016/0166-8641(93)90044-E
[11] O. Okunev and K. Tamano, *Lindelöf powers and products of function spaces*. Proc. Amer. Math. Soc. 124(1996), 2905–2916. https://doi.org/10.1090/S0002-9939-96-03629-5
[12] I. Sánchez and M. Tkachenko, *Products of bounded subsets of paratopological groups*. Topol. Appl. 190(2015), 42–58. https://doi.org/10.1016/j.topol.2015.03.017

Department of Mathematics, Ben-Gurion University of the Negev, Beer Sheva, P.O.B. 653, Israel
e-mail: arkady@math.bgu.ac.il

Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Del. Iztapalapa, C.P. 09340, Mexico City, Mexico
e-mail: mich@xanum.uam.mx