Canad. Math. Bull. Vol. 23 (2), 1980

A GENERALIZATION OF THE LAX-MILGRAM LEMMA

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Let H be a real Hilbert space with its dual space H'. The norm and inner product in H are denoted by $\|\cdot\|$ and $\langle . , . \rangle$ respectively. We denote by $\langle . , . \rangle$, the pairing between H' and H.

If $a(u, v)$ is a bilinear form and F is a real-valued continuous functional on H, then we consider $I[v]$, a functional defined by

$$I[v] = a(v, v) - 2F(v), \qquad \text{for all} \quad v \in H.$$

It has been shown by Noor and Whiteman [5], that under certain conditions on $a(u, v)$ and F, the minimum of $I(v)$ on H can be characterized by

$$(1) \qquad a(u, v) = \langle F'(u), v \rangle, \qquad \text{for all} \quad v \in H,$$

where $F'(u)$ is the Fréchet derivative of F at $u \in H$.

For a linear continuous functional F, solving equation (1) is equivalent to finding $u \in H$ such that

$$a(u, v) = \langle F, v \rangle, \qquad \text{for all} \quad v \in H,$$

and this is the well known Lax-Milgram lemma [2].

The motivation of this paper is to show that under certain conditions, there does exist a unique solution of a more general equation of which (1) is a special case. Our result can be considered as a representation theorem analogous to the Lax-Milgram lemma for a class of nonlinear problems.

DEFINITION 1. The operator $T: H \to H'$ is called *antimonotone*, if

$$\langle Tu - Tv, u - v \rangle \leq 0, \qquad \text{for all} \quad u, v \in H,$$

and is said to be *Lipschitz continuous*, if there exists a constant $\gamma > 0$ such that

$$\|Tu - Tv\| \leq \gamma \|u - v\|, \qquad \text{for all} \quad u, v \in H.$$

DEFINITION 2. A bilinear form $a(u, v)$ on H is said to be *coercive* [3] and *continuous*, if there exist constants $\rho > 0$, $\mu > 0$ such that

$$a(v, v) \geq \rho \|v\|^2, \qquad \text{for all} \quad v \in H,$$

Received by the editors October 25, 1978.

https://doi.org/10.4153/CMB-1980-024-8 Published online by Cambridge University Press

and

$$|a(u, v)| \leq \mu \|u\| \|v\|, \qquad \text{for all} \quad u, v \in H.$$

In particular, it follows that $\rho \leq \mu$, see [4]. If $a(u, v)$ is continuous coercive bilinear form, then by the Riesz-Fréchet representation theorem [1], we have

$$a(u, v) = \langle Tu, v \rangle, \qquad \text{for all} \quad v \in H.$$

It has been shown [4] that $\|T\| \leq \mu$. Finally, we define Λ, a canonical isomorphism from H' onto H by

$$\langle f, v \rangle = (\Lambda f, v), \qquad \text{for all} \quad v \in H, f \in H'. \tag{2}$$

Then $\|\Lambda\|_{H'} = \|\Lambda^{-1}\|_H = 1$.

We make the following hypothesis.

Condition N. We assume that $\gamma < \rho$, where γ is the Lipschitz constant of the nonlinear operator A and ρ is the coercivity constant.

We now state and prove the main result.

THEOREM 1. *Let $a(u, v)$ be a coercive continuous bilinear form and A is a Lipschitz continuous antimonotone operator. If condition N holds, then there exists a unique $u \in H$ such that*

$$a(u, v) = \langle A(u), v \rangle, \qquad \text{for all} \quad v \in H. \tag{3}$$

Moreover, if $a(u, v)$ is a symmetric positive bilinear form and $A(u) = F'(u)$, the Fréchet derivative of F at u, then solving (3) is equivalent to finding $\text{Min}_{v \in H} \{a(v, v) - 2F(v)\}$, *as shown in* [5].

We need the following lemma, which is essentially due to Noor [4]. We include its proof for the sake of completeness.

LEMMA 1. *Let ξ be a number such that $0 < \xi < 2(\rho - \gamma/\mu^2 - \gamma^2)$ and $\gamma\xi < 1$. Then there exists a θ with $0 < \theta < 1$ such that*

$$\|\Phi(u_1) - \Phi(u_2)\| \leq \theta \|u_1 - u_2\|, \qquad \textit{for all} \quad u_1, u_2 \in H,$$

where for $u \in H$, $\Phi(u) \in H'$ is defined by

$$\langle \Phi(u), v \rangle = (u, v) - \xi a(u, v) + \xi \langle A(u), v \rangle, \qquad \textit{for all} \quad v \in H. \tag{4}$$

Proof. For all $u_1, u_2 \in H$,

$$\langle \Phi(u_1) - \Phi(u_2), v \rangle = (u_1 - u_2, v) - \xi a(u_1 - u_2, v) + \xi \langle A(u_i) - A(u_2), v \rangle, \qquad \text{for all} \quad v \in H.$$

$$= (u_1 - u_2, v) - \xi \langle T(u_1 - u_2), v \rangle + \xi \langle A(u_1) - A(u_2), v \rangle$$

$$= (u_1 - u_2, v) - \xi (\Lambda T(u_1 - u_2), v) + \xi (\Lambda A(u_1) - \Lambda A(u_2), v), \text{ by (2).}$$

$$= (u_1 - u_2 - \xi \Lambda T(u_1 - u_2), v) + \xi (\Lambda A(u_1) - \Lambda A(u_2), v).$$

Thus

$$|\langle\Phi(u_1)-\Phi(u_2), v\rangle| \le \|u_1-u_2-\xi\Lambda T(u_1-u_2)\|\,\|v\|+\xi\,\|A(u_1)-A(u_2)\|\,\|v\|.$$

Now by $\|T\|\le\mu$ and the coercivity of $a(u, v)$, it follows that

$$\|u_1-u_2-\xi\Lambda T(u_1-u_2)\|^2 \le \|u_1-u_2\|^2+\xi^2\,\|T\|^2\,\|u_1-u_2\|^2-2\xi a(u_1-u_2, u_1-u_2),$$
$$\le(1+\xi^2\mu^2-2\xi\rho)\,\|u_1-u_2\|^2.$$

Hence

$$|\langle\Phi(u_1)-\Phi(u_2), v\rangle| \le (\sqrt{1+\xi^2\mu^2-2\xi\rho})\,\|u_1-u_2\|\,\|v\|+\xi\,\|A(u_1)-A(u_2)\|\,\|v\|,$$
$$\le\{(\sqrt{1+\xi^2\mu^2-2\xi\rho})+\xi\gamma\}\,\|u_1-u_2\|\,\|v\|,$$

by the Lipschitz continuity of A.

$$=\theta\,\|u_1-u_2\|\,\|v\|,$$

where $\theta=\sqrt{1+\xi^2\mu^2-2\xi\rho}+\gamma\xi<1$ for $0<\xi<2(\rho-\gamma/\mu^2-\gamma^2)$, and $\xi\gamma<1$, because $\gamma<\rho$ by condition N.

Thus for all $u_1, u_2\in H$,

$$\|\Phi(u_1)-\Phi(u_2)\|_{H'}=\sup_{v\in H}\frac{|\langle\Phi(u_1)-\Phi(u_2), v\rangle|}{\|v\|}$$
$$\le\theta\,\|u_1-u_2\|.$$

Proof of theorem 1. *Uniqueness.*

Let u_1, u_2 be two solutions in H of

$$a(u_1, v)=\langle A(u_1), v\rangle \qquad \text{for all} \quad v\in H,$$
$$a(u_2, v)=\langle A(u_2), v\rangle \qquad \text{for all} \quad v\in H.$$

Thus by subtracting and taking v as (u_1-u_2), we get

$$a(u_1-u_2, u_1-u_2)=\langle A(u_1)-A(u_2), u_1-u_2\rangle.$$

By the coercivity of $a(u, v)$ and the antimonotonicity of A, it follows that there exists $\rho>0$ such that

$$\rho\,\|u_1-u_2\|^2\le a(u_1-u_2, u_1-u_2)$$
$$=\langle A(u_1)-A(u_2), u_1-u_2\rangle$$
$$\le 0.$$

Hence $u_1=u_2$, the uniqueness.

Existence. For a fixed ξ as in lemma 1 and $u\in H$, define $\Phi(u)\in H'$, by (4). Thus by the Riesz-Fréchet theorem, there exists a unique $w\in H$ such that

$$(w, v)=\langle\Phi(u), v\rangle \qquad \text{for all} \quad v\in H,$$

https://doi.org/10.4153/CMB-1980-024-8 Published online by Cambridge University Press

and w is given by

$$w = \Lambda\Phi(u) = Tu,$$

which defines a map from H into itself.

Now for all $u_1, u_2 \in H$,

$$\begin{aligned}\|Tu_1 - Tu_2\| &= \|\Lambda\Phi(u_1) - \Lambda\Phi(u_2)\| \\ &\leq \|\Phi(u_1) - \Phi(u_2)\| \\ &\leq \theta\, \|u_1 - u_2\|, \text{ by lemma 1.}\end{aligned}$$

Since $\theta < 1$, Tu is a contraction and has a fixed point $Tu = u \in H$, which satisfies

$$\begin{aligned}(u, v) &= \langle \Phi(u), v\rangle \\ &= (u, v) - \xi a(u, v) + \xi\langle A(u), v\rangle\end{aligned}$$

Thus for $\xi > 0$, we have

$$a(u, v) = \langle A(u), v\rangle \qquad \text{for all} \quad v \in H.$$

REMARK 1. It is obvious that for $A(u) = F^1(u)$, the existence of a unique solution of (1) follows under the assumptions of theorem 1.

If A is independent of u, i.e., $Au = f$ (say), then the Lipschitz constant γ is zero. Consequently theorm 1 is exactly the same as one proved by Lax and Milgram [2].

Furthermore, for the special case $a(u, v) = (u, v)$, theorem 1 reduces to:

THEOREM 2. *If A is Lipschitz continuous antimonotone operator with Lipschitz constant $\gamma < 1$, then there exists a unique solution $u \in H$* such that

$$(u, v) = \langle A(u), v\rangle \qquad \text{for all} \quad v \in H.$$

Theorem 2 shows that the Riesz–Fréchet theorem also holds for a class of monotone operators on H, which includes the Fréchet derivatives of nonlinear functionals as a special case.

We give another proof of theorem 1 based on the iteration scheme similar to Picard's and also derive a bound for the error.

We define the iteration u_n by the following scheme

$$a(u_{n+1}, v) = \langle A(u_n), v\rangle \qquad \text{for all} \quad v \in H. \tag{5}$$

THEOREM 3. *If $a(u, v)$ is a positive definite bilinear form on H and A is a Lipschitz continuous operator such that condition N holds, then the iteration u_n defined by* (5) *converges strongly to u, the solution of* (3) *in H. Moreover, the bound for the error, for any $u_0 \in H$, is given by*

$$\|u_n - u\| \leq \frac{\alpha^n}{1-\alpha}\|u_1 - u_0\|, \qquad \textit{for} \quad n = 0, 1, 2, \ldots.$$

where $\alpha = \gamma/\rho$.

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Proof. By the coercivity (positive definiteness) of $a(u, v)$, it follows that

$$\begin{aligned}\rho\,\|u_{n+1}-u_n\|^2 &\le a(u_{n+1}-u_n,\, u_{n+1}-u_n)\\ &= \langle A(u_n)-A(u_{n-1}),\, u_{n+1}-u_n\rangle, \text{ by (5).}\\ &\le \|A(u_n)-A(u_{n-1})\|,\end{aligned}$$

by the Cauchy–Schwarz inequality.

$$\le \gamma\,\|u_n-u_{n-1}\|\,\|u_{n+1}-u_n\|,$$

by the Lipschitz continuity of A.

Thus

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \frac{\gamma}{\rho}\|u_n-u_{n-1}\|\\ &= \alpha\,\|u_n-u_{n-1}\|,\end{aligned}$$

where $\alpha=\gamma/\rho<1$ by condition N.

Continuing in this way, we obtain

$$\|u_{n+1}-u_n\|\le\alpha^n\,\|u_1-u_0\|.$$

Hence, by the repeated use of the triangle inequality, it follows that

$$\begin{aligned}\|u_{n+k}-u_n\| &\le (\alpha^{n+k-1}+\cdots+\alpha^n)\,\|u_1-u_0\|,\\ &\le \frac{\alpha^n}{1-\alpha}\|u_1-u_0\|.\end{aligned}$$

Since $\alpha<1$, it follows that u_n is a Cauchy sequence and has a limit point such that $u_n\to u\in H$, the unique solution of (3). Also at the same time it implies that

$$u_n\to u \text{ in } H \text{ strongly.}$$

REMARK 2. Theorem 3 holds for any general complete normed space. Note that it also shows the existence of a unique solution of (3).

REMARK 3. We note that if $a(u, v)$ is a positive definite bilinear form on H, then from (1), it follows that for all $u\in H$,

$$\begin{aligned}\rho\,\|u\|^2 &\le a(u,u)=\langle F'(u),u\rangle\\ &\le \|F'(u)\|_{H'}\,\|u\|,\end{aligned}$$

by the Cauchy–Schwarz inequality.

Thus

$$\|u\|\le\frac{1}{\rho}\|F'(u)\|_{H'}.$$

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This expresses the continuous dependence of u on the Fréchet derivative $F'(u)$. For the linear functional F, it follows that

$$\|u\| \leq \frac{1}{\rho} \|F\|_{H'},$$

a well known result, see Strang and Fix [6, page 16].

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https://doi.org/10.4153/CMB-1980-024-8 Published online by Cambridge University Press