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GENERALISED PARTIAL TRANSFORMATION SEMIGROUPS

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1. Introduction

It is well-known that for any set X, $\mathscr{P}_X$, the semigroup of all partial transformations on X, can be embedded in $\mathscr{T}_{X\cup a}$ for some $a \notin X$ (see for example Clifford and Preston (1967) and Ljapin (1963)). Recently Magill (1967) has considered a special case of what we call '*generalised partial transformation semigroups*'. We show here that any such semigroup can always be embedded in a full transformation smigroup in which the operation is not in general equal to the usual composition of mappinas. We then examine conditions under which such a semigroup, $(\mathscr{T}_X, \theta)$, is isomorphic to the semigroup, under composition, of all transformations on the *same* set X.

The results in this paper from part of the author's thesis written under the guidance of Professor G. B. Preston. I wish to express my gratitude to him for his guidance during this work.

2. The embedding

We assume a familiarity with the notation of Clifford and Preston (1961 and 1967). If X, Y are any two sets, let $P(X, Y)$ denote the set of all mapping with domain in X and range in Y, and choose $\theta \in P(Y, X)$. Define an operation $*$ on S, any non-empty subset of $P(X, Y)$ by

$$\alpha * \beta = \alpha\theta\beta \text{ for } \alpha, \beta \text{ in } S.$$

Any such system (S, X, Y, θ) will be a semigroup, *a generalised partial transformation semigroup*, provided $\alpha\theta\beta \in S$ for all α, β in S. Magill [4] has considered (S, X, Y, θ) when this is a semigroup and dom θ, the domain of θ, equals Y. The notation (S, X, Y, θ) will be abbreviated to (S, θ) whenever convenient and if no confusion will arise. In particular, we abbreviate $(\mathscr{T}_X, X, X, \theta)$ to $(\mathscr{T}_X, \theta)$ if and only if $\theta \in \mathscr{T}_X$. We now define two mappings: we will show that the first embeds (S, X, Y, θ) in $(\mathscr{T}_{X\cup a}, \theta_1)$ when $|X| \geqq |Y|$, and that the second embeds (S, X, Y, θ)

in $(\mathscr{T}_{Y\cup a},\theta_2)$ when $|Y|>|X|$, where $a\notin X\cup Y$ and $\theta_1\in\mathscr{T}_{X\cup a}$, $\theta_2\in\mathscr{T}_{Y\cup a}$, and $a\theta_i=a$, $i=1,2$.

Definition 2.1. If $|Y|\leqq|X|$, let λ_1 be a $1-1$ mapping of Y into X. For each $\alpha\in S$, define α_1 and θ_1 in $\mathscr{T}_{X\cup a}$ by

$$\begin{aligned} x\alpha_1 &= x\alpha\lambda_1, && x\in\operatorname{dom}\alpha\\ &= a, && \text{otherwise}\end{aligned}$$

and

$$\begin{aligned} x\theta_1 &= x\lambda_1^{-1}\theta, && x\in(\operatorname{dom}\theta)\lambda_1\\ &= a, && \text{otherwise.}\end{aligned}$$

Hence we have the diagram

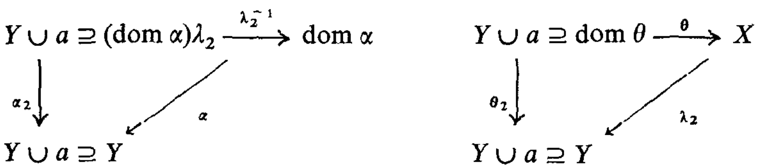
Definition 2.2. If $|X|<|Y|$, let λ_2 be a 1-1 mapping of X into Y. For each $\alpha\in S$, define α_2 and θ_2 in $\mathscr{T}_{Y\cup a}$ by

$$\begin{aligned} y\alpha_2 &= y\lambda_2^{-1}\alpha, && y\in(\operatorname{dom}\alpha)\lambda_2\\ &= a, && \text{otherwise,}\end{aligned}$$

and

$$\begin{aligned} y\theta_2 &= y\theta\lambda_2, && y\in\operatorname{dom}\theta\\ &= a, && \text{otherwise.}\end{aligned}$$

We then have the diagram



Then $\alpha_i,\theta_i, i=1,2$ are well-defined since the corresponding λ_i are 1-1. Now define the mapping $\omega(\lambda_1),\omega(\lambda_2)$ where for α in S

$$\alpha\omega(\lambda_1)=\alpha_1 \text{ as in Definition 2.1 if } |Y|\leqq|X|, \text{ and}$$

$$\alpha\omega(\lambda_2)=\alpha_2 \text{ as in Definition 2.2 if } |X|<|Y|.$$

Theorem 2.3. *Suppose* (S,X,Y,θ) *is a semigroup.*

(i) *if* $|Y|\leqq|X|$ *then* $\omega(\lambda_1)\colon(S,\theta)\cong(S\omega(\lambda_1),\theta_1)$

(ii) *if* $|Y|>|X|$ *then* $\omega(\lambda_2)\colon(S,\theta)\cong(S\omega(\lambda_2),\theta_2)$.

PROOF. (i) Let $\alpha, \beta \in S$. If $\alpha\theta\beta \neq \square$, let $x \in \operatorname{dom}(\alpha\theta\beta)$. Then $x\alpha \in \operatorname{dom}\theta$ and $x\alpha\theta \in \operatorname{dom}\beta$. Using Definition 2.1, it follows that

$$\begin{aligned} x(\alpha\theta\beta)_1 &= x\alpha\theta\beta \cdot \lambda_1 \\ &= x\alpha\theta \cdot \beta_1 \\ &= x\alpha\lambda_1 \cdot \lambda_1^{-1}\theta \cdot \beta_1 \\ &= x\alpha_1\theta_1\beta_1 . \end{aligned}$$

Now suppose $x \notin \operatorname{dom}(\alpha\theta\beta)$, that is, $x(\alpha\theta\beta)_1 = a$. Then one of only three cases can occur: either $x \notin \operatorname{dom}\alpha$, or $x \in \operatorname{dom}\alpha$ but $x\alpha \notin \operatorname{dom}\theta$, or $x\alpha \in \operatorname{dom}\theta$ but $x\alpha\theta \notin \operatorname{dom}\beta$. In each of these cases $x\alpha_1\theta_1\beta_1 = a$. Hence $\alpha_1\theta_1\beta_1 = (\alpha\theta\beta)_1$ and so $\omega(\lambda_1)$ is a morphism. Furthermore if $x\alpha_1 = x\beta_1$ for all x in $X \cup a$, then $\operatorname{dom}\alpha = \operatorname{dom}\beta$ and $x\alpha\lambda_1 = x\beta\lambda_1$ for all x in $\operatorname{dom}\alpha$. Hence $\alpha = \beta$ and so $\omega(\lambda_1)$ is 1-1. The proof of part (ii) is similar to that given above for part (i).

We shall say that $\alpha \in \mathscr{P}_X$ [*strictly*] *fixes* a *in* X if $a\alpha = a$ [$a\alpha = a$ and $x\alpha = a$ implies $x = a$]. We denote by $F(X, a)$ [$F(X, \sigma a)$] the subsemigroup of all elements in $\mathscr{T}_X$ [strictly] fixing a in X. The following result shows that Theorem 2.3 generalises the usual method of embedding $(\mathscr{P}_X, \iota_X)$ in $(\mathscr{T}_{X\cup a}, \iota_X)$.

COROLLARY 2.4. *Any semigroup* (S, X, Y, θ) *where* θ *maps* Y 1-1 *onto* X *can be embedded in* $(\mathscr{T}_{X\cup a}, \iota_{X\cup a})$ *for some* a *not in* X. *In particular,*

$$\omega(\iota_X) \colon (\mathscr{P}_X, \iota_X) \cong F(X \cup a, a). \tag{1}$$

PROOF. If $|X| = |Y|$, put $\lambda_1 = \theta^{-1}$ in Definition 2.1. Then since $\theta_1 = \iota_{X\cup a}$, the assertion follows by Theorem 2.3 (i). If $X = Y$, $\theta = \iota_X$, and $S = \mathscr{P}_X$, then $S\omega(\iota_X) = F(X \cup a, a)$ by Definition 2.1 and so again by Theorem 2.3 (i), (1) holds.

We have thus shown that any generalised partial transformation semigroup can be embedded in $(\mathscr{T}_X, \theta)$ for some X and some $\theta \in \mathscr{T}_X$. It is now natural to ask whether the semigroup $(\mathscr{T}_X, \theta)$ can always be embedded in $(\mathscr{T}_X, \iota_X)$. The next result provides a partial answer to this. Following Magill (1967) X_a denotes the mapping in $\mathscr{T}_X$ with domain X and range a.

THEOREM 2.5. *If there exists an isomorphism* ϕ *from* $(\mathscr{T}_X, \theta)$ *onto* $(\mathscr{T}_X, \iota_X)$ *then* $\theta \in \mathscr{G}_X$ *and there exists* g *in* $\mathscr{G}_X$ *such that* $\alpha\phi = g^{-1}\theta\alpha g$ *for all* $\alpha \in \mathscr{T}_X$.

PROOF. Let $a \in X$ and $\alpha\phi = X_a$, so that $\alpha\theta\alpha = \alpha$. Fix an element u in X and let $\lambda = (X_{u\alpha})\phi$. We now have

$$\lambda = (X_{u\alpha})\phi = (X_{u\alpha}\theta\alpha)\phi = \lambda X_a .$$

Since $\lambda \in \mathscr{T}_X$, this implies $\lambda = X_a$ and so $X_{u\alpha} = \alpha$. Hence α is a constant mapping.

In a similar fashion we can show that if $X_a\phi = \alpha$ then $\alpha = X_b$ for some b in X. Now define g in $\mathscr{T}_X$ by

$$ag = b \text{ if and only if } X_a\phi = X_b.$$

Then $g \in \mathscr{G}_X$ since ϕ maps the set $K = \{X_a : a \in X\}$ 1-1 onto itself, and we have $X_a\phi = X_a g$ for all a in X. Now let $a \in X$, $a = bg$, and $\alpha \in \mathscr{T}_X$. Then we have

$$X_a \cdot \alpha\phi = X_b\phi \cdot \alpha\phi = (X_{b\theta\alpha})\phi = X_b \cdot \theta\alpha g = X_a \cdot g^{-1}\theta\alpha g.$$

Hence $\alpha\phi = g^{-1}\theta\alpha g$. It now only remains to show that $\theta \in \mathscr{G}_X$. To do this note that $\iota_X\phi = g^{-1}\theta g$ and there exists α such that $\iota_X = \alpha\phi = g^{-1}\theta\alpha g$. Hence $\theta\alpha = \iota_X$ and so $(\alpha\theta)\phi = g^{-1}\theta \cdot \alpha\theta g = g^{-1}\theta g$. And hence also $\alpha\theta = \iota_X$. That is, $\theta \in \mathscr{G}_X$.

We note that the mapping: $\alpha\phi = h^{-1}\alpha\theta h$ defined for all $\alpha \in \mathscr{T}_X$ and some h and θ in $\mathscr{G}_X$ also embeds $(\mathscr{T}_X, \theta)$ in $(\mathscr{T}_X, \iota_X)$ but that this mapping is in fact described in the theorem above; that is, when $g = \theta h$.

REMARK 2.6. This result solves the problem completely when X is finite. For then $\mathscr{T}_X\phi = \mathscr{T}_X$ for all 1-1 mappings ϕ and so the theorem applies. The case when X is infinite and ϕ properly embeds $(\mathscr{T}_X, \theta)$ into $(\mathscr{T}_X, \iota_X)$ remains open. We note however that in that case if λ, μ are any two mappings such that $\mu\lambda = \iota_X$, θ_1 a mapping onto X, and θ_2 any 1-1 mapping, then ϕ, ψ defined by

$$\alpha\phi = \lambda\theta_1\alpha\mu, \quad \alpha\psi = \lambda\alpha\theta_2\mu \tag{2}$$

are isomorphism of $(\mathscr{T}_X, \theta_1)$ into $(\mathscr{T}_X, \iota_X)$ and $(\mathscr{T}_X, \theta_2)$ into $(\mathscr{T}_X, \iota_X)$ respectively. This is so by the following well-known lemma.

LEMMA 2.7. *Let* $\alpha, \beta \in \mathscr{P}_X$.
(i) *If* $\gamma \in \mathscr{T}_X$, *then* γ *is* 1-1 *if and only if* $\alpha\gamma = \beta\gamma$ *implies* $\alpha = \beta$.
(ii) *If* $\gamma \in \mathscr{P}_X$, *then* γ *is onto* X *if and only if* $\gamma\alpha = \gamma\beta$ *implies* $\alpha = \beta$.

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