Glasgow Math. J. **57** (2015) 543–554. © Glasgow Mathematical Journal Trust 2014.
doi:10.1017/S0017089514000470.

ON THE COMPLEXITY OF FINDING A NECESSARY AND SUFFICIENT CONDITION FOR BLASCHKE-OSCILLATORY EQUATIONS

JANNE HEITTOKANGAS AND ATTE REIJONEN

*Department of Physics and Mathematics, University of Eastern Finland,
P.O. Box 111, 80101 Joensuu, Finland*
e-mails: janne.heittokangas@uef.fi, atte.reijonen@uef.fi

(Received 27 September 2013; revised 12 February 2014; accepted 4 March 2014; first published online 17 December 2014)

Abstract. If $A(z)$ belongs to the Bergman space $B^{\frac{1}{2}}$, then the differential equation $f'' + A(z)f = 0$ is Blaschke-oscillatory, meaning that the zero sequence of every nontrivial solution satisfies the Blaschke condition. Conversely, if $A(z)$ is analytic in the unit disc such that the differential equation is Blaschke-oscillatory, then $A(z)$ almost belongs to $B^{\frac{1}{2}}$. It is demonstrated that certain "nice" Blaschke sequences can be zero sequences of solutions in both cases when $A \in B^{\frac{1}{2}}$ or $A \notin B^{\frac{1}{2}}$. In addition, no condition regarding only the number of zeros of solutions is sufficient to guarantee that $A \in B^{\frac{1}{2}}$.

2010 *Mathematics Subject Classification.* Primary 34M10; Secondary 30J10.

1. Introduction. In 1982, Pommerenke [**17**] proved that if $A(z)$ is analytic in the unit disc $\mathbb{D}$ (denoted by $A \in H(\mathbb{D})$ for short) and

$$\int_{\mathbb{D}} |A(z)|^{\frac{1}{2}}\, dm(z) < \infty, \tag{1}$$

where $dm(z) = rdrd\theta$ is the Lebesgue area measure, then all solutions of

$$f'' + A(z)f = 0 \tag{2}$$

belong to the Nevanlinna class N [**4**]. This in turn implies that (2) is Blaschke-oscillatory, meaning that the zero sequence $\{z_n\}$ of any nontrivial solution of (2) satisfies the Blaschke condition

$$\sum_n (1 - |z_n|) < \infty. \tag{3}$$

The converse direction is considered in [**10**, Theorems 2 and 4 (b)]: If $A \in H(\mathbb{D})$ and if either (2) is Blaschke-oscillatory or (2) possesses a nontrivial solution in N, then

$$\int_{D(0,r)} |A(z)|^{\frac{1}{2}}\, dm(z) = O\left(\log^2 \frac{e}{1-r}\right), \quad r \to 1^-. \tag{4}$$

Moreover, if the solution $f \in N$ has the property that $f' \in N$, then (4) improves to

$$\int_{D(0,r)} |A(z)|^{\frac{1}{2}}\, dm(z) = O\left(\log \frac{e}{1-r}\right), \quad r \to 1^-. \tag{5}$$

The problem of finding a necessary and sufficient condition for (2) to be Blaschke-oscillatory is still open. To illustrate the complexity of this problem, and also to obtain partial results in this direction, we will concentrate on prescribed zero sequences.

In 1959, Šeda [**12**, **19**] proved that if $\{z_n\}_{n=1}^{\infty}$ is a given sequence of pairwise distinct points in $\mathbb{C}$ with no finite limit points, then there is an entire function $A(z)$ such that the differential equation (2) has a solution with zeros precisely at the points z_n. Here, we consider prescribed Blaschke sequences having certain constraints. For example, we say that an arbitrary sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ is *separated*, if

$$\delta := \inf_{n \neq k} \left| \frac{z_n - z_k}{1 - \overline{z}_n z_k} \right| > 0, \tag{6}$$

and *uniformly separated*, if

$$\delta_u := \inf_k \prod_{n \neq k} \left| \frac{z_n - z_k}{1 - \overline{z}_n z_k} \right| > 0. \tag{7}$$

It is known that a uniformly separated sequence is a Blaschke sequence, while a separated sequence need not be one.

The convergence condition (3) guarantees that the corresponding Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}$$

converges in compact subsets of $\mathbb{D}$, and hence is an analytic function with zeros precisely at the points z_n. Since, by [**2**, Theorem 3], a uniformly separated sequence is interpolating, a Blaschke product with a uniformly separated zero sequence is known as interpolating. At times (3) needs to be strengthened, for example, to

$$\sum_n (1 - |z_n|)^{\alpha} < \infty, \tag{8}$$

where $\alpha \in (0, 1]$. For example, if (8) holds for $\alpha \in (0, 1/2)$, then the corresponding Blaschke product B has the property that B' belongs to the Hardy space $H^{1-\alpha}$, and hence to N [**18**, Theorem 2].

For $p > 0$, the Bergman space B^p consists of all functions $f \in H(\mathbb{D})$ satisfying

$$\int_{\mathbb{D}} |f(z)|^p dm(z) < \infty.$$

The discussion above leads us to consider the following problem.

Problem. Which assumptions on a given Blaschke sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ will guarantee that there exists a function $A \in B^{\frac{1}{2}}$ such that the differential equation (2) has a solution with zeros precisely at the points z_n?

Clearly the zeros must be pairwise distinct and the possible limit points must be on $\partial\mathbb{D}$. Assuming, for example, that $\{z_n\}_{n=1}^{\infty}$ is separated will guarantee both requirements. We have the following partial result.

THEOREM 1 ([**10**], Theorem 19). *Let $\{z_n\}_{n=1}^{\infty}$ be a uniformly separated sequence of nonzero points in $\mathbb{D}$ satisfying the condition* (8) *for some $\alpha \in (0,1)$. Then there exists a function $A \in B^{\frac{1}{2}}$ such that the differential equation* (2) *has a solution with zeros precisely at the points z_n.*

We sketch the proof as follows. Take $f = Be^g$ for a candidate solution, where B is a Blaschke product with zeros at the points z_n, and the function $g \in H(\mathbb{D})$ has the interpolation property

$$g'(z_n) = -\frac{B''(z_n)}{2B'(z_n)}, \quad n \in \mathbb{N}. \tag{9}$$

The latter guarantees that

$$A(z) = -\frac{B''(z)}{B(z)} - 2g'(z)\frac{B'(z)}{B(z)} - g'(z)^2 - g''(z) \tag{10}$$

is analytic in $\mathbb{D}$, and that f solves (2). After a suitable g is found, then the growth of $A(z)$ in (10) is estimated [**10**, p. 78].

Theorem 1 in the case $\alpha = 1$ extends to the following result.

THEOREM 2. *Let $\{z_n\}_{n=1}^{\infty}$ be a uniformly separated sequence of nonzero points in $\mathbb{D}$. Then there exists a function $A \in H(\mathbb{D})$ satisfying* (5) *such that the differential equation* (2) *has a solution with zeros precisely at the points z_n.*

Although not entirely removable, the assumption (8) for $\alpha \in (0,1)$ in Theorem 1 can be slightly weakened to

$$\sum_n h(1-|z_n|) < \infty, \tag{11}$$

where $h(x)$ is a positive continuous function on $(0,1)$ tending to zero as $x \to 0^+$, and satisfies the following conditions:

(i) $h(x)/x$ is decreasing and $h(x)$ is increasing on $(0,1)$;

(ii) $\int_0^1 (1-r)^{-\frac{1}{2}} h(1-r)^{-\frac{1}{2}}\, r dr < \infty$.

Then the choice $h(x) = x \log^p \frac{e^p}{x}$, $p > 2$, shows that the following result is an improvement of Theorem 1.

THEOREM 3. *Let $h(t)$ be a function as above, and let $\{z_n\}_{n=1}^{\infty}$ be a uniformly separated sequence of nonzero points in $\mathbb{D}$ satisfying* (11). *Then there exists a function $A \in B^{\frac{1}{2}}$ such that the differential equation* (2) *has a solution with zeros precisely at the points z_n.*

Next we turn our attention to the separation of the zero sequences. If $\{z_n\}_{n=1}^{\infty}$ is a union of two exponential sequences [**4**, p. 156] approaching pairwise to one another exponentially, then it is proved in [**8**, Theorem 5] that $A(z)$ cannot belong to the Korenblum space $\mathcal{A}^{-\infty}$ [**9**, p. 110]. In particular, both of the two component sequences are uniformly separated, and yet all solutions of (2) are "far away" from the class N as being of infinite order of growth.

https://doi.org/10.1017/S0017089514000470 Published online by Cambridge University Press

In the recent study **[3]**, it is illustrated that the growth of $A(z)$ within the Korenblum space is restricted by the (weighted) separation of the zeros of any solution, and conversely. Here, we assume the usual separation, but require the separation constant $\delta \in (0, 1)$ in (6) to be close enough to the constant one in terms of

$$(2\pi + 1)\frac{\sqrt{1-\delta}}{(1-\sqrt{1-\delta})^2} < 1. \tag{12}$$

This makes $\{z_n\}_{n=1}^{\infty}$ to be interpolating for some Bergman space B^p with $p > 1$ [**5**, p. 192]. Now the assumption on uniform separation in Theorem 1 can be relaxed at the expense of strengthening the Blaschke condition.

THEOREM 4. *Let* $\{z_n\}_{n=1}^{\infty}$ *be a separated sequence of nonzero points in* $\mathbb{D}$ *such that the separation constant* δ *in* (6) *satisfies* (12). *Suppose in addition that* (8) *holds for* $\alpha \in (0, 1/2]$. *Then there exists a function* $A \in B^{\frac{1}{2}}$ *such that the differential equation* (2) *has a solution with zeros precisely at the points* z_n.

REMARK. The assumption (12) in Theorem 4 can in fact be replaced by a weaker condition that the upper uniform density D^+ of $\{z_n\}_{n=1}^{\infty}$ is less than one. See [**5**, pp. 171–172] for more details.

So far we have concentrated in one prescribed zero sequence only. Next we will widen our point of view to concern all solutions. First, an example of a nonoscillatory differential equation of the form (2) is constructed in [**10**, Section 4.3] such that all nontrivial solutions of (2) are of unbounded characteristic. In this case it is clear that $A \notin B^{\frac{1}{2}}$ [**17**]. Second, by a more careful analysis on an example, which can be found in [**10**, **13**], we will prove that the solutions of (2) can have infinite uniformly separated and sparse zero sequences even if $A \notin B^{\frac{1}{2}}$. This result also shows that the requirement $\alpha \in (0, 1/2]$ in Theorem 4 is essential.

THEOREM 5. *There exists a function* $A \in H(\mathbb{D})$ *with* $A \notin B^{\frac{1}{2}}$ *such that the differential equation* (2) *has the following properties:*

(a) *There exists a zero-free solution base.*
(b) *There are solutions with infinitely many zeros.*
(c) *Every infinite zero sequence satisfies* (8) *for every* $\alpha \in (1/2, 1]$.
(d) *Every infinite zero sequence is uniformly separated.*
(e) *There are infinite separated zero sequences whose separation constant* δ *satisfies* (12).
(f) *Every solution and all of their derivatives belong to* N.

Note that even if the differential equation (2) is disconjugate, it is still possible that $A \notin B^{\frac{1}{2}}$. This can be seen by means of the gap series

$$g(z) = \sum_{n=0}^{\infty} \left(\frac{2^n}{n}\right)^2 z^{2^n}, \quad z \in \mathbb{D}.$$

First, by [**1**, Proposition 2.1], we infer $g \notin B^{\frac{1}{2}}$. Second, using [**20**, Theorem 1 (II)] together with the standard representation

$$g(z) = g(0) + \int_0^z g'(\xi)\, d\xi$$

https://doi.org/10.1017/S0017089514000470 Published online by Cambridge University Press

yields $g \in H_2^\infty$. Finally, denoting $A(z) = g(z)/\|g\|_{H_2^\infty}$, we see that (2) is disconjugate by the proof of [**16**, Theorem I], while $A \notin B^{\frac{1}{2}}$.

Summarizing, we conclude that no condition regarding only the number of zeros of solutions of (2) is sufficient to guarantee that $A \in B^{\frac{1}{2}}$.

2. Proof of Theorem 2. One of the key results in proving Theorem 1 is

$$\int_{\mathbb{D}} \left| \frac{B^{(k)}(z)}{B(z)} \right|^{\frac{1}{k}} dm(z) < \infty, \tag{13}$$

valid for an interpolating Blaschke product B with zeros $\{z_n\}$ satisfying (8) for some $\alpha \in (0, 1)$ [**10**, Theorem 15]. We will require a variant of this result.

LEMMA 1. *Let B be an interpolating Blaschke product. Then*

$$\int_{D(0,r)} \left| \frac{B^{(k)}(z)}{B(z)} \right|^{\frac{1}{k}} dm(z) = O\left(\log \frac{e}{1-r}\right), \quad r \to 1^-,$$

for all $k \in \mathbb{N}$.

Proof. If B is an arbitrary Blaschke product, then Cauchy's formula yields

$$|B^{(k)}(z)| = O\left((1-|z|)^{-k}\right), \quad |z| \to 1^-,$$

for every $k \in \mathbb{N}$. In particular,

$$\int_0^{2\pi} |B^{(k)}(re^{i\theta})|^{\frac{1}{k}} \, d\theta = O\left(\frac{1}{1-r}\right), \quad k \in \mathbb{N}. \tag{14}$$

Suppose that B is interpolating, and denote the pseudo-hyperbolic disc with centre $a \in \mathbb{D}$ and radius $r \in (0, 1)$ by

$$\Delta(a, r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}, \tag{15}$$

where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$. Let $K = \bigcup_{n\in\mathbb{N}} \Delta(z_n, \delta_u/2)$, where $\{z_n\}_{n=1}^\infty$ is the zero sequence of B and δ_u is the interpolation constant defined in (7). Now, by using (14) instead of [**7**, Theorem 1.1] in the proof of [**10**, Theorem 15], we obtain

$$\int_{D(0,r)\setminus K} \left| \frac{B^{(k)}(z)}{B(z)} \right|^{\frac{1}{k}} dm(z) = O\left(\log \frac{e}{1-r}\right), \quad r \to 1^-.$$

Integration over the discs $\Delta(z_n, \delta_u/2)$ can be done in a similar way as in the proof of [**10**, Theorem 15]. Thus the assertion follows. □

We proceed to prove Theorem 2. Let B be an interpolating Blaschke product having zeros at the points z_n. By the proof of [**11**, Theorem 4.2], we can find $g \in H(\mathbb{D})$ satisfying (9). Indeed, the derivative of g is given by

$$g'(z) = \sum_{k=1}^\infty \sigma_k \frac{C_k(z)}{B'(z_k)} \frac{|z_k|^2 - 1}{(1 - \overline{z}_k z)^2}, \tag{16}$$

https://doi.org/10.1017/S0017089514000470 Published online by Cambridge University Press

where

$$C_k(z) = B(z)\frac{1-\bar{z}_k z}{z_k - z} \quad \text{and} \quad \sigma_k = -\frac{B''(z_k)}{2B'(z_k)}.$$

Now $f = Be^g$ solves (2), where $A(z)$ is given by (10) and belongs to $H(\mathbb{D})$.

Next, we proceed to prove that

$$\int_{D(0,r)} |g'(z)|\, dm(z) = O\left(\log \frac{e}{1-r}\right), \quad r \to 1^-, \tag{17}$$

and

$$\int_{D(0,r)} |g''(z)|^{\frac{1}{2}}\, dm(z) = O\left(\log \frac{e}{1-r}\right), \quad r \to 1^-. \tag{18}$$

Following [**11**], we know that there exist constants $M_1, M_2 > 0$ such that

$$|\sigma_k| \le \frac{M_1}{1-|z_k|^2} \quad \text{and} \quad \frac{1}{|B'(z_k)|} \le M_2(1-|z_k|^2) \tag{19}$$

for all $k \in \mathbb{N}$. Hence

$$|g'(z)| \le R \sum_{k=1}^{\infty} \frac{1-|z_k|^2}{|1-\bar{z}_k z|^2}, \tag{20}$$

where the constant $R > 0$ is independent of z, and so

$$\begin{aligned}\int_0^{2\pi} |g'(re^{i\theta})|\, d\theta &\le R \sum_{k=1}^{\infty} \frac{1-|z_k|^2}{1-|z_k|^2 r^2} \int_0^{2\pi} \frac{1-|z_k|^2 r^2}{|1-\bar{z}_k t e^{i\theta}|^2}\, d\theta \\ &\le 2\pi R \sum_{k=1}^{\infty} \frac{1-|z_k|^2}{1-|z_k|^2 r^2} = O\left(\frac{1}{1-r}\right), \quad r \to 1^-.\end{aligned}$$

This implies (17). Now the Hölder inequality and the proof of [**4**, Theorem 5.5] yield

$$\int_0^{2\pi} |g''(re^{i\theta})|^{\frac{1}{2}}\, d\theta \le (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} |g''(re^{i\theta})|\, d\theta\right)^{\frac{1}{2}} = O\left(\frac{1}{1-r}\right), \quad r \to 1^-,$$

and so (18) follows.

Finally, by (10) it is clear that

$$|A(z)|^{\frac{1}{2}} \le \left|\frac{B''(z)}{B(z)}\right|^{\frac{1}{2}} + \sqrt{2}|g'(z)|^{\frac{1}{2}} \left|\frac{B'(z)}{B(z)}\right|^{\frac{1}{2}} + |g'(z)| + |g''(z)|^{\frac{1}{2}}. \tag{21}$$

Thus, the assertion follows by the Hölder inequality, (17), (18) and Lemma 1.

3. Proof of Theorem 3. We begin by introducing two lemmas in which $h(t)$ is a positive continuous function defined on $(0, 1)$ and tends to zero as $t \to 0^+$.

https://doi.org/10.1017/S0017089514000470 Published online by Cambridge University Press

LEMMA 2. *Let $k \in \mathbb{N}$, and let B be a Blaschke product formed with zeros $\{z_n\}_{n=1}^{\infty}$ satisfying* (11) *such that $h(x)/x$ is decreasing and $h(x)$ is increasing on* $(0, 1)$*, and that*

$$\int_0^1 (1-r)^{\frac{1-k}{k}} h(1-r)^{-\frac{1}{k}}\, r dr < \infty. \tag{22}$$

Then

$$\int_{\mathbb{D}} |B^{(k)}(re^{i\theta})|^{\frac{1}{k}}\, dm(z) < \infty. \tag{23}$$

Proof. By [**6**, Theorem 3.1] the assertion holds for $k = 1$. Hence, by $k - 1$ applications of the proof of [**4**, Theorem 5.5], we can easily see that

$$\int_0^{2\pi} |B^{(k)}(re^{i\theta})|\, d\theta = O\left(\frac{1}{(1-\rho)^{k-1} h(1-\rho)}\right)$$

for $\rho \in (r, 1)$. Thus the Hölder inequality yields

$$\begin{aligned}\int_0^{2\pi} |B^{(k)}(re^{i\theta})|^{\frac{1}{k}}\, d\theta &\leq (2\pi)^{\frac{1}{k}} \left(\int_0^{2\pi} |B^{(k)}(re^{i\theta})|\, d\theta\right)^{\frac{1}{k}}\\ &= O\left(\frac{1}{(1-\rho)^{\frac{k-1}{k}} h(1-\rho)^{\frac{1}{k}}}\right),\end{aligned}$$

and so the assertion follows. □

REMARK. As noted in [**6**], since our sequences already satisfy the Blaschke condition, the condition (11) provides with further information on the rate of increase of the zeros only if $h(x) \geq x$ as $x \to 0^+$. This means that if (22) is valid for some $k \in \mathbb{N}$, then (22) is valid when k is replaced with any integer $j \in \{1, \ldots, k\}$.

By using Lemma 2, we can rewrite the proof of [**10**, Theorem 15], and achieve the following result.

LEMMA 3. *Let B and h be as in Lemma 2, and, in addition, suppose that B is interpolating. Then* (13) *holds.*

We proceed to prove Theorem 3. Let B be an interpolating Blaschke product having zeros at the points z_n satisfying (11). As in the proof of Theorem 2, we can find a function $g \in H(\mathbb{D})$ such that $f = Be^g$ solves (2), where $A(z)$ is given by (10) and belongs to $H(\mathbb{D})$.

By (20) and [**6**, Lemma 2.1], we have

$$\int_0^{2\pi} |g'(z)|\, d\theta \leq R \sum_{k=1}^{\infty} \frac{1-|z_k|^2}{1-|z_k|^2 r^2} \int_0^{2\pi} \frac{1-|z_k|^2 r^2}{|1-\bar{z}_k re^{i\theta}|^2}\, d\theta = \frac{2\pi R}{h(1-r)}.$$

Hence, the Hölder inequality and the proof of [**4**, Theorem 5.5] yield

$$\int_0^{2\pi} |g''(z)|^{\frac{1}{2}}\, d\theta \le (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} |g''(z)|^{\frac{1}{2}}\, d\theta \right)^{\frac{1}{2}}$$
$$= O\left(\frac{1}{(1-r)^{\frac{1}{2}} h(1-r)^{\frac{1}{2}}} \right), \quad r \to 1^-.$$

Using the condition (ii) related to (11), we obtain $g' \in B^1$ and $g'' \in B^{\frac{1}{2}}$. Thus the assertion follows by (21), the Hölder inequality and Lemma 3.

4. Proof of Theorem 4. We require a new lemma on logarithmic derivatives of noninterpolating Blaschke products.

LEMMA 4. *Suppose that B is a Blaschke product with a zero sequence $\{z_n\}_{n=1}^{\infty}$. Then* (13) *holds if*

$$\sum_{n=1}^{\infty} (1-|z_n|) \log \frac{e}{1-|z_n|} < \infty \tag{24}$$

in the case $k=1$, and if (8) *holds for $\alpha \in (0, 1/k]$ in the case $k \ge 2$.*

Proof. Since

$$\left| \frac{B'(z)}{B(z)} \right| \le \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|1-\bar{z}_n z||z_n - z|} = \sum_{n=1}^{\infty} \left| \frac{\varphi'_{z_n}(z)}{\varphi_{z_n}(z)} \right|,$$

we have

$$\int_{\mathbb{D}} \left| \frac{B'(z)}{B(z)} \right| dm(z) \le \sum_{n=1}^{\infty} \int_{\mathbb{D} \setminus \Delta(z_n, \frac{1}{2})} \left| \frac{\varphi'_{z_n}(z)}{\varphi_{z_n}(z)} \right| dm(z) + \sum_{n=1}^{\infty} \int_{\Delta(z_n, \frac{1}{2})} \left| \frac{\varphi'_{z_n}(z)}{\varphi_{z_n}(z)} \right| dm(z)$$
$$=: S_1 + S_2,$$

where $\Delta(a, r)$ is the pseudo-hyperbolic disc defined in (15). By the condition (24), it follows that

$$S_1 \le 2 \sum_{n=1}^{\infty} \int_{\mathbb{D}} |\varphi'_{z_n}(z)|\, dm(z)$$
$$= 2 \sum_{n=1}^{\infty} \int_0^1 \frac{1-|z_n|^2}{1-|z_n|^2 r^2} \left(\int_0^{2\pi} \frac{1-|z_n|^2 r^2}{|1-\bar{z}_n r e^{i\theta}|}\, d\theta \right) r d\theta dr$$
$$= 4\pi \sum_{n=1}^{\infty} (1-|z_n|^2) \int_0^1 \frac{r dr}{1-|z_n|^2 r^2}$$
$$\le C_1 \sum_{n=1}^{\infty} (1-|z_n|) \log \frac{e}{1-|z_n|} < \infty$$

https://doi.org/10.1017/S0017089514000470 Published online by Cambridge University Press

for some constant $C_1 > 0$. On the other hand, by a change of variable, we have

$$\begin{aligned}
S_2 &\le \sum_{n=1}^{\infty} \int_{\Delta(z_n,\frac{1}{2})} \frac{1-|z_n|^2}{(1-|z_n|)^2|\varphi_{z_n}(z)|}\, dm(z) \\
&\le 2\sum_{n=1}^{\infty} \frac{1}{1-|z_n|} \int_{\Delta(z_n,\frac{1}{2})} \frac{dm(z)}{|\varphi_{z_n}(z)|} \\
&= 2\sum_{n=1}^{\infty} \frac{1}{1-|z_n|} \int_{|w|<\frac{1}{2}} \frac{1}{|w|} |\varphi_{z_n}'(w)|^2\, dm(w) \\
&= 2\sum_{n=1}^{\infty} \frac{1}{1-|z_n|} \int_{|w|<\frac{1}{2}} \frac{1}{|w|} \frac{(1-|z_n|^2)^2}{|1-\overline{z}_n w|^4}\, dm(w) \\
&\le 2^7 \sum_{n=1}^{\infty} (1-|z_n|) \int_{|w|<\frac{1}{2}} \frac{dm(w)}{|w|} < \infty.
\end{aligned}$$

Hence, the assertion for $k = 1$ is proved.

If the condition (8) holds for some $\alpha = 1/k$, where $k \ge 2$, then also (24) holds. Hence, we make an induction assumption that (13) holds for some $k \ge 1$, and so we have to prove that (13) holds also for the index $k + 1$.

By [**15**, p. 44], we have

$$\frac{B^{(k+1)}(z)}{B(z)} = \sum_{n=2}^{k+1} \sum^{*} M \prod_{j=1}^{m} \frac{B^{(n_j)}(z)}{B(z)} - \sum_{n=1}^{\infty}\sum_{j=0}^{k} \frac{(1-|z_n|^2)\overline{z}_n^j k!}{(1-\overline{z}_n z)^{j+1}(z_n - z)^{k-j+1}},$$

where M is a constant and the sum $\sum^*$ is taken over all m-tuples $(n_1, n_2, \ldots, n_m)$ for which $n_1 + n_2 + \cdots + n_m = k + 1$, and $n_j \ge 1$ for $1 \le j \le m$. Next, we use the general form of the Hölder inequality with conjugate indices $p_j = \frac{k+1}{n_j}$ for which

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1$$

to conclude that

$$\int_{\mathbb{D}} \prod_{j=1}^{m} \left| \frac{B^{(n_j)}(z)}{B(z)} \right|^{\frac{1}{k+1}} dm(z) \le \prod_{j=1}^{m} \left(\int_{\mathbb{D}} \left| \frac{B^{(n_j)}(z)}{B(z)} \right|^{\frac{1}{n_j}} dm(z) \right)^{\frac{n_j}{k+1}} < \infty.$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{D}} \left| \frac{B^{(k+1)}(z)}{B(z)} \right|^{\frac{1}{k+1}} dm(z) \\
&\quad \le C_2 + C_3 \sum_{n=1}^{\infty}\sum_{j=0}^{k} \int_{\mathbb{D}} \left(\frac{1-|z_n|^2}{|1-\overline{z}_n z|^{k+2}|\varphi_{z_n}(z)|^{k-j+1}} \right)^{\frac{1}{k+1}} dm(z),
\end{aligned}$$

where $C_2, C_3 > 0$ are constants. It suffices to consider the term with index $j = 0$ since the associated integrand takes the largest value for every z with index $j = 0$, because

https://doi.org/10.1017/S0017089514000470 Published online by Cambridge University Press

$|\varphi_{z_n}(z)| < 1$. Now,

$$\begin{aligned}
\int_{\mathbb{D}} \left(\frac{1-|z_n|^2}{|1-\overline{z}_n z|^{k+2} |\varphi_{z_n}(z)|^{k+1}} \right)^{\frac{1}{k+1}} dm(z) &= \int_{\mathbb{D}\setminus\Delta(z_n,\frac{1}{2})} + \int_{\Delta(z_n,\frac{1}{2})} \\
&\leq 2\int_{\mathbb{D}} \left(\frac{1-|z_n|^2}{|1-\overline{z}_n z|^{k+2}} \right)^{\frac{1}{k+1}} dm(z) + \frac{2^{\frac{1}{k+1}}}{1-|z_n|} \int_{\Delta(z_n,\frac{1}{2})} \frac{dm(z)}{|\varphi_{z_n}(z)|} \\
&\leq 2(1-|z_n|^2)^{\frac{1}{k+1}} \int_{\mathbb{D}} \frac{dm(z)}{|1-\overline{z}_n z|^{\frac{k+2}{k+1}}} + \frac{2^{\frac{1}{k+1}}}{1-|z_n|} \int_{|w|<\frac{1}{2}} \frac{1}{|w|} |\varphi_{z_n}'(z)|^2 \, dm(w) \\
&\leq C_4 (1-|z_n|)^{\frac{1}{k+1}},
\end{aligned}$$

where $C_4 > 0$ is a constant. This proves (13) for the index $k+1$, and hence the assertion follows by the principle of mathematical induction. □

Remark. For $f \in H(\mathbb{D})$, $k \in \mathbb{N}$ and $p > 0$, we recall from [**14**, pp. 179–180] the asymptotic comparability

$$\int_0^{2\pi} \left|D^k f(re^{i\theta})\right|^p d\theta \asymp \int_0^{2\pi} \left|f^{(k)}(re^{i\theta})\right|^p d\theta, \quad r \to 1^-,$$

where $D^k f$ is the fractional derivative of f of order k. Now Lemma 4 can be seen as an improvement of a special case of [**14**, Theorem 3.1. (2)(3)], where $\alpha = 0$, $\beta = k$, $p = 1/k$, and the fractional derivative is replaced with the usual derivative of the same order k.

We proceed to prove Theorem 4. Let B be a Blaschke product having zeros at the points z_n satisfying the given assumptions. Then, by [**5**, p. 192], the sequence $\{z_n\}_{n=1}^{\infty}$ is interpolating for some Bergman space B^p with $p > 1$. Hence we can find a function $g \in H(\mathbb{D})$ satisfying (9) such that $g' \in B^p$ and that $f = Be^g$ solves (2), where $A(z)$ is given by (10) and belongs to $H(\mathbb{D})$. On the other hand, by [**5**, p. 80], it is clear that $g'' \in B^{\frac{1}{2}}$. Thus the assertion follows by (21), the Hölder inequality and Lemma 4.

5. Proof of Theorem 5. Let $g(z) = \log(1-z)$ and $h(z) = z/(1-z)$. We see that the functions

$$f_j(z) = \exp\left(g(z) + (-1)^j h(z)\right), \quad j = 1, 2,$$

are linearly independent solutions of (2), where

$$A(z) = -\left(g''(z) + g'(z)^2 + h'(z)^2\right) = -(1-z)^{-4}$$

https://doi.org/10.1017/S0017089514000470 Published online by Cambridge University Press

belongs to $H(\mathbb{D})$ and satisfies

$$\int_{D(0,r)} |A(z)|^{\frac{1}{2}}\, dm(z) = \int_0^r \int_0^{2\pi} \frac{1}{|1-te^{i\theta}|^2}\, d\theta\, t dt$$
$$= \int_0^r \frac{1}{1-t^2} \int_0^{2\pi} \frac{1-t^2}{|1-te^{i\theta}|^2}\, d\theta\, t dt$$
$$= \pi \int_0^r \frac{2t}{1-t^2}\, dt = \pi \log \frac{1}{1-r^2}.$$

Thus $A \notin B^{\frac{1}{2}}$. Since the base functions f_1 and f_2 have no zeros, we turn our attention to their linear combinations. For simplicity, we consider the special case $f = f_2 - Cf_1$, $C > 0$, leaving the general solution $cf_1 + df_2$, $c, d \in \mathbb{C}$, to the reader.

By the $2\pi i$-periodicity of the exponential function, the solution f has zeros exactly at points z_n satisfying $h(z_n) = D + \pi n i$, where $D = (\log C)/2$. Since h maps $\mathbb{D}$ onto the half-plane $\Re z > -1/2$, we need to assume $C > 1/e$ in order for the solution f to have zeros in the first place. Under the assumption $C > 1/e$ the zeros of f are precisely at the points

$$z_n = \frac{D + \pi n i}{1 + D + \pi n i} = \frac{D(1+D) + \pi^2 n^2 + \pi n i}{(1+D)^2 + \pi^2 n^2}, \quad n \in \mathbb{Z}. \tag{25}$$

We see that

$$\sum_{n=-\infty}^{\infty} (1-|z_n|^2)^\alpha = \sum_{n=-\infty}^{\infty} \left(\frac{1+2D}{(1+D)^2 + \pi^2 n^2} \right)^\alpha < \infty$$

holds for every $\alpha > 1/2$. In addition, (8) is not valid for $\alpha \in (0, 1/2]$. On the other hand, by a direct calculation, we obtain

$$1 - \left| \frac{z_n - z_k}{1 - \overline{z}_n z_k} \right|^2 = \frac{(1+2D)^2}{\pi^2 (n-k)^2 + (1+2D)^2}, \tag{26}$$

so that

$$\sum_{n \neq k} \left(1 - \left| \frac{z_n - z_k}{1 - \overline{z}_n z_k} \right| \right) \leq \frac{(1+2D)^2}{\pi^2} \sum_{n \neq k} \frac{1}{(n-k)^2}$$
$$= \frac{2(1+2D)^2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{(1+2D)^2}{3}$$

for every $k \in \mathbb{Z}$. In particular, the upper bound is independent on k. This implies (7), and so the sequence $\{z_n\}_{n\in\mathbb{Z}}$ is uniformly separated.

Since $\{z_n\}_{n\in\mathbb{Z}}$ is uniformly separated, it is also separated. From (26) we see that the separation constant δ defined in (6) is $\delta = \pi/\sqrt{\pi^2 + (1+2D)^2}$. If the constant $C > 1/e$ is chosen close enough to $1/e$, then this δ clearly satisfies the condition (12).

Finally, since $e^h \in N$, it follows that every solution f belongs to N. Moreover, the functions g, h as well as their derivatives belong to the union of all Hardy spaces, and hence in N, so that $f^{(n)} \in N$ for all $n \in \mathbb{N}$.

ACKNOWLEDGEMENT. The research reported in this paper was supported in part by the Academy of Finland, Project #268009, the Faculty of Science and Forestry of the University of Eastern Finland, Project #930349, and the Finnish Academy of Science and Letters.

REFERENCES

1. S. Buckley, P. Koskela and D. Vukotić, Fractional integration, differentiation, and weighted Bergman spaces, *Math. Proc. Camb. Phil. Soc.* **126**(2) (1999), 369–385.

2. L. Carleson, An interpolation problem for bounded analytic functions, *Am. J. Math.* **80** (1958), 921–930.

3. M. Chuaqui, J. Gröhn, J. Heittokangas and J. Rättyä, Zero separation results for solutions of second order linear differential equations, *Adv. Math.* **245** (2013), 382–422.

4. P. Duren, *Theory of H^p spaces* (Academic Press, New York, 1970).

5. P. Duren and A. Schuster, *Bergman Spaces, Mathematical Surveys and Monographs, vol. 100* (American Mathematical Soc., Providence, 2004).

6. E. Fricain and J. Mashreghi, Integral means of the derivatives of Blaschke products, *Glasg. Math. J.* **50**(2) (2008), 233–249.

7. Y. Gotoh, On integral means of the derivatives of Blaschke products, *Kodai Math. J.* **30**(1) (2007), 147–155.

8. J. Gröhn and J. Heittokangas, New findings on Bank-Sauer approach in oscillation theory, *Constr. Approx.* **35** (2012), 345–361.

9. H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman spaces, Graduate Texts in Mathematics, vol. 199* (Springer, New York, 2000).

10. J. Heittokangas, A survey on Blaschke-oscillatory differential equations, with updates, in *Blaschke products and their applications, Fields Inst. Commun. vol. 65* (Fricain E. and Mashreghi J., Editors) (Springer, New York, 2013), 43–98.

11. J. Heittokangas, Solutions of $f'' + A(z)f = 0$ in the unit disc having Blaschke sequences as the zeros, *Comput. Methods Funct. Theory* **5**(1) (2005), 49–63.

12. J. Heittokangas and I. Laine, Solutions of $f'' + A(z)f = 0$ with prescribed sequences of zeros, *Acta Math. Univ. Comen.* **124**(2) (2005), 287–307.

13. J. Heittokangas, R. Korhonen and J. Rättyä, Linear differential equations with coefficients in weighted Bergman and Hardy spaces, *Trans. Am. Math. Soc.* **360** (2008), 1035–1055.

14. H. Kim, Derivatives of Blaschke products, *Pac. J. Math.* **114**(1) (1984), 175–190.

15. C. Linden, H^p-derivatives of Blaschke products, *Mich. Math. J.* **23** (1976), 43–51.

16. Z. Nehari, The Schwarzian derivative and Schlicht functions, *Bull. Am. Math. Soc.* **55** (1949), 545–551.

17. Ch. Pommerenke, On the mean growth of the solutions of complex linear differential equations in the disk, *Complex Var. Theory Appl.* **1**(1) (1982), 23–38.

18. D. Protas, Blaschke products with derivative in H^p and B^p, *Mich. Math. J.* **20** (1973), 393–396.

19. V. Šeda, On some properties of solutions of the differential equation $y'' + Ay = 0$ with solutions having the prescribed zeros, *Acta Fac. Nat. Univ. Comen. Math.* **4** (1959), 223–253. (Slovak)

20. S. Yamashita, Gap series and α-Bloch functions, *Yokohama Math. J.* **28** (1980), 31–36.