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ZERO-FREE REGIONS FOR POLYNOMIALS AND SOME GENERALIZATIONS OF ENESTRÖM–KAKEYA THEOREM

BY

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ABSTRACT. In this paper we shall use matrix methods to obtain several generalizations of a well known result of Eneström and Kakeya about the location of the zeros of polynomials. We shall also obtain zero-free regions of polynomials having complex coefficients. Finally we prove some results concerning the zeros of a class of polynomials.

If $P(z)=\sum_{j=0}^{n} a_jz^j$ is a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then according to a well known result due to Eneström and Kakeya, the polynomial $P(z)$ does not vanish in $|z|>1$.

We may apply this result to $P(z/t)$ to obtain the following more general

THEOREM A. *If $P(z)=\sum_{j=0}^{n} a_jz^j$ is a polynomial of degree n such that*

$$a_n \geq ta_{n-1} \geq \cdots \geq t^{n-1}a_1 \geq t^n a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leq 1/t$.

In the literature [1, 2, 4–7] there exist some extensions of the Eneström–Kakeya theorem. If we drop the restriction that the coefficients are all positive, then the following result which is implicit in [8, p. 137] holds.

THEOREM B. *If $P(z)=\sum_{j=0}^{n} a_jz^j$ is a polynomial of degree n with complex coefficients such that for some $t>0$*

$$|a_n| \geq t^{n-j}\,|a_j|, \qquad j=0,1,2,\ldots,n-1,$$

then $P(z)$ has all its zeros in $|z| \leq (1/t)K_1$ where K_1 is the greatest positive root of the trinomial equation

$$K^{n+1}-2K^n+1=0.$$

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In this paper we generalize Theorem B to lacunary type polynomials and thereby give an independent proof of Theorem B as well. As an application of this theorem, we shall obtain zero-free regions of polynomials having complex coefficients. We also obtain a generalization and a refinement of Theorem 2 of [1]. Finally we shall present certain generalizations of Theorem A.

We start by proving

THEOREM 1. *Let* $P(z)=a_nz^n+a_pz^p+\cdots+a_1z+a_0$, $a_p\neq 0$, $0\leq p\leq n-1$, *be a polynomial of degree n with complex coefficients such that for some* $t>0$

$$|a_n|\geq t^{n-j}\,|a_j|, \qquad j=0,1,2,\ldots,p,$$

then $P(z)$ *has all its zeros in* $|z|\leq(1/t)K_1$ *where* K_1 *is the greatest positive root of the equation*

$$K^{n+1}-K^n-K^{p+1}+1=0 \tag{1}$$

The polynomial $P(z)=(tz)^n-(tz)^p-\cdots-(tz)-1$ *shows that the result is best possible.*

For the proof of this theorem, we shall use the following result due to Gershgorin [3] (see also [10]).

LEMMA 1. *Let* $A=[a_{ij}]$ *be an* $n\times n$ *complex matrix and let* R_i *be the sum of the moduli of the off diagonal elements in the* ith *row. Then each eigenvalue of A lies in the union of the circles*

$$|z-a_{ii}|\leq R_i, \qquad i=1,2,\ldots,n.$$

The analogous result holds if the columns of A are considered.

LEMMA 2. *Let* $P(z)=a_nz^n+a_pz^p+\cdots+a_1z+a_0$, $0\leq p\leq n-1$, *be a polynomial of degree n with complex coefficients. Then for every positive real number r, all the zeros of* $P(z)$ *lie in the circle*

$$|z|\leq \text{Max}\left\{r, \sum_{j=0}^{p}|a_j/a_n|\frac{1}{r^{n-j-1}}\right\}.$$

Proof of Lemma 2. The companion matrix of the polynomial $P(z)$ is

$$C=\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0/a_n \\ 1 & 0 & \cdots & 0 & -a_1/a_n \\ 0 & 1 & \cdots & 0 & -a_2/a_n \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & -a_p/a_n \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

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We take a matrix $P = \operatorname{diag}(r^{n-1}, r^{n-2}, \ldots, r, 1)$ where r is a positive real number and form the matrix

$$P^{-1}CP = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0/a_n r^{n-1} \\ r & 0 & \cdots & 0 & -a_1/a_n r^{n-2} \\ 0 & r & \cdots & 0 & -a_2/a_n r^{n-3} \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & -a_p/a_n r^{n-p-1} \\ \vdots & & & & \\ 0 & 0 & \cdots & r & 0 \end{bmatrix}.$$

Applying Lemma 1 to the columns, it follows that the eigen values of $P^{-1}CP$ lie in the circle

$$|z| \le \operatorname{Max}\left\{ r, \sum_{j=0}^{p} |a_j/a_n| \frac{1}{r^{n-j-1}} \right\}. \tag{2}$$

Since the matrix $P^{-1}CP$ is similar to the matrix C and the eigen values of C are the zeros of $P(z)$, it follows that all the zeros of $P(z)$ lie in the circle defined by (2). This completes the proof of Lemma 2.

Proof of Theorem 1. Since by hypothesis

$$|a_j/a_n| \le \frac{1}{t^{n-j}}, \qquad j = 0, 1, 2, \ldots, p,$$

it follows by Lemma 2 that for every positive real number r, all the zeros of $P(z)$ lie in the circle

$$|z| \le \operatorname{Max}\left\{ r, r \sum_{j=0}^{p} \frac{1}{(rt)^{n-j}} \right\}. \tag{3}$$

We choose r such that

$$\sum_{j=0}^{p} \frac{1}{(rt)^{n-j}} = 1,$$

which gives

$$(rt)^p + (rt)^{p-1} + \cdots + (rt) + 1 = (rt)^n.$$

Equivalently

$$(rt)^{n+1} - (rt)^n - (rt)^{p+1} + 1 = 0.$$

Replacing rt by K, it follows from (3) that all the zeros of $P(z)$ lie $|z| \le (1/t)K_1$, where K_1 is the greatest positive root of the equation defined by (1) and the theorem is proved.

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As an application of Theorem B we prove

THEOREM 2. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients, then for every real* $a>0$, $P(z)$ *does not vanish in the disk*

$$|z-ae^{i\alpha}|<\frac{a}{2n},$$

where $\operatorname{Max}_{|z|=a}|P(z)|=|P(ae^{i\alpha})|$.

For the proof of Theorem 2, we also need the following lemma [9].

LEMMA 3. *Let* $P(z)$ *be a polynomial of degree* $n\geq 1$, *then*

$$\operatorname*{Max}_{|z|=r}|P'(z)|\leq (n/r)\operatorname*{Max}_{|z|=r}|P(z)|.$$

The next lemma is obtained by the repeated application of Lemma 3.

LEMMA 4. *Let* $P(z)$ *be a polynomial of degree* $n\geq 1$, *then*

$$\operatorname*{Max}_{|z|=r}|P^{(k)}(z)|\leq\frac{n(n-1)\cdots(n-k+1)}{r^k}\operatorname*{Max}_{|z|=r}|P(z)|,$$

$k=1,2,\ldots,n.$

Proof of Theorem 2. Let a be a positive real number and $w=ae^{i\alpha}$, so that $|w|=a$. Consider the polynomial

$$\begin{aligned}F(z)&=P\left(\frac{wz}{n}+w\right)\\&=P(w)+(w/n)P'(w)z+(w/n)^2P''(w)\frac{z^2}{2!}+\cdots+(w/n)^nP^{(n)}(w)\frac{z^n}{n!},\end{aligned}$$

then

$$\begin{aligned}G(z)=z^nF(1/z)&=P(w)z^n+(w/n)P'(w)z^{n-1}+\cdots+(w/n)^n\frac{P^{(n)}(w)}{n!}\\&=\sum_{k=0}^{n}(w/n)^k\frac{P^{(k)}(w)}{k!}z^{n-k}.\end{aligned}$$

Since $|w|=a$, we have by Lemma 4

$$\begin{aligned}|P(w)|=|P(ae^{i\alpha})|&=\operatorname*{Max}_{|z|=a}|P(z)|\\&\geq\frac{a^k}{n(n-1)\cdots(n-k+1)}\operatorname*{Max}_{|z|=a}|P^{(k)}(z)|\\&\geq\frac{a^k}{n(n-1)\cdots(n-k+1)}|P^{(k)}(w)|\\&\geq\frac{|w|^k\,|P^{(k)}(w)|}{n^kk!}\quad\text{for all } k=1,2,\ldots,n.\end{aligned}$$

That is

$$|P(w)| \geq \left|(w/n)^k \frac{P^{(k)}(w)}{k!}\right|, \qquad k = 1, 2, \ldots, n.$$

This shows that the polynomial $G(z)$ satisfies the conditions of Theorem B with $t = 1$ and therefore, it follows that all the zeros of $G(z)$ lie in $|z| \leq K_1$, where K_1 is the greatest positive root of the equation

$$K^{n+1} - 2K^n + 1 = 0. \tag{4}$$

Since $G(z) = z^n F(1/z)$, we conclude that all the zeros of $F(z)$ lie in $|z| \geq 1/K_1$. That is $F(z) = P((w/n)z + w)$ does not vanish in $|z| < 1/K_1$. Replacing z by $(n/w)(z - w)$, it follows that $P(z)$ does not vanish in the disk $|z - w| < |w|/nK_1$. Now using the fact that all the zeros of the trinomial equation defined by (4) lie in $|z| < 2$, we conclude that the polynomial $P(z)$ does not vanish in $|z - w| < |w|/2n$. This completes the proof of Theorem 2.

Next we prove the following result which is both a generalization and a refinement of Theorem 2 of [1].

THEOREM 3. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n > 1$, such that for some $t > 0$*

$$ta_j \geq a_{j-1} > 0, \qquad j = 2, 3, \ldots, n, \tag{5}$$

a_0 may be a real or a complex number, then $P(z)$ cannot have a zero of order greater or equal to 2 of modulus greater than $t(n-1)/n$. In other words, all the zeros of $P(z)$ of modulus greater than $t(n-1)/n$ are simple.

Proof of Theorem 3. Since $a_j > 0$, $j = 1, 2, \ldots, n$ and

$$P'(z) = \sum_{j=0}^{n-1} (j+1)a_{j+1} z^j = \sum_{j=0}^{n-1} b_j z^j \quad \text{(say)},$$

it follows that $P'(z)$ is a polynomial of degree $n - 1$ with real positive coefficients. Now by (5)

$$ta_{j+1} \geq a_j, \qquad \text{for all } j = 1, 2, \ldots, n-1,$$

also

$$\frac{n-1}{n} \geq \frac{j}{j+1} \quad \text{for all } j = 1, 2, \ldots, n-1,$$

therefore, we have

$$\frac{t(n-1)}{n}(j+1)a_{j+1} \geq ja_j, \qquad j = 1, 2, \ldots, n-1.$$

That is

$$\frac{t(n-1)}{n} b_j \geq b_{j-1}, \quad \text{for all } j = 1, 2, \ldots, n-1.$$

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Applying Theorem A (with t replaced by $n/t(n-1)$), we conclude that all the zeros of $P'(z)$ lie in $|z|\le t(n-1)/n$. This shows that all the zeros of $P(z)$ of modulus greater than $t(n-1)/n$ are simple and the Theorem 3 is proved.

COROLLARY 1. *All the zeros of $P(z)$ of Theorem 3 of modulus t are simple.*

We now prove the following generalization of Theorem A.

THEOREM 4. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some $k=0,1,\ldots,n$ and for some $t>0$*

$$t^n|a_n|\le t^{n-1}|a_{n-1}|\le\cdots\le t^k|a_k|\ge t^{k-1}|a_{k-1}|\ge\cdots\ge t|a_1|\ge|a_0|,$$

then $P(z)$ has all its zeros in the circle

$$|z|\le t\{(2t^k|a_k|/t^n|a_n|)-1\}+2\sum_{j=0}^{n}|a_j-|a_j||/|a_n|\,t^{n-j-1}.$$

Proof of Theorem 4. Consider the polynomial

$$\begin{aligned}F(z)&=(t-z)P(z)\\&=-a_nz^{n+1}+(ta_n-a_{n-1})z^n+\cdots+(ta_1-a_0)z+ta_0.\end{aligned}$$

Applying Lemma 2 to the polynomial $F(z)$, which is of degree $n+1$, with $p=n$ and $r=t$, it follows that all the zeros of $F(z)$ lie in the circle

$$\begin{aligned}|z|&\le \mathrm{Max}\left\{t,\sum_{j=0}^{n}|ta_j-a_{j-1}|/t^{n-j}|a_n|\right\}\qquad(a_{-1}=0)\\&=\sum_{j=0}^{n}|ta_j-a_{j-1}|/t^{n-j}|a_n|,\end{aligned}\tag{6}$$

since

$$t=\left|\sum_{j=0}^{n}\frac{(ta_j-a_{j-1})}{t^{n-j}a_n}\right|\le\sum_{j=0}^{n}\frac{|ta_j-a_{j-1}|}{t^{n-j}|a_n|}.$$

Now

$$\begin{aligned}\sum_{j=0}^{n}\frac{|ta_j-a_{j-1}|}{t^{n-j}|a_n|}&\le\sum_{j=0}^{n}\frac{|t|a_j|-|a_{j-1}||}{t^{n-j}|a_n|}+\sum_{j=0}^{n}\frac{|t(a_j-|a_j|)-(a_{j-1}-|a_{j-1}|)|}{t^{n-j}|a_n|}\\&=\sum_{j=0}^{k}\frac{t|a_j|-|a_{j-1}|}{t^{n-j}|a_n|}+\sum_{j=k+1}^{n}\frac{|a_{j-1}|-t|a_j|}{t^{n-j}|a_n|}\\&\quad+\sum_{j=0}^{n}\frac{|t(a_j-|a_j|)-(a_{j-1}-|a_{j-1}|)|}{t^{n-j}|a_n|}\\&=t\left\{\frac{2t^k|a_k|}{t^n|a_n|}-1\right\}+\sum_{j=0}^{n}\frac{|t(a_j-|a_j|)-(a_{j-1}-|a_{j-1}|)|}{t^{n-j}|a_n|}\\&\le t\left\{\frac{2t^k|a_k|}{t^n|a_n|}-1\right\}+2\sum_{j=0}^{n}\frac{|a_j-|a_j||}{t^{n-j-1}|a_n|},\end{aligned}$$

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therefore, it follows from (6) that all the zeros of $F(z)$ lie in

$$|z| \le t\left\{\frac{2t^k\,|a_k|}{t^n\,|a_n|} - 1\right\} + 2\sum_{j=0}^{n} \frac{|a_j - |a_j||}{t^{n-j-1}\,|a_n|}.$$

Since all the zeros of $P(z)$ are also the zeros of $F(z)$, the theorem is proved.

Applying Theorem 4 with $t = 1$ and $k = 0$ to the polynomial $z^n P(1/z)$, we get the following

COROLLARY 2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real positive coefficients such that*

$$a_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0 > 0,$$

then $P(z)$ does not vanish in

$$|z| < \frac{a_0}{(2a_n - a_0)}.$$

The polynomial $P(z) = z^n + \cdots + z + 1$ shows that the result is best possible.

REMARK. It is interesting to examine the bound if in addition to the hypothesis of Theorem 4, the coefficients a_j of $P(z)$ are such that

$$|\arg a_j - \beta| \le \alpha \le \pi/2, \qquad j = 0, 1, 2, \ldots, n,$$

for some real β. In this case it is easy to verify that

$$|ta_j - a_{j-1}| \le |t\,|a_j| - |a_{j-1}||\cos\alpha + (t\,|a_j| + |a_{j-1}|)\sin\alpha,$$

$j = 0, 1, 2, \ldots, n$. Using these observations in (6) and proceeding similarly as in the proof of Theorem 4, it follows that all the zeros of $P(z)$ lie in

$$|z| \le t\left\{\left(\frac{2t^k\,|a_k|}{t^n\,|a_n|} - 1\right)\cos\alpha + \sin\alpha\right\} + 2\sin\alpha \sum_{j=0}^{n-1} \frac{|a_j|}{t^{n-j-1}\,|a_n|}.$$

For $k = n$ and $\alpha = \beta = 0$, this reduces to Theorem A. Also for $k = n$ and $t = 1$ this reduces to a result proved by Govil and Rahman [4, Theorem 2].

Finally we state the following two generalizations of Theorem A. As their proofs are almost similar to the proof of Theorem 4, we omit the details.

THEOREM 5. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If* Re $a_j = \alpha_j$, Im $a_j = \beta_j$, $j = 0, 1, \ldots, n$ *and for some* $t > 0$

$$0 < t^n\alpha_n \le \cdots \le t^k\alpha_k \ge t^{k-1}\alpha_{k-1} \ge \cdots \ge t\alpha_1 \ge \alpha_0 \ge 0,$$

where $0 \le k \le n$, then all the zeros of $P(z)$ lie in the circle

$$|z| \le t\left(\frac{2t^k\alpha_k}{t^n\alpha_n} - 1\right) + \frac{2}{\alpha_n}\sum_{j=0}^{n} \frac{|\beta_j|}{t^{n-j-1}}.$$

THEOREM 6. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients. If* $\operatorname{Re} a_j=\alpha_j$, $\operatorname{Im} a_j=\beta_j$, $j=0,1,2,\ldots,n$ *and a positive number t can be found such that*

$$0\le\alpha_0\le t\alpha_1\le\cdots\le t^k\alpha_k\ge t^{k+1}\alpha_{k+1}\ge\cdots\ge t^n\alpha_n>0,\qquad 0\le k\le n$$

and

$$0\le\beta_0\le t\beta_1\le\cdots\le t^r\beta_r\ge t^{r+1}\beta_{r+1}\ge\cdots\ge t^n\beta_n\ge 0,\qquad 0\le r\le n,$$

then all the zeros of $P(z)$ *lie in the circle*

$$|z|\le\frac{t}{|a_n|}\{2(t^{k-n}\alpha_k+t^{r-n}\beta_r)-(\alpha_n+\beta_n)\}.$$

If we take $k=r=n$ in Theorem 6, we obtain

COROLLARY 3. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients. If* $\operatorname{Re} a_j=\alpha_j$, $\operatorname{Im} a_j=\beta_j$, $j=0,1,\ldots,n$ *and for some positive number t*

$$0\le\alpha_0\le t\alpha_1\le\cdots\le t^n\alpha_n,\qquad 0\le\beta_0\le t\beta_1\le\cdots\le t^n\beta_n,$$

then all the zeros of $P(z)$ *lie in the circle*

$$|z|\le t\left(\frac{\alpha_n+\beta_n}{|a_n|}\right)\le 2^{1/2}t.$$

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