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RINGS WITH NO NILPOTENT ELEMENTS AND WITH THE MAXIMUM CONDITION ON ANNIHILATORS

BY
W. H. CORNISH AND P. N. STEWART

1. **Introduction.** Rings (all of which are assumed to be associative) with no non-zero nilpotent elements will be called *reduced rings; R* is a reduced ring if and only if $x^2=0$ implies $x=0$, for all $x \in R$. In 2. we prove that the following conditions on an annihilator ideal I of a reduced ring are equivalent: I is a maximal annihilator, I is prime, I is a minimal prime, I is completely prime. A characterization of reduced rings with the maximum condition on annihilators is given in 3.

Let R be a ring in which $xy=0$ if and only if $yx=0$, for all $x, y \in R$. If $x \in R$ and $x^n=0$ for some integer $n \geq 1$, then any product of elements of R involving n occurrences of x must be 0. To see this let n_i, $i=1, \ldots, k$ be positive integers such that $n_1+\cdots+n_k=n$ and let r_i, $i=1, \ldots, k-1$ be elements of R. Then in succession we obtain: $x^n=0$, $r_1x^n=0$, $x^{n-n_1}r_1x^{n_1}=0, \ldots, x^{n_k}r_{k-1}x^{n_{k-1}}\cdots r_1x^{n_1}=0$. It follows that if $x^n=0$, then $X^n=(0)$ where X is the ideal of R which is generated by x. Therefore, the set of nilpotent elements of R is an ideal N, N is the prime radical of R, and R/N is a reduced ring.

In fact, for R to be a reduced ring it is necessary and sufficient that R be semi-prime (that is, R have no non-zero nilpotent ideals) and $xy=0$ if and only if $yx=0$, for all $x, y \in R$. This follows because in a reduced ring R, if $xy=0$ then $(yx)^2=y(xy)x=0$ and so $yx=0$. Of course, all commutative semi-prime rings are reduced rings.

Finally we note that every ring R contains a unique smallest ideal I such that R/I is a reduced ring. For details see the discussion of the generalized nil radical in Divinsky [1].

2. **Maximal annihilators and minimal primes.** An ideal P of a ring R is *prime* if and only if $P \neq R$ and $aRb \subseteq P$ implies that $a \in P$ or $b \in P$, for all $a, b \in R$; P is *completely prime* if $P \neq R$ and $ab \in P$ implies that $a \in P$ or $b \in P$, for all $a, b \in R$.

Let R be a ring in which $xy=0$ if and only if $yx=0$, for all $x, y \in R$. If $S \subseteq R$ we shall denote the *annihilator* of S by S^*; that is,

$$S^* = \{r \in R : rs = 0 \quad \text{for all} \quad s \in S\}.$$

Because of the condition on R, an annihilator S^* is a two-sided ideal of R. If $y \in R$ we shall denote $\{y\}^*$ by y^*. An annihilator S^* is *maximal* if and only if $S^* \neq R$ and $S^* \subseteq T^* \neq R$ implies that $S^*=T^*$, for all $T \subseteq R$. If $S^* \neq R$ then there

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is a $y \in S$ such that $y^* \neq R$. Clearly $S^* \subseteq y^*$ so all maximal annihilators are of the form y^* for some $y \in R$.

PROPOSITION 2.1. *Let R be a reduced ring and $S \subseteq R$. Then the following are equivalent:*

(i) *S^* is a maximal annihilator,*

(ii) *S^* is prime,*

(iii) *S^* is a minimal prime,*

(iv) *S^* is completely prime.*

Proof. (i)→(ii) Select $y \in S$ such that $S^*=y^*$. Since $y^* \subseteq (y^2)^*$ and $y^3 \neq 0$, $y^*=(y^2)^*$. If $a \in R$ and $(ya)^*=R$ then $y^2a=0$ so $a \in (y^2)^*=y^*$; thus, if $a \notin y^*$ $y^*=(ya)^*$. It follows that y^* is completely prime, so of course $S^*=y^*$ is prime.

(ii)→(iii) Suppose that Q is a prime ideal and $Q \subseteq S^*$. Since S^* is a prime ideal, $S^* \neq R$; so we may choose a non-zero $y \in S$. If $a \in S^*$ in succession we obtain: $ay=0$, $Ray=(0)$, $yRa=(0) \subseteq Q$, $y \in Q$ or $a \in Q$. Since $Q \subseteq S^*$ and $y^2 \neq 0$, $y \notin Q$. Therefore $a \in Q$ so $Q=S^*$.

(iii)→(iv) Suppose that $ab \in S^*$. In succession we obtain, for each $y \in S$: $aby=0$, $bya=0$, $Rbya=(0)$, $aRby=(0)$. Thus $aRb \subseteq S^*$ and since S^* is prime, $a \in S^*$ or $b \in S^*$.

(iv)→(i) Suppose that $S^* \subseteq T^* \neq R$. Since $T^* \neq R$ there is a non-zero $y \in T$. If $a \in T^*$ then $ay=0 \in S^*$, so $a \in S^*$ or $y \in S^*$. Because $S^* \subseteq T^*$ and $y^2 \neq 0$, $y \notin S^*$. Therefore $a \in S^*$ so $S^*=T^*$.

3. **Reduced rings with the maximum condition on annihilators.** For any two sets A and B, let $A-B=\{x \in A : x \notin B\}$. We require the following rather technical lemma.

LEMMA 3.1. *Let R be a ring and P_i, $i=1,\ldots,n$ any prime ideals of R such that for all k, $l \leq n$, $P_l \nsubseteq P_k$ if $l \neq k$.*

If $a \in R$ and L is a left ideal of R such that for some k, $0 \leq k \leq n$:

$$a \notin P_i \quad \text{if} \quad 1 \leq i \leq k$$
$$a \in P_j \quad \text{if} \quad n \geq j \geq k+1$$
$$L \nsubseteq P_j \quad \text{if} \quad n \geq j \geq k+1,$$

then there is a $d \in R-\bigcup_{i=1}^{n} P_i$ such that $d-a \in L \cap [\bigcap_{i=1}^{k} P_i]$.

Notice that if $k=0$, $L \cap [\bigcap_{i=1}^{k} P_i]=L$.

Proof. Let $j \geq k+1$. By assumption $L \nsubseteq P_j$ and $P_i \nsubseteq P_j$ for $i \neq j$, so

$$L \cap \left[\bigcap_{i \neq j} P_i\right] \nsubseteq P_j$$

because P_j is a prime ideal. Thus we may choose $u_j \in (L \cap [\bigcap_{i \neq j} P_i])-P_j$.

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Let

$$d = a + \sum_{j=k+1}^{n} u_j.$$

Now, $u_j \in L \cap [\bigcap_{i=1}^k P_i]$ for all $j \geq k+1$ so $d-a \in L \cap [\bigcap_{i=1}^k P_i]$.

If $d \in P_i$ for some $i \leq k$, then $a = d - \sum_{j=k+1}^n u_j \in P_i$ contrary to assumption.

If $d \in P_l$ for some $l \geq k+1$, then $u_l = d - a - \sum_{j=k+1, j\neq l}^n u_j \in P_l$ contrary to the way in which u_l was chosen.

Therefore, $d \in R - \bigcup_{i=1}^n P_i$.

An element d of a ring R is *regular* if and only if for every $r \in R$, $rd=0$ or $dr=0$ implies that $r=0$. A ring R is an *integral domain* if and only if every non-zero element of R is regular. Finally, R is a ring with *max*$-a$ (*the maximum condition on annihilators*) if and only if every non-empty set of annihilators has a maximal element.

THEOREM 3.2. *For any ring $R \neq (0)$ the following are equivalent:*

(i) *R is a reduced ring with max$-a$,*

(ii) *R has only a finite number of distinct minimal prime ideals P_i, $i=1, \ldots, n$; $\bigcap_{i=1}^n P_i = (0)$, and all elements in $R - \bigcup_{i=1}^n P_i$ are regular,*

(iii) *R has a finite number of completely prime ideals Q_i, $i=1, \ldots, k$ such that $\bigcap_{i=1}^k Q_i = (0)$,*

(iv) *R is isomorphic to a subring of a direct product of a finite number of integral domains.*

Proof. (i)$\to$(ii) Choose a non-zero $y \in R$. Since $y^2 \neq 0$, $y^* \neq R$; so y^* is contained in a maximal annihilator of R. Thus R has maximal annihilators.

Let $P_i = y_i^*$, $i=1, \ldots, k+1$ be maximal annihilator ideals of R. Suppose that $y_{k+1} \in [\bigcap_{i=1}^k P_i]^*$. Then $P_{k+1} = y_{k+1}^* \supseteq [\bigcap_{i=1}^k P_i]^{**} \supseteq \bigcap_{i=1}^k P_i$. Since, by 2.1, P_{k+1} is prime, $P_j \subseteq P_{k+1}$ for some $j \leq k$. By the maximality of P_j, $P_j = P_{k+1}$. Therefore, if the annihilators P_i, $i=1, \ldots, k+1$ are distinct, $y_{k+1} \notin [\bigcap_{i=1}^k P_i]^*$ and consequently $[\bigcap_{i=1}^{k+1} P_i]^* \supsetneq [\bigcap_{i=1}^k P_i]^*$.

Since R is a ring with *max*$-a$, there are only a finite number $P_i = y_i^*$, $i=1, \ldots, n$ of distinct maximal annihilators, and by 2.1 they are all minimal prime ideals.

If $x \in R$ and $x \neq 0$ then $x^* \subseteq P_j$ for some $j \leq n$, so if $x \in \bigcap_{i=1}^n P_i$ then $y_j \in x^* \subseteq P_j = y_j^*$ and hence $y_i^2 = 0$. Since R is a reduced ring, $\bigcap_{i=1}^n P_i = (0)$.

It follows that for any prime ideal P of R, $P_j \subseteq P$ for some $j \leq n$. Thus P_i, $i = 1, \ldots, n$ are the only minimal prime ideals of R.

If $y, z \in R$, $yz=0$ and $z \neq 0$ then $z^* \neq R$ so $y \in z^* \subseteq P_j$ for some $j \leq n$. Therefore, if $y \in R - \bigcup_{i=1}^n P_i$ then y is regular.

(ii)$\to$(iii) It is sufficient to prove that each $P_j, j \leq n$, is completely prime.

First notice that $R - \bigcup_{i=1}^n P_i$ = the set of regular elements of R. This follows because we are assuming that all elements in $R - \bigcup_{i=1}^n P_i$ are regular; and no element in $\bigcup_{i=1}^n P_i$ can be regular because for each $j \leq n$, $P_j[\bigcap_{i\neq j} P_i] \subseteq \bigcap_{i=1}^n P_i = (0)$, and $\bigcap_{i \neq j} P_i \neq (0)$ since the minimal prime ideals P_i, $i=1, \ldots, n$ are distinct.

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Suppose that $a, b \in R-P_j$ for some $j \leq n$. Taking $L=R$ in 3.1 we find regular elements $d, d_1 \in R-\bigcup_{i=1}^n P_i$ such that $d-a, d_1-b \in P_j$. Now $(d-a)b=db-ab \in P_j$ and $d(d_1-b)=dd_1-db \in P_j$; so if $ab \in P_j$ then $db \in P_j$ and $dd_1 \in P_j$. But $dd_1 \notin P_j$ because dd_1 is regular, so $ab \in R-P_j$. Therefore each $P_j, j \leq n$, is completely prime.

(iii)→(iv) The ring R is isomorphic to a subdirect product of the integral domains R/Q_i, $i=1, \ldots, k$.

(iv)→(i) A finite direct product of integral domains has no non-zero nilpotent elements and only a finite number of annihilators. Both properties are inherited by subrings.

We note that these results can be applied to obtain the following version of Goldie's Theorem for reduced rings (see [2] for definitions).

THEOREM 3.3 (Goldie). *A ring $R \neq (0)$ has a classical left quotient ring which is isomorphic to a finite direct product of division rings if and only if R is a reduced ring with max−a and Rd is essential for each regular $d \in R$.*

To summarise: if $R \neq (0)$ is a reduced ring with *max−a*, then R has only a finite number of distinct minimal prime ideals P_i, $i=1, \ldots, n$ and

$$P_i = y_i{}^*, \; i = 1, \ldots, n$$

are maximal annihilators,

$$P_i \qquad , i = 1, \ldots, n$$

are completely prime,

$$\bigcap_{i=1}^{n} P_i = (0), \text{ and}$$

$$R - \bigcup_{i=1}^{n} P_i = \text{the set of regular elements of } R.$$

If R satisfies the conditions of 3.3, then

$$Q(R) \cong \prod_{i=1}^{n} Q(R)/Q(R)P_i \cong \prod_{i=1}^{n} Q(R/P_i)$$

where for any ring A, $Q(A)$ denotes a classical left quotient ring of A. The last isomorphism is due to Goldie, a proof can be found in Lambek [2, 4.6].

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THE FLINDERS UNIVERSITY OF SOUTH AUSTRALIA,
BEDFORD PARK, SOUTH AUSTRALIA
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HALIFAX, NOVA SCOTIA