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THE RESIDUAL SPECTRUM OF G_2

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ABSTRACT. We completely determine the residual spectrum of the split exceptional group of type G_2, thus completing the work of Langlands and Moeglin-Waldspurger on the part of the residual spectrum attached to the trivial character of the maximal torus. We also give the Arthur parameters for the residual spectrum coming from Borel subgroups. The interpretation in terms of Arthur parameters explains the "bizarre" multiplicity condition in Moeglin-Waldspurger's work. It is related to the fact that the component group of the Arthur parameter is non-abelian.

1. **Introduction.** Let F be a number field and $(\mathbb{A})$ its ring of adeles. Let G be a reductive group. A central problem in the theory of automorphic forms is to decompose the right regular representation of $G(\mathbb{A})$ acting on the Hilbert space $L^2\big(G(F)\backslash G(\mathbb{A})\big)$. It has the continuous spectrum and the discrete spectrum:

$$L^2\big(G(F)\backslash G(\mathbb{A})\big) = L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big) \oplus L^2_{\text{cont}}\big(G(F)\backslash G(\mathbb{A})\big).$$

We are mainly interested in the discrete spectrum. Langlands [L4] described, using his theory of Eisenstein series, an orthogonal decomposition:

$$L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big) = \bigoplus_{(M,\pi)} L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(M,\pi)},$$

where (M,π) is a Levi subgroup with a cuspidal automorphic representation π taken modulo conjugacy. (Here we normalize π so that the action of the maximal split torus in the center of G at the archimedean places is trivial.) $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(M,\pi)}$ is a space of residues of Eisenstein series associated to (M,π). Here we note that the subspace

$$\bigoplus_{(G,\pi)} L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(G,\pi)},$$

is the space of cuspidal representations $L^2_{\text{cusp}}\big(G(F)\backslash G(\mathbb{A})\big)$. Its orthogonal complement in $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ is called the residual spectrum and we denote it by $L^2_{\text{res}}\big(G(F)\backslash G(\mathbb{A})\big)$. Therefore we have an orthogonal decomposition

$$L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A}) = L^2_{\text{cusp}}\big(G(F)\backslash G(\mathbb{A})\big) \oplus L^2_{\text{res}}\big(G(F)\backslash G(\mathbb{A})\big).$$

Arthur described a conjectural decomposition of this space as follows:

$$L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big) = \bigoplus_{\psi} L^2\big(G(F)\backslash G(\mathbb{A})\big)_{\psi},$$

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where ψ runs, modulo conjugacy, through the set of morphism:

$$\psi: L_F \times \mathrm{SL}(2,\mathbb{C}) \longmapsto G^*,$$

where L_F is the conjectural tannakian group, G^* is the Langlands' L-group and ψ satisfies certain conditions; in particular, ψ restricted to $\mathrm{SL}(2,\mathbb{C})$ is algebraic, ψ restricted to L_F parametrize a cuspidal tempered representation of a Levi subgroup and the image of ψ is not included in proper Levi subgroups. The space $L^2\big(G(F)\backslash G(\mathbb{A})\big)_\psi$ is defined by local data (See Section 3.7).

The purpose of this paper is to determine explicitly $L^2_{\mathrm{res}}\big(G(F)\backslash G(\mathbb{A})\big)$ for G the split exceptional group of type G_2. There are 3 Levi subgroups modulo conjugacy. Let M_1 be the Levi of the maximal parabolic subgroup P_1 attached to the long simple root and M_2 be the Levi of the maximal parabolic subgroup P_2 attached to the short simple root. Let $M_0 = T$ be the maximal split torus. Let

$$L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_i} = \bigoplus L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(M_i,\pi)}.$$

Theorems 3.6.1, Theorem 4.2 and Theorem 5.1 describe the decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_i}$ for $i = 0, 1, 2$, respectively. Due to the lack of information on the poles of the adjoint cube L-function of GL_2 (symmetric cube L-function of GL_2 twisted by a character) and insufficient bound for Fourier coefficients of cuspidal representations of GL_2, our results on $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_i}$, $i = 1, 2$, are incomplete. Also we only consider the K-finite, K_∞-invariant subspace of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$. We also give the Arthur parameter for automorphic representations in $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$ and verify Arthur's conjecture, reformulated by Moeglin [M3].

In order to obtain the decomposition for $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$, we use the inner product formula (3.1) of the pseudo-Eisenstein series as in [M-W1]. For $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_i}$, $i = 1, 2$, we use the method in [Ki], where we calculated the residual spectrum of Sp_4. We use the notation of [M-W1]. Let us explain in detail in each of the cases.

The most interesting one is the analysis of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$. It was Langlands [L4, Appendix 3] who first calculated its K-fixed subspace. It is dimension 2. One is spherical and the other is a very interesting residual automorphic representation. Its Archimedean component is infinite dimensional, of class one and is not tempered. Moeglin and Waldspurger [M-W1, Appendix III] calculated K-finite, K_∞-invariant subspace V of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,1)}$, where 1 is the trivial character of T, whose cuspidal exponents are short roots. They found surprising results that only those which satisfy a certain condition appear in $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$. Let us explain in detail. Let $J_v = \{\pi_{1v}, \pi_{2v}\}$ be a set of irreducible representations of G_v, where π_{1v} is spherical. For S a finite set of finite places, set $\pi^S = \otimes_{v\notin S}\pi_{1v} \otimes \otimes_{v\in S}\pi_{2v}$. Then

$$V = \bigoplus_{S,\mathrm{card}(S)\neq 1} \pi^S.$$

The condition $\mathrm{card}(S) \neq 1$ is quite surprising (In Sp_4 case [Ki], we have the condition "card(S) even"). We can interpret their results in terms of Arthur parameters. In fact, the condition comes from the Springer correspondence (see Section 3.7 for more details) and is related to the fact that $A(u)$ is non-abelian, *i.e.*, S_3, the symmetric group on 3 letters. Recall that the Springer correspondence is an injective map from the set of irreducible characters of W, the Weyl group of G, into the set of pairs (O,η), where O is a unipotent orbit and η is an irreducible character of $A(u) = C(u)/C(u)^0$, $u \in O$ and $C(u)$ is the centralizer of u. Let Springer(O) be the set of characters of $A(u)$ which are in the image of Springer correspondence. Then J_v is associated with $\mathrm{Springer}\big(G_2(a_1)\big)$, where $G_2(a_1)$ is the sub-regular unipotent orbit of G_2 ([Ca, p.401]). We note that Moeglin [M1] proved that the residual spectrum attached to the trivial character of the torus is parametrized by distinguished unipotent orbits O and Springer(O) and we can expect that the same thing would happen for all split groups.

Among non-trivial characters of $T(\mathbb{A})/T(F)$, modulo conjugacy, the following characters of the torus contribute to the residual spectrum (see Section 3.6). Under the identification, $M_1 \simeq \mathrm{GL}_2$, where M_1 is the Levi subgroup of P_1, $\chi = \chi(\mu,\nu)$, μ and ν are grössencharacters of F.

(1) $\mu = \nu, \mu^2 = 1, \mu \neq 1$

(2) $\mu^3 = 1, \mu \neq 1, \nu = \mu^2$.

For Case (2), there is only one residual spectrum, which is the global Langlands' quotient of $\mathrm{Ind}_{P_1}^G \exp\big(\beta_4, H_{P_1}()\big) \otimes (\mathrm{Ind}_{B_0}^{\mathrm{GL}_2}\chi)$. Case (1) is more interesting. In this case the Eisenstein series has a pole at β_2, the sum of two simple positive roots. If μ_v is not trivial, then the character $\chi_v \otimes \exp\big(\beta_2, H_B(\ )\big)$ is regular and we can apply Rodier's result ([R]) to analyze the image of the intertwining operator $R(\rho_2,\beta_2,\rho_2\chi)$. In particular, it is irreducible. If μ_v is trivial, then the image of $R(\rho_2,\beta_2,\rho_2\chi)$ is the same as the one Moeglin and Waldspurger found [M-W1, Appendix III]. It is the sum of two irreducible representations. Let $J(\chi_v) = \{\pi_{1v},\pi_{2v}\}$ is the set of irreducible components. We put $\pi_{2v} = 0$ if $\chi_v \neq 1$. For S a finite set of finite places, set $\pi^S = \otimes_{v\notin S}\pi_{1v} \otimes \otimes_{v\in S}\pi_{2v}$. Then the residual spectrum attached to the character (1) is given by

$$J(\chi) = \bigoplus_S \pi^S.$$

There is no condition on S.

For $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1}$, the Levi factor is $M_1 = \mathrm{GL}_2$. We have to look at Eisenstein series associated to cuspidal representations of GL_2. The L-functions in the constant terms of Eisenstein series are the adjoint cube L-function and Hecke L-function. We analyze the poles and the irreducibility of the images of local intertwining operators as in [Ki]. However, the pole of the adjoint cube L-function of GL_2 is not known. We assume the location of poles. Also we need to assume certain estimates of Fourier coefficients of cuspidal representations of GL_2. More precisely, if $\pi_v = \pi(\mu|\ |^r, \mu|\ |^{-r})$ is a complementary series representation of GL_2, we assume that $r < \frac{1}{6}$. Right now the best known result is that $r < \frac{1}{5}$ due to Shahidi [S3]. Assuming these facts, we obtain a decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1}$, parametrized by cuspidal representations π of GL_2 with trivial central characters and $L(\frac{1}{2},\pi,r_3^0) \neq 0$ and by monomial representations

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of GL_2. Recall that a cuspidal representation σ of GL_2 is called monomial if $\sigma \simeq \sigma \otimes \eta$ for a quadratic grössencharacter η of F.

For $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_2}$, the Levi factor is $M_2 = \mathrm{GL}_2$. The L-functions in the constant term of the Eisenstein series are just Jacquet-Langlands' L-function, its twist by a character, and a Hecke L-function. In this case, assuming the fact on the estimates of Fourier coefficients of cuspidal representations of GL_2, we obtain a decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_2}$, parametrized by cuspidal representations π of GL_2 with trivial central characters and $L(\frac{1}{2},\pi) \neq 0$.

F. Shahidi brought to our attention the paper by Li and Schwermer [Li-Sc] who studied the poles of Eisenstein series attached to the maximal parabolic subgroups over $\mathbb{Q}$. After this paper was accepted, the author received a preprint, "The residual spectrum of the group of type G_2," by S. Zampera. She obtained similar results. But her result does not have the interpretation in terms of Arthur parameters.

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2. **Some facts about G_2; roots and parabolic subgroups.** Let G be a split group of type G_2. Fix a Cartan subgroup T in G and let $B = TU$ be a Borel subgroup of G. Let K_∞ be the standard maximal compact subgroup in $G(\mathbb{A}_\infty)$ and $K_v = G(\mathcal{O}_v)$ for finite v. The product $K = K_\infty \times \prod K_v$ is a maximal compact subgroup in $G(\mathbb{A})$.

We follow Moeglin and Waldspurger [M-W1, Appendix III]. Let β_1 be a long simple root and β_6 a short one. Let

$$\beta_2 = \beta_1 + \beta_6, \quad \beta_3 = 2\beta_1 + 3\beta_6, \quad \beta_4 = \beta_1 + 2\beta_6, \quad \beta_5 = \beta_1 + 3\beta_6$$

then $\{\beta_1, \ldots, \beta_6\}$ is a set of positive roots.

Let $\beta_i^\vee$ be a corresponding coroot of β_i for $i = 1, \ldots, 6$. Then

$$\beta_2^\vee = 3\beta_1^\vee + \beta_6^\vee, \quad \beta_3^\vee = 2\beta_1^\vee + \beta_6^\vee, \quad \beta_4^\vee = 3\beta_1^\vee + 2\beta_6^\vee, \quad \beta_5^\vee = \beta_1^\vee + \beta_6^\vee.$$

Let P_1 the maximal parabolic subgroup generated by β_1 (long root) and P_2 be the maximal parabolic subgroup generated by β_6 (short root). Then we have Levi decompositions (Shahidi [S4]):

$$P_1 = M_1N_1, \quad P_2 = M_2N_2, \quad M_1 \simeq \mathrm{GL}_2, \quad M_2 \simeq \mathrm{GL}_2.$$

Under the identification $M_1 \simeq \mathrm{GL}_2$,

$$\begin{aligned} &\beta_1^\vee(t) = \mathrm{diag}(t, t^{-1}),\ \beta_6^\vee(t) = \mathrm{diag}(t^{-1}, t^2),\ \beta_2^\vee(t) = \mathrm{diag}(t^2, t^{-1}) \\ &\beta_3^\vee(t) = \mathrm{diag}(t, 1),\ \beta_4^\vee(t) = \mathrm{diag}(t, t),\ \beta_5^\vee(t) = \mathrm{diag}(1, t). \end{aligned} \tag{2.1}$$

Under the identification $M_2 \simeq \mathrm{GL}_2$,

$$\beta_1^\vee(t) = \mathrm{diag}(1,t),\ \beta_6^\vee(t) = \mathrm{diag}(t,t^{-1}),\ \beta_2^\vee(t) = \mathrm{diag}(t,t^2) \tag{2.2}$$
$$\beta_3^\vee(t) = \mathrm{diag}(t,t),\ \beta_4^\vee(t) = \mathrm{diag}(t^2,t),\ \beta_5^\vee(t) = \mathrm{diag}(t,1).$$

Let $X(T)$ (resp. $X^*(T)$) be the character (resp. cocharacter) group of T. Since G is simply connected,

$$X(T) = \mathbb{Z}\beta_3 + \mathbb{Z}\beta_4, \quad X^*(T) = \mathbb{Z}\beta_1^\vee + \mathbb{Z}\beta_6^\vee.$$

Let

$$\mathfrak{a}^* = X(T) \otimes \mathbb{R}, \quad \mathfrak{a}_{\mathbb{C}}^* = X(T) \otimes \mathbb{C}, \quad \mathfrak{a} = X^*(T) \otimes \mathbb{R} = \mathrm{Hom}\big(X(T), \mathbb{R}\big)$$
$$\mathfrak{a}_{\mathbb{C}} = X^*(T) \otimes \mathbb{C}.$$

Then $\{\beta_3, \beta_4\}$ and $\{\beta_1^\vee, \beta_6^\vee\}$ are the pair of dual bases for $\mathfrak{a}^*$ and $\mathfrak{a}$, respectively.

The positive Weyl Chamber in $\mathfrak{a}^*$ is

$$C^+ = \{\Lambda \in \mathfrak{a}^* \mid \langle \Lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha \text{ positive roots }\}$$
$$= \{a\beta_3 + b\beta_4 \mid a, b > 0\}.$$

The half sum of positive roots is $\rho_B = \beta_3 + \beta_4$.

We list the elements of the Weyl group together with their actions on the positive roots.

w	decomposition	β_1	β_2	β_3	β_4	β_5	β_6
1		β_1	β_2	β_3	β_4	β_5	β_6
ρ_1	ρ_1	$-\beta_1$	β_6	β_5	β_4	β_3	β_2
ρ_2	$\rho_1\rho_6\rho_1$	$-\beta_3$	$-\beta_2$	$-\beta_1$	β_6	β_5	β_4
ρ_3	$\rho_1\rho_6\rho_1\rho_6\rho_1$	$-\beta_5$	$-\beta_4$	$-\beta_3$	$-\beta_2$	$-\beta_1$	β_6
ρ_4	$\rho_6\rho_1\rho_6\rho_1\rho_6$	β_1	$-\beta_6$	$-\beta_5$	$-\beta_4$	$-\beta_3$	$-\beta_2$
ρ_5	$\rho_6\rho_1\rho_6$	β_3	β_2	β_1	$-\beta_6$	$-\beta_5$	$-\beta_4$
ρ_6	ρ_6	β_5	β_4	β_3	β_2	β_1	$-\beta_6$
$\sigma(\frac{\pi}{3})$	$\rho_6\rho_1$	$-\beta_5$	$-\beta_6$	β_1	β_2	β_3	β_4
$\sigma(\frac{2\pi}{3})$	$\rho_6\rho_1\rho_6\rho_1$	$-\beta_3$	$-\beta_4$	$-\beta_5$	$-\beta_6$	β_1	β_2
$\sigma(\pi)$	$\rho_6\rho_1\rho_6\rho_1\rho_6\rho_1$	$-\beta_1$	$-\beta_2$	$-\beta_3$	$-\beta_4$	$-\beta_5$	$-\beta_6$
$\sigma(\frac{4\pi}{3})$	$\rho_1\rho_6\rho_1\rho_6$	β_5	β_6	$-\beta_1$	$-\beta_2$	$-\beta_3$	$-\beta_4$
$\sigma(\frac{5\pi}{3})$	$\rho_1\rho_6$	β_3	β_4	β_5	β_6	$-\beta_1$	$-\beta_2$

3. **Decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$.** We fix an additive character $\psi = \otimes_v \psi_v$ of $\mathbb{A}/F$ and let $\xi(z,\mu)$ be the Hecke L-function with the ordinary Γ-factor so that it satisfies the functional equation $\xi(z,\mu) = \epsilon(z,\mu)\xi(1-z,\mu^{-1})$, where $\epsilon(z,\mu) = \prod_v \epsilon(z,\mu_v,\psi_v)$ is

the usual ϵ-factor (see, for example, [Go, p158]). If μ is the trivial character μ_0, then we write simply $\xi(z)$ for $\xi(z,\mu_0)$. We have the Laurent expansion of $\xi(z)$ at $z=1$:

$$\xi(z)=\frac{c(F)}{z-1}+a+\cdots.$$

3.1. *Definition of Eisenstein Series.* For a unitary character χ of $T(\mathbb{A})/T(F)$ and for each $\Lambda\in\mathfrak{a}_{\mathbb{C}}^*$, let $I(\Lambda,\chi)=\mathrm{Ind}_B^G\,\chi\otimes\exp\bigl(\Lambda,H_B(\ )\bigr)$ be the induced representation, where H_B is the homomorphism $H_B\colon T(\mathbb{A})\longmapsto\mathfrak{a}$ defined by

$$\exp\langle\chi,H_B(t)\rangle=\prod_v|\chi_v(t_v)|_v.$$

We form the Eisenstein series:

$$E(g,f,\Lambda)=\sum_{\gamma\in B(F)\backslash G(F)}f(\gamma g),$$

where $f\in I(\Lambda,\chi)$. The Eisenstein series converges absolutely for $\mathrm{Re}\,\Lambda\in C^++\rho_B$ and extends to a meromorphic function of Λ. It is an automorphic form and the constant term of $E(g,f,\Lambda)$ along B is given by

$$E_0(g,f,\Lambda)=\int_{U(F)\backslash U(\mathbb{A})}E(ug,f,\Lambda)\,du=\sum_{w\in W}M(w,\Lambda,\chi)f(g),$$

where W is the Weyl group and for sufficiently regular Λ,

$$M(w,\Lambda,\chi)f(g)=\int_{U_w(\mathbb{A})}f(w^{-1}ug)\,du,$$

where $U_w=U\cap w\bar{U}w^{-1}$, $\bar{U}$ is the unipotent radical opposed to U. Then $M(w,\Lambda,\chi)$ defines a linear map from $I(\Lambda,\chi)$ to $I(w\Lambda,w\chi)$ and satisfies the functional equation of the form

$$M(w_1w_2,\Lambda,\chi)=M(w_1,w_2\Lambda,w_2\chi)M(w_2,\Lambda,\chi).$$

The Eisenstein series satisfies the functional equation

$$E\bigl(g,M(w,\Lambda,\chi)f,w\Lambda\bigr)=E(g,f,\Lambda).$$

Let S be a finite set of places of F, including all the archimedean places such that for every $v\notin S$, χ_v and ψ_v are unramified and if $f=\otimes f_v$, for $v\notin S$, f_v is the unique K_v-fixed function normalized by $f_v(e_v)=1$. We have

$$M(w,\Lambda,\chi)=\bigotimes_v M(w,\Lambda,\chi_v).$$

Then by applying Gindikin-Karpelevic method (Langlands [L4]), we can see that for $v\notin S$,

$$M(w,\Lambda,\chi_v)f_v=\prod_{\alpha>0,w\alpha<0}\frac{L(\langle\Lambda,\alpha^\vee\rangle,\chi_v\circ\alpha^\vee)}{L(\langle\Lambda,\alpha^\vee\rangle+1,\chi_v\circ\alpha^\vee)}\tilde{f}_v,$$

where $L(s,\eta_v)$ is the local Hecke L-function attached to a character η_v of $F_v^\times$ and $s \in \mathbb{C}$ and $\tilde{f}_v$ is the K_v-fixed function in the space of $I(w\Lambda, w\chi)$ satisfying $\tilde{f}_v(e_v) = 1$. For any v, let

$$r_v(w) = \prod_{\alpha>0, w\alpha<0} \frac{L(\langle\Lambda,\alpha^\vee\rangle, \chi_v\circ\alpha^\vee)}{L(\langle\Lambda,\alpha^\vee\rangle+1, \chi_v\circ\alpha^\vee)\epsilon(\langle\Lambda,\alpha^\vee\rangle, \chi_v\circ\alpha^\vee, \psi_v)}.$$

We normalize the intertwining operator $M(w,\Lambda,\chi_v)$ for all v by

$$M(w,\Lambda,\chi_v) = r_v(w)R(w,\Lambda,\chi_v).$$

Let $R(w,\Lambda,\chi) = \otimes_v R(w,\Lambda,\chi_v)$ and $R(w,\Lambda,\chi)$ satisfies the functional equation

$$R(w_1w_2,\Lambda,\chi) = R(w_1, w_2\Lambda, w_2\chi)R(w_2,\Lambda,\chi).$$

We know, by Winarsky [Wi] for p-adic cases and Shahidi [S2, p110] for real and complex cases, that

$$M(w,\Lambda,\chi_v) \prod_{\alpha>0, w\alpha<0} L(\langle\Lambda,\alpha^\vee\rangle, \chi_v\circ\alpha^\vee)^{-1}$$

is holomorphic for any v. So for any v, $R(w,\Lambda,\chi_v)$ is holomorphic for Λ with $\mathrm{Re}(\langle\Lambda,\alpha^\vee\rangle) > -1$, for all positive α with $w\alpha < 0$.

We note that a character χ of $T(F)\backslash T(\mathbb{A})$ defines a cuspidal representation of T. For any $w \in W$, $wTw^{-1} = T$ and so $(T, w\chi)$ is conjugate to (T,χ).

Let $I(\chi)$ be the set of entire functions f of Paley-Wiener type such that $f(\Lambda) \in I(\Lambda,\chi)$ for each Λ. Let

$$\theta_f(g) = \left(\frac{1}{2\pi i}\right)^2 \int_{\mathrm{Re}\,\Lambda=\Lambda_0} E\big(g, f(\Lambda), \Lambda\big)\, d\Lambda,$$

where $\Lambda_0 \in \rho_B + C^+$. Then we have

LEMMA ([L4]). *$L^2\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$ spanned by θ_f for all $f \in I(w\chi)$ as $w\chi$ runs through all distinct conjugates of χ.*

$L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$ is the discrete part of $L^2\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$. It is the set of iterated residues of $E\big(g, f(\Lambda), \Lambda\big)$ of order 2.

In order to decompose $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$, we use the inner product formula of two pseudo-Eisenstein series: Let χ and χ' be conjugate characters and $f \in I(\chi), f' \in I(\chi')$. Then

$$\langle\theta_f, \theta_{f'}\rangle = \frac{1}{(2\pi i)^2}\int_{\mathrm{Re}\,\Lambda=\Lambda_0} \sum_{w\in W(\chi,\chi')} \big(M(w,\Lambda)f(\Lambda), f'(-w\bar{\Lambda})\big)\, d\Lambda,$$

where $W(\chi,\chi') = \{w \in W \mid w\chi = \chi'\}$. Let $\{d\chi \mid d \in D\}$ be the set of distinct conjugates of χ.

In order to deal with the distinct conjugates of χ simultaneously, we consider, for $f \in I(\chi)$,

$$A(f,f';\Lambda) = \sum_{d\in D}\sum_{w\in W(\chi,d\chi)} \big(M(w,\Lambda,\chi)f(\Lambda), f'_d(-w\bar{\Lambda})\big),$$

where $f'_d \in I(d\chi)$. Since $W = \bigcup_{d\in D} W(\chi, d\chi)$, for simplicity, we write it as

$$A(f,f';\Lambda) = \sum_{w\in W} \big(M(w,\Lambda,\chi)f(\Lambda), f'(-w\bar{\Lambda})\big). \tag{3.1}$$

We also have the adjoint formula for the intertwining operators

$$\begin{aligned} \big(M(w,\Lambda,\chi)f(\Lambda), f'(-w\bar{\Lambda})\big) &= \big(f(\Lambda), M(w^{-1}, -w\bar{\Lambda}, w\chi)f'(-w\bar{\Lambda})\big) \\ \big(R(w,\Lambda,\chi)f(\Lambda), f'(-w\bar{\Lambda})\big) &= \big(f(\Lambda), R(w^{-1}, -w\bar{\Lambda}, w\chi)f'(-w\bar{\Lambda})\big). \end{aligned} \tag{3.2}$$

We use this adjoint formula and calculate the residue of $A(f,f';\Lambda)$ to obtain the residual spectrum $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$.

Let $S_i = \{\Lambda \in \mathfrak{a}^*_{\mathbb{C}} : \langle \Lambda, \beta_i^\vee\rangle = 1\}$ and we introduce a coordinate on S_i as follows: $\Lambda = zu_i + \frac{\beta_i}{2}$, where $u_1 = \beta_4, u_2 = \beta_5, u_3 = \beta_6, u_4 = \beta_1, u_5 = \beta_2, u_6 = \beta_3$. We note that $\langle u_i, \beta_i^\vee\rangle = 0$.

In order to get discrete spectrum, we have to deform the contour $\operatorname{Re}\Lambda = \Lambda_0$ to $\operatorname{Re}\Lambda = 0$. Since the poles of the functions $M(w,\Lambda,\chi)$ all lie on S_i which is defined by real equations, we can represent the process of deforming the contour and the singular hyperplanes S_i as dashed lines by the following diagram in the real plane as in [L4, Appendix 3].

The integral at $\operatorname{Re}\Lambda = 0$,

$$\frac{1}{(2\pi i)^2}\int_{\operatorname{Re}\Lambda=0} A(f,f';\Lambda)\,d\Lambda$$

gives the continuous spectrum of dimension 2. As can be seen in the diagram, if we move the contour along the dotted line indicated we may pick up residues at the points λ_i, $i = 1,\ldots,6$:

$$\frac{1}{2\pi i}\int_{\operatorname{Re}\Lambda=\lambda_i} \operatorname{Res}_{S_i} A(f,f';\Lambda)\,d\Lambda,$$

where $\Lambda \in S_i$. Then we deform the contours $\operatorname{Re}\Lambda = \lambda_i$ to $\operatorname{Re}\Lambda = \frac{\beta_i}{2}$, *i.e.*, the origin of S_i. The integrals at $\operatorname{Re}\Lambda = \frac{\beta_i}{2}$,

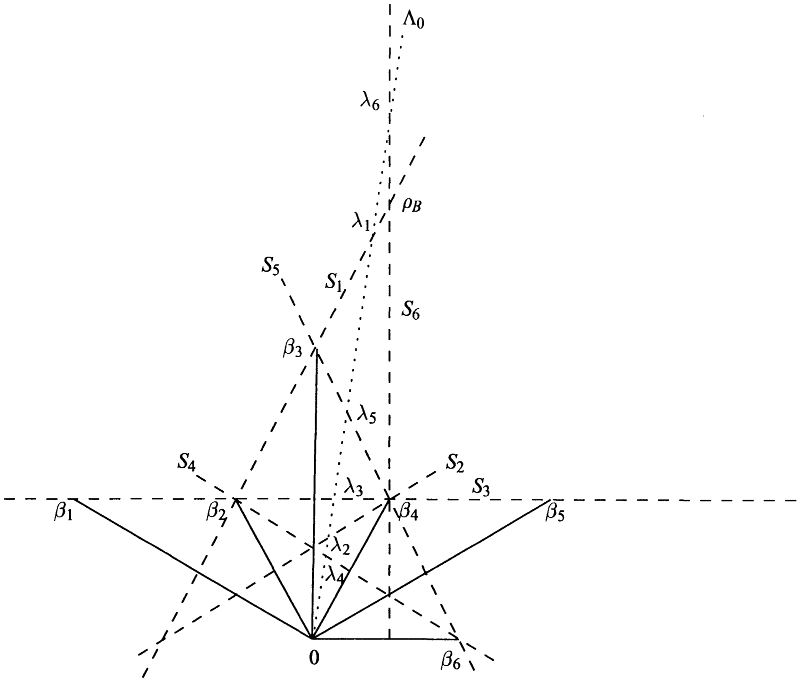
$$\frac{1}{2\pi i}\int_{\operatorname{Re}\Lambda=\frac{\beta_i}{2}} \operatorname{Res}_{S_i} A(f,f';\Lambda)\,d\Lambda$$

give the continuous spectrum of dimension 1. The square integrable residues which arise during the deformation span the discrete spectrum.

As we see in the diagram, we have to consider

$\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_1} A(f,f';\Lambda)$

$\operatorname{Res}_{\beta_2}\operatorname{Res}_{S_1} A(f,f';\Lambda)$



$\mathrm{Res}_{\frac{1}{3}\beta_3}\,\mathrm{Res}_{S_2}\,A(f,f';\Lambda)$
$\mathrm{Res}_{\beta_4}\,\mathrm{Res}_{S_5}\,A(f,f';\Lambda)$
$\mathrm{Res}_{\rho_B}\,\mathrm{Res}_{S_6}\,A(f,f';\Lambda)$
$\mathrm{Res}_{\beta_4}\,\mathrm{Res}_{S_6}\,A(f,f';\Lambda)$
$\mathrm{Res}_{\frac{1}{3}\beta_5}\,\mathrm{Res}_{S_6}\,A(f,f';\Lambda)$.

For χ a character of $T(\mathbb{A})/T(F)$, let $\chi_i = \chi \circ \beta_i^\vee$.
Then

$$\chi_2 = \chi_1^3\chi_6, \chi_3 = \chi_1^2\chi_6, \chi_4 = \chi_1^3\chi_6^2, \chi_5 = \chi_1\chi_6.$$

Set

$$M^i(w,\Lambda,\chi) = \frac{\xi(2)}{c(F)}\,\mathrm{Res}_{S_i}\,M(w,\Lambda,\chi).$$

Let $W_i = \{w \in W \mid w\beta_i < 0\}$ for $i = 1,\ldots,6$. Then

$$\mathrm{Res}_{S_i}\,A(f,f',\Lambda) = \sum_{w\in W_i}\big(M^i(w,\Lambda,\chi)f(\Lambda), f'(-w\bar{\Lambda})\big).$$

3.2. *Calculation of* $\mathrm{Res}_{S_1}\,A(f,f';\Lambda)$. $M(w,\Lambda,\chi)$ has a pole at S_1 if $\chi_1 = 1$.
On S_1,

$$\langle\Lambda,\beta_2^\vee\rangle = z+\frac{3}{2}, \langle\Lambda,\beta_3^\vee\rangle = z+\frac{1}{2}.$$
$$\langle\Lambda,\beta_4^\vee\rangle = 2z, \langle\Lambda,\beta_5^\vee\rangle = z-\frac{1}{2}, \langle\Lambda,\beta_6^\vee\rangle = z-\frac{3}{2}.$$

LEMMA 3.2.1.

$$M^1(\rho_1,\Lambda,\chi) = R(\rho_1,\Lambda,\chi)$$
$$M^1(\rho_2,\Lambda,\chi) = \frac{\xi(z+\frac{1}{2},\chi_6)\, R(\rho_2,\Lambda,\chi)}{\xi(z+\frac{5}{2},\chi_6)\varepsilon(z+\frac{1}{2},\chi_6)\varepsilon(z+\frac{3}{2},\chi_6)}$$
$$M^1(\rho_3,\Lambda,\chi) = \frac{\xi(2z,\chi_6^2)\xi(z-\frac{1}{2},\chi_6)\quad R(\rho_3,\Lambda,\chi)}{\xi(z+\frac{5}{2},\chi_6)\xi(2z+1,\chi_6^2)\varepsilon(z+\frac{3}{2},\chi_6)\varepsilon(z+\frac{1}{2},\chi_6)\varepsilon(2z,\chi_6^2)\varepsilon(z-\frac{1}{2},\chi_6)}$$
$$M^1\Big(\sigma\Big(\frac{\pi}{3}\Big),\Lambda,\chi\Big) = \frac{\xi(z+\frac{3}{2},\chi_6)}{\xi(z+\frac{5}{2},\chi_6)\varepsilon(z+\frac{3}{2},\chi_6)} R\Big(\sigma\Big(\frac{\pi}{3}\Big),\Lambda,\chi\Big)$$
$$M^1\Big(\sigma\Big(\frac{2\pi}{3}\Big),\Lambda,\chi\Big)$$
$$= \frac{\xi(z+\frac{1}{2},\chi_6)\xi(2z,\chi_6^2)\quad R\big(\sigma(\frac{2\pi}{3}),\Lambda,\chi\big)}{\xi(z+\frac{5}{2},\chi_6)\xi(2z+1,\chi_6^2)\varepsilon(z+\frac{3}{2}\chi_6)\varepsilon(z+\frac{1}{2},\chi_6)\varepsilon(z+\frac{1}{2},\chi_6)\varepsilon(2z,\chi_6^2)}$$
$$M^1\big(\sigma(\pi),\Lambda,\chi\big) = \frac{\xi(2z,\chi_6^2)\xi(z-\frac{3}{2},\chi_6)}{\xi(z+\frac{5}{2},\chi_6)\xi(2z+1,\chi_6^2)}$$
$$\cdot\frac{R\big(\sigma(\pi),\Lambda,\chi\big)}{\varepsilon(z+\frac{3}{2},\chi_6)\varepsilon(z+\frac{1}{2},\chi_6)\varepsilon(2z,\chi_6^2)\varepsilon(z-\frac{1}{2},\chi_6)\varepsilon(z-\frac{3}{2},\chi_6)}$$

PROPOSITION 3.2.2. *If χ is not trivial, then* $\mathrm{Res}_{S_1} A(f,f';\Lambda)$ *has a pole at* $z=\frac{1}{2}$, *i.e.* $\Lambda=\beta_2$ *when* $\chi_6^2=1$, $\chi_6\neq 1$ *and the residue is given by*

$$\mathrm{Res}_{\beta_2}\mathrm{Res}_{S_1} A(f,f';\Lambda) = c_1\big(f(\beta_2), R(\rho_2,\beta_2,\rho_2\chi)\big)$$
$$\left(R(\rho_5,\text{“}\beta_2\text{''},\chi)f'(\beta_2) + c_2R\Big(\sigma\Big(\frac{\pi}{3}\Big),\beta_4,\rho_3\chi\Big)f'(\beta_4) + c_3R\Big(\rho_6,\beta_4,\frac{2\pi}{3}\chi\Big)f'(\beta_4)\right),$$

where c_1, c_2, c_3 *are constants and* $R(\rho_5, \text{“}\beta_2\text{''}, \chi) = \otimes_v R(\rho_5, \text{“}\beta_2\text{''}, \chi_v)$ *and* $R(\rho_5, \text{“}\beta_2\text{''}, \chi_v)$ *is the value at* β_2 *of the restriction* $R(\rho_5,\Lambda,\chi_v)|_{S_1}$. *It is an isomorphism from* $I(\beta_2,\chi)$ *to* $I(\beta_2,\rho_5\chi)$.

PROOF. The local intertwining operators are holomorphic at $z=\frac{1}{2}$ except $R\big(\sigma(\pi),\Lambda,\chi_v\big)$ when $\chi_v = 1$. However, Moeglin-Waldspurger [M-W1, Appendix III] showed that the restriction $R\big(\sigma(\pi),\Lambda,\chi_v\big)|_{S_1}$ is holomorphic at $z=\frac{1}{2}$: Let $\chi_v=1$. By the cocycle relation, $R\big(\sigma(\pi),\Lambda,\chi_v\big) = R(\rho_5,\rho_2\Lambda,\chi_v)R(\rho_2,\Lambda,\chi_v)$. The pole is contained in $R(\rho_5,\rho_2\Lambda,\chi_v)$. However, the restriction $R(\rho_5,\rho_2\Lambda,\chi_v)|_{S_1}$ of $R(\rho_5,\rho_2\Lambda,\chi_v)$ onto S_1 is holomorphic at β_2. If χ_v is not trivial, $R(\rho_5,\rho_2\Lambda,\chi_v)$ is holomorphic at β_2.

Therefore, all the poles are contained in the normalizing factors. So

$$\operatorname{Res}_{\beta_2}\operatorname{Res}_{S_1} A(f,f';\Lambda) = \frac{c(F)\xi(1,\chi_6)}{\xi(2)\xi(3,\chi_6)\varepsilon(1,\chi_6)\varepsilon(2,\chi_6)}\big(R(\rho_3,\beta_2,\chi)f(\beta_2),f'(\beta_4)\big)$$
$$+\left(R\left(\sigma\left(\frac{2\pi}{3}\right),\beta_2,\chi\right)f(\beta_2),f'(\beta_4)\right)$$
$$+\frac{c(F)\xi(2,\chi_6)}{\xi(2)\xi(3,\chi_6)\varepsilon(2,\chi_6)}\Big(R\big(\sigma(\pi),\beta_2,\chi\big)f(\beta_2),f'(\beta_2)\Big).$$

In order to use the adjoint formula (3.2), we note that $\rho_3 = \rho_2\sigma(\frac{\pi}{3})$ and $\sigma(\frac{2\pi}{3})^{-1} = \sigma(\frac{4\pi}{3}) = \rho_2\rho_6$. We also note that $\sigma(\pi) = \rho_2\rho_5 = \rho_5\rho_2$ and so we have $R\big(\sigma(\pi),\Lambda,\chi_v\big) = R(\rho_2,\rho_5\Lambda,\rho_5\chi_v)R(\rho_5,\Lambda,\chi_v)$. By the same reasoning as above, if $\chi_v = 1$, $R(\rho_5,\Lambda,\chi_v)|_{S_1}$ is holomorphic at β_2 and we denote its value by $R(\rho_5,\text{“}\beta_2\text{”},\chi_v)$. If χ_v is not trivial, then $R(\rho_5,\Lambda,\chi_v)$ is holomorphic at β_2 and in this case, $R(\rho_5,\beta_2,\chi_v) = R(\rho_5,\text{“}\beta_2\text{”},\chi_v)$.

PROPOSITION 3.2.3. *If χ is trivial, then* $\operatorname{Res}_{S_1} A(f,f';\Lambda)$ *has a pole at $z = \frac{1}{2}$, i.e. at β_2 and $z = \frac{3}{2}$, i.e., at β_3.*

(1) $\operatorname{Res}_{\beta_2}\operatorname{Res}_{S_1} A(f,f';\Lambda)$ *was calculated by Moeglin-Waldspurger.*

(2) $\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_1} A(f,f';\Lambda) = 0$.

PROOF. (1) See [M-W1, Appendix IV].

(2) At $z = \frac{3}{2}$, $M^1(\rho_3,\Lambda,\chi)$ and $M^1\big(\sigma(\pi),\Lambda,\chi\big)$ have simple poles. So the residue is given by

$$\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_1} A(f,f';\Lambda) = \frac{\xi(3)c(F)}{\xi(4)^2}\bigg(\Big(R(\rho_3,\beta_3,\chi) - R\big(\sigma(\pi),\beta_3,\chi\big)\Big)f(\beta_3),f'(\beta_3)\bigg).$$

But $R\big(\sigma(\pi),\beta_3,\chi\big) = R(\rho_3\rho_6,\beta_3,\chi) = R(\rho_3,\beta_3,\chi)R(\rho_6,\beta_3,\chi)$ and $R(\rho_6,\beta_3,\chi) = id$ by the following Lemma. So $\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_1} A(f,f';\Lambda) = 0$.

LEMMA 3.2.4. *Suppose a character χ_v satisfies $\rho_6\chi_v = \chi_v$. Then for any real number t, the normalized intertwining operator $R(\rho_6,t\beta_3,\chi_v)$ is a self-intertwining operator of $I(t\beta_3,\chi_v)$. It acts like the identity.*

PROOF. Since $\rho_6\beta_3 = \beta_3$, it is a self-intertwining operator of $I(t\beta_3,\chi_v)$. Since $\langle\beta_3,\beta_6^\vee\rangle = 0$, $R(\rho_6,t\beta_3,\chi_v)$ is actually an intertwining operator for the Levi subgroup $M_2 \simeq \mathrm{GL}_2$, where $P_2 = M_2N_2$ is the maximal parabolic subgroup attached to β_6. Under the identification $M_2 \simeq \mathrm{GL}_2$, $R(\rho_6,t\beta_3,\chi_v)$ is an intertwining operator of $\mathrm{Ind}_{B_0}^{\mathrm{GL}_2}\mu|\ |^t \times \mu|\ |^t$. Since $\mathrm{Ind}_{B_0}^{\mathrm{GL}_2}\mu|\ |^t \times \mu|\ |^t$ is irreducible, $R(\rho_6,t\beta_3,\chi_v)$ acts like a scalar. But it acts like the identity on the K-fixed vector. Therefore it is the identity.

3.3. *Calculation of* $\operatorname{Res}_{S_2} A(f,f';\Lambda)$. $M(w,\Lambda,\chi)$ has a pole at S_2 if $\chi_2 = 1$, *i.e.*, $\chi_1^3\chi_6 = 1$.

On S_2,

$$\langle\Lambda,\beta_1^\vee\rangle = -z+\frac{1}{2}, \langle\Lambda,\beta_3^\vee\rangle = z+\frac{1}{2}$$
$$\langle\Lambda,\beta_4^\vee\rangle = 3z+\frac{1}{2}, \langle\Lambda,\beta_5^\vee\rangle = 2z, \langle\Lambda,\beta_6^\vee\rangle = 3z-\frac{1}{2}.$$

LEMMA 3.3.1.

$$M^2(\rho_2,\Lambda,\chi)=\frac{\xi(-z+\frac12,\chi_1)\xi(z+\frac12,\chi_1^{-1})\quad R(\rho_2,\Lambda,\chi)}{\xi(-z+\frac32,\chi_1)\xi(z+\frac32,\chi_1^{-1})\varepsilon(-z+\frac12,\chi_1)\varepsilon(z+\frac12,\chi_1^{-1})}$$

$$M^2(\rho_3,\Lambda,\chi)=\frac{\xi(-z+\frac12,\chi_1)\xi(z+\frac12,\chi_1^{-1})\xi(3z+\frac12,\chi_1^{-3})\xi(2z,\chi_1^{-2})}{\xi(-z+\frac32,\chi_1)\xi(z+\frac32,\chi_1^{-1})\xi(3z+\frac32,\chi_1^{-3})\xi(2z+1,\chi_1^{-2})}\cdot\frac{R(\rho_3,\Lambda,\chi)}{\varepsilon(-z+\frac12,\chi_1)\varepsilon(z+\frac12,\chi_1^{-1})\varepsilon(3z+\frac12,\chi_1^{-3})\varepsilon(2z,\chi_1^{-2})}$$

$$M^2(\rho_4,\Lambda,\chi)=\frac{\xi(z+\frac12,\chi_1^{-1})\xi(2z,\chi_1^{-2})\xi(3z-\frac12,\chi_1^{-3})}{\xi(z+\frac32,\chi_1^{-1})\xi(2z+1,\chi_1^{-2})\xi(3z+\frac32,\chi_1^{-3})}\cdot\frac{R(\rho_4,\Lambda,\chi)}{\varepsilon(z+\frac12,\chi_1^{-1})\varepsilon(3z+\frac12,\chi_1^{-3})\varepsilon(2z,\chi_1^{-2})\varepsilon(3z-\frac12,\chi_1^{-3})}$$

$$M^2\Big(\sigma\Big(\frac{\pi}{3}\Big),\Lambda,\chi\Big)=\frac{\xi(-z+\frac12,\chi_1)}{\xi(-z+\frac32,\chi_1)\varepsilon(-z+\frac12,\chi_1)}R\Big(\sigma\Big(\frac{\pi}{3}\Big),\Lambda,\chi\Big)$$

$$M^2\Big(\sigma\Big(\frac{2\pi}{3}\Big),\Lambda,\chi\Big)=\frac{\xi(-z+\frac12,\chi_1)\xi(z+\frac12,\chi_1^{-1})\xi(3z+\frac12,\chi_1^{-3})}{\xi(-z+\frac32,\chi_1)\xi(z+\frac32,\chi_1^{-1})\xi(3z+\frac32,\chi_1^{-3})}\cdot\frac{R\big(\sigma(\frac{2\pi}{3}),\Lambda,\chi\big)}{\varepsilon(-z+\frac12,\chi_1)\varepsilon(z+\frac12,\chi_1^{-1})\varepsilon(3z+\frac12,\chi_1^{-3})}$$

$$M^2\big(\sigma(\pi),\Lambda,\chi\big)=\frac{\xi(-z+\frac12,\chi_1)\xi(z+\frac12,\chi_1^{-1})\xi(2z,\chi_1^{-2})\xi(3z-\frac12,\chi_1^{-3})}{\xi(-z+\frac32,\chi_1)\xi(z+\frac32,\chi_1^{-1})\xi(2z+1,\chi_1^{-2})\xi(3z+\frac32,\chi_1^{-3})}\cdot\frac{R\big(\sigma(\pi),\Lambda,\chi\big)}{\varepsilon(-z+\frac12,\chi_1)\varepsilon(z+\frac12,\chi_1^{-1})\varepsilon(2z,\chi_1^{-2})\varepsilon(3z+\frac12,\chi_1^{-3})\varepsilon(3z-\frac12,\chi_1^{-3})}$$

PROPOSITION 3.3.2. $\mathrm{Res}_{S_2} A(f,f';\Lambda)$ *has a simple pole at* $z=\frac16$, *i.e.* $\Lambda=\frac13\beta_3$ *when* $\chi_1^3=1$, *i.e.,* $\chi_6=1$ *and* $\mathrm{Res}_{\frac13\beta_3}\mathrm{Res}_{S_2} A(f,f';\Lambda)=0$.

PROOF. All the local intertwining operators are holomorphic at $z=\frac16$. So all the poles are contained in the normalizing factors. Therefore,

$$\begin{aligned}&\mathrm{Res}_{\frac13\beta_3}\mathrm{Res}_{S_2} A(f,f';\Lambda)\\&=(*)\Big(\Big(R\Big(\rho_3,\frac13\beta_3,\chi\Big)-R\Big(\sigma(\pi),\frac13\beta_3,\chi\Big)\Big)f\Big(\frac13\beta_3\Big),f'\Big(\frac13\beta_3\Big)\Big)\\&\quad+(**)\Big(\Big(-R\Big(\rho_4,\frac13\beta_3,\chi\Big)+R\Big(\sigma\Big(\frac{2\pi}{3}\Big),\frac13\beta_3,\chi\Big)\Big)f\Big(\frac13\beta_3\Big),f'\Big(\frac13\beta_3\Big)\Big),\end{aligned}$$

with $(*)$ and $(**)$ being constants depending on χ_1. Here $R\big(\sigma(\pi),\frac13\beta_3,\chi\big)=R(\rho_3,\frac13\beta_3,\chi)R(\rho_6,\frac13\beta_3,\chi)$ and $R(\rho_4,\frac13\beta_3,\chi)=R\big(\sigma(\frac{2\pi}{3}),\frac13\beta_3,\chi\big)R(\rho_6,\frac13\beta_3,\chi)$. Since $\rho_6\chi=\chi$, by Lemma 3.2.4, $R(\rho_6,\frac13\beta_3,\chi)=id$. Therefore, $\mathrm{Res}_{\frac13\beta_3}\mathrm{Res}_{S_2} A(f,f';\Lambda)=0$.

3.4. *Calculation of* $\mathrm{Res}_{S_5} A(f,f';\Lambda)$. $M(w,\Lambda,\chi)$ has a pole at S_5 if $\chi_5 = 1$, *i.e.* $\chi_1\chi_6 = 1$. On S_5,

$$\langle \Lambda, \beta_1^\vee \rangle = z - \frac{1}{2}, \langle \Lambda, \beta_2^\vee \rangle = 2z, \langle \Lambda, \beta_3^\vee \rangle = z + \frac{1}{2}$$
$$\langle \Lambda, \beta_4^\vee \rangle = z + \frac{3}{2}, \langle \Lambda, \beta_6^\vee \rangle = -z + \frac{3}{2}.$$

LEMMA 3.4.1.

$$M^5(\rho_3,\Lambda,\chi) = \frac{\xi(z-\frac{1}{2},\chi_1)\xi(2z,\chi_1^2) \quad R(\rho_3,\Lambda,\chi)}{\xi(z+\frac{5}{2},\chi_1)\xi(2z+1,\chi_1^2)\varepsilon(z-\frac{1}{2},\chi_1)\varepsilon(2z,\chi_1^2)\varepsilon(z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{3}{2},\chi_1)}$$

$$M^5(\rho_4,\Lambda,\chi) = \frac{\xi(2z,\chi_1^2)\xi(z+\frac{1}{2},\chi_1)\xi(-z+\frac{3}{2},\chi_1^{-1})}{\xi(2z+1,\chi_1^2)\xi(z+\frac{5}{2},\chi_1)\xi(-z+\frac{5}{2},\chi_1^{-1})} \cdot \frac{R(\rho_4,\Lambda,\chi)}{\varepsilon(2z,\chi_1^2)\varepsilon(z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{3}{2},\chi_1)\varepsilon(-z+\frac{3}{2},\chi_1^{-1})}$$

$$M^5(\rho_5,\Lambda,\chi) = \frac{\xi(z+\frac{3}{2},\chi_1)\xi(-z+\frac{3}{2},\chi_1^{-1}) \quad R(\rho_5,\Lambda,\chi)}{\xi(z+\frac{5}{2},\chi_1)\xi(-z+\frac{5}{2},\chi_1^{-1})\varepsilon(z+\frac{3}{2},\chi_1)\varepsilon(-z+\frac{3}{2},\chi_1^{-1})}$$

$$M^5\big(\sigma(\pi),\Lambda,\chi\big) = \frac{\xi(z-\frac{1}{2},\chi_1)\xi(2z,\chi_1^2)\xi(-z+\frac{3}{2},\chi_1^{-1})}{\xi(z+\frac{5}{2},\chi_1)\xi(2z+1,\chi_1^2)\xi(-z+\frac{5}{2},\chi_1^{-1})} \cdot \frac{R\big(\sigma(\pi),\Lambda,\chi\big)}{\varepsilon(z-\frac{1}{2},\chi_1)\varepsilon(2z,\chi_1^2)\varepsilon(z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{3}{2},\chi_1)\varepsilon(-z+\frac{3}{2},\chi_1^{-1})}$$

$$M^5\Big(\sigma\Big(\frac{4\pi}{3}\Big),\Lambda,\chi\Big) = \frac{\xi(z+\frac{1}{2},\chi_1)\xi(-z+\frac{3}{2},\chi_1^{-1})}{\xi(z+\frac{5}{2},\chi_1)\xi(-z+\frac{5}{2},\chi_1^{-1})} \cdot \frac{R\big(\sigma(\frac{4\pi}{3}),\Lambda,\chi\big)}{\varepsilon(z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{3}{2},\chi_1)\varepsilon(-z+\frac{3}{2},\chi_1^{-1})}$$

$$M^5\Big(\sigma\Big(\frac{5\pi}{3}\Big),\Lambda,\chi\Big) = \frac{\xi(-z+\frac{3}{2},\chi_1^{-1})}{\xi(-z+\frac{5}{2},\chi_1^{-1})\varepsilon(-z+\frac{3}{2},\chi_1^{-1})} R\Big(\sigma\Big(\frac{5\pi}{3}\Big),\Lambda,\chi\Big)$$

PROPOSITION 3.4.2. *If χ is not trivial,* $\mathrm{Res}_{S_5} A(f,f';\Lambda)$ *has a simple pole at* $\Lambda = \beta_4$, *i.e.,* $z = \frac{1}{2}$, *when* $\chi_1^2 = 1$, $\chi_1 \neq 1$ *and its residue is given by*

$$\mathrm{Res}_{\beta_4} \mathrm{Res}_{S_5} A(f,f';\Lambda) = c_1\big(R(\rho_3,\beta_4,\chi)f(\beta_4),f'(\beta_2)\big) + c_2\Big(\Big(R(\rho_4,\beta_4,\chi) + R\big(\sigma(\pi),\beta_4,\chi\big)\Big)f(\beta_4),f'(\beta_4)\Big),$$

where $c_1 = \frac{c(F)\xi(0,\chi_1)}{\xi(2)\xi(3,\chi_1)\varepsilon(2,\chi_1)}$ *and* $c_2 = \frac{c(F)\xi(1,\chi_1)^2}{\xi(2)\xi(3,\chi_1)\xi(2,\chi_1)\varepsilon(1,\chi_1)^2}$.

PROOF. All the local intertwining operators are holomorphic at $z = \frac{1}{2}$. Therefore all the poles are contained in the normalizing factors. Our assertion follows from the straightforward computation.

PROPOSITION 3.4.3. *If χ is trivial,* $\mathrm{Res}_{S_5} A(f,f';\Lambda)$ *has a triple pole at* $z=\frac{1}{2}$. *Moeglin-Waldspurger calculated the residue.*

3.5. *Calculation of* $\mathrm{Res}_{S_6} A(f,f';\Lambda)$. $M(w,\Lambda,\chi)$ has a pole at S_6 if $\chi_6=1$. On S_6,

$$\langle\Lambda,\beta_1^\vee\rangle = z-\frac{1}{2},\quad \langle\Lambda,\beta_2^\vee\rangle = 3z-\frac{1}{2},\quad \langle\Lambda,\beta_3^\vee\rangle = 2z,$$
$$\langle\Lambda,\beta_4^\vee\rangle = 3z+\frac{1}{2},\quad \langle\Lambda,\beta_5^\vee\rangle = z+\frac{1}{2}.$$

LEMMA 3.5.1.

$$M^6(\rho_4,\Lambda,\chi) = \frac{\xi(3z-\frac{1}{2},\chi_1^3)\xi(2z,\chi_1^2)\xi(z+\frac{1}{2},\chi_1)}{\xi(3z+\frac{3}{2},\chi_1^3)\xi(2z+1,\chi_1^2)\xi(z+\frac{3}{2},\chi_1)} \cdot \frac{R(\rho_4,\Lambda,\chi)}{\varepsilon(3z-\frac{1}{2},\chi_1^3)\varepsilon(2z,\chi_1^2)\varepsilon(3z+\frac{1}{2},\chi_1^3)\varepsilon(z+\frac{1}{2},\chi_1)}$$

$$M^6(\rho_5,\Lambda,\chi) = \frac{\xi(3z+\frac{1}{2},\chi_1^3)\xi(z+\frac{1}{2},\chi_1)\quad R(\rho_5,\Lambda,\chi)}{\xi(3z+\frac{3}{2},\chi_1^3)\xi(z+\frac{3}{2},\chi_1)\varepsilon(3z+\frac{1}{2},\chi_1^3)\varepsilon(z+\frac{1}{2},\chi_1)}$$

$$M^6(\rho_6,\Lambda,\chi) = R(\rho_6,\Lambda,\chi)$$

$$M^6\big(\sigma(\pi),\Lambda,\chi\big) = \frac{\xi(z-\frac{1}{2},\chi_1)\xi(3z-\frac{1}{2},\chi_1^3)\xi(2z,\chi_1^2)}{\xi(z+\frac{3}{2},\chi_1)\xi(3z+\frac{3}{2},\chi_1^3)\xi(2z+1,\chi_1^2)} \cdot \frac{R\big(\sigma(\pi),\Lambda,\chi\big)}{\varepsilon(z-\frac{1}{2},\chi_1)\varepsilon(3z-\frac{1}{2},\chi_1^3)\varepsilon(2z,\chi_1^2)\varepsilon(3z+\frac{1}{2},\chi_1^3)\varepsilon(z+\frac{1}{2},\chi_1)}$$

$$M^6\Big(\sigma\Big(\frac{4\pi}{3}\Big),\Lambda,\chi\Big) = \frac{\xi(2z,\chi_1^2)\xi(3z+\frac{1}{2},\chi_1^3)\xi(z+\frac{1}{2},\chi_1)\quad R\big(\sigma(\frac{4\pi}{3}),\Lambda,\chi\big)}{\xi(2z+1,\chi_1^2)\xi(3z+\frac{3}{2},\chi_1^3)\xi(z+\frac{3}{2},\chi_1)\varepsilon(2z,\chi_1^2)\varepsilon(3z+\frac{1}{2},\chi_1^3)\varepsilon(z+\frac{1}{2},\chi_1)}$$

$$M^6\Big(\sigma\Big(\frac{5\pi}{3}\Big),\Lambda,\chi\Big) = \frac{\xi(z+\frac{1}{2},\chi_1)}{\xi(z+\frac{3}{2},\chi_1)\varepsilon(z+\frac{1}{2},\chi_1)} R\Big(\sigma\Big(\frac{5\pi}{3}\Big),\Lambda,\chi\Big).$$

All the local intertwining operators are holomorphic at $z=\frac{1}{6}, z=\frac{1}{2}, z=\frac{3}{2}$ and so all the poles are contained in the normalizing factors.

PROPOSITION 3.5.2. *If χ is not trivial,* $\mathrm{Res}_{S_6} A(f,f';\Lambda)$ *has a simple pole at* $\Lambda=\frac{1}{3}\beta_5$, *i.e.,* $z=\frac{1}{6}$ *if* $\chi_1^3=1$. $\mathrm{Res}_{\frac{1}{3}\beta_5}\mathrm{Res}_{S_6} A(f,f';\Lambda)=0$.

PROOF.

$$\mathrm{Res}_{\frac{1}{3}\beta_5}\mathrm{Res}_{S_6} A(f,f';\Lambda)$$
$$= c_1\bigg(\Big(-R\Big(\rho_4,\frac{1}{3}\beta_5,\chi\Big)+R\Big(\sigma\Big(\frac{4\pi}{3}\Big),\frac{1}{3}\beta_5,\chi\Big)\Big)f\Big(\frac{1}{3}\beta_5\Big),f'\Big(\frac{1}{3}\beta_3\Big)\bigg)$$
$$+c_2\bigg(\Big(R\Big(\rho_5,\frac{1}{3}\beta_5,\chi\Big)-R\Big(\sigma(\pi),\frac{1}{3}\beta_5,\chi\Big)\Big)f\Big(\frac{1}{3}\beta_5\Big),f'\Big(\frac{1}{3}\beta_5\Big)\bigg)$$

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where c_1, c_2 depend on χ_1. But

$$R\Big(\rho_4, \frac{1}{3}\beta_5, \chi\Big) = R\Big(\rho_6, -\frac{1}{3}\beta_3, \chi\Big) R\Big(\sigma\Big(\frac{4\pi}{3}\Big), \frac{1}{3}\beta_5, \chi\Big)$$
$$R\Big(\sigma(\pi), \frac{1}{3}\beta_5, \chi\Big) = R\Big(\rho_5, \frac{1}{3}\beta_5, \chi\Big) R\Big(\rho_2, \frac{1}{3}\beta_5, \chi\Big).$$

Here $\sigma(\frac{4\pi}{3})\chi = \chi$ and $\rho_6\rho_1\chi = \rho_1\chi$ and

$$R\Big(\rho_2, \frac{1}{3}\beta_5, \chi\Big) = R\Big(\rho_1, \frac{1}{3}\beta_3, \rho_1\chi\Big) R\Big(\rho_6, \frac{1}{3}\beta_3, \rho_1\chi\Big) R\Big(\rho_1, \frac{1}{3}\beta_5, \chi\Big).$$

By Lemma 3.2.4, $R(\rho_6, -\frac{1}{3}\beta_3, \chi)$ and $R(\rho_6, \frac{1}{3}\beta_3, \rho_1\chi)$ are the identity. By the cocycle relation of the normalized intertwining operators, $R(\rho_2, \frac{1}{3}\beta_5, \chi)$ is also the identity. Therefore, the residue is zero.

PROPOSITION 3.5.3. *$A(f, f'; \Lambda)$ has a simple pole at $\Lambda = \beta_4$, i.e., $z = \frac{1}{2}$ if $\chi_1^3 = 1$, $\chi_1^2 \neq 1$ or $\chi_1^2 = 1$, $\chi_1 \neq 1$.*

(1) $\chi_1^3 = 1$, $\chi_1^2 \neq 1$. The residue is given by

$$\mathrm{Res}_{\beta_4} \mathrm{Res}_{S_6} A(f, f'; \Lambda) = c\big(f(\beta_4), R(\rho_4, \beta_4, \rho_4\chi)(f'(\beta_4) + R(\rho_1, \beta_4, \chi)f'(\beta_4)\big),$$

where $c = \frac{c(F)\xi(1,\chi_1^2)\xi(1,\chi_1)}{\xi(2)\xi(2,\chi_1^2)\xi(2,\chi_1)\varepsilon(1,\chi_1^2)\varepsilon(1,\chi_1)}$.

(2) $\chi_1^2 = 1$, $\chi_1 \neq 1$. The residue is given by

$$\mathrm{Res}_{\beta_4} \mathrm{Res}_{S_6} A(f, f'; \Lambda) = c_1\Big(\Big(R(\rho_4, \beta_4, \chi) + R\big(\sigma(\pi), \beta_4, \chi\big)\Big)f(\beta_4), f'(\beta_4)\Big)$$
$$+ c_2\Big(R\Big(\sigma\Big(\frac{4\pi}{3}\Big), \beta_4, \chi\Big)f(\beta_4), f'(\beta_2)\Big),$$

where $c_1 = \frac{\xi(1,\chi_1)\xi(1,\chi_1) \quad c(F)}{\xi(2,\chi_1)\xi(2,\chi_1)\varepsilon(1,\chi_1)\varepsilon(2,\chi_1)\varepsilon(1,\chi_1)\xi(2)}$ and $c_2 = \frac{c(F)\xi(1,\chi_1)}{\xi(2)\xi(3,\chi_1)\varepsilon(2,\chi_1)\varepsilon(1,\chi_1)}$.

PROOF. It follows from Lemma 3.5.1 and straightforward computation.

PROPOSITION 3.5.4. *If χ is trivial, $\mathrm{Res}_{S_6} A(f, f'; \Lambda)$ has a triple pole at $\Lambda = \beta_4$. Moeglin-Waldspurger calculated the residue.*

PROPOSITION 3.5.5. *If χ is trivial, $\mathrm{Res}_{S_6} A(f, f'; \Lambda)$ has a simple pole at $\Lambda = \rho_B$, i.e., $z = \frac{3}{2}$. The residue is given by*

$$\mathrm{Res}_{\rho_B} \mathrm{Res}_{S_6} A(f, f'; \Lambda) = \frac{c(F)\xi(\frac{5}{2})}{\xi(\frac{9}{2})\xi(4)}\Big(R\big(\sigma(\pi), \rho_B, \chi\big)f(\rho_B), f'(\rho_B)\Big).$$

This gives the constant.

3.6. *Conclusion.* Let $J(\chi)$ be the subspace of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$, which is K-finite and K_∞-invariant.

3.6.1. χ is trivial. We can see from the above calculation (Propositions 3.2.3, 3.3.2, 3.4.3, 3.5.4, 3.5.5) that Moeglin and Waldspurger obtained all the residual spectrum attached to the trivial character of the torus. We summarize their results; $J(1)$ is isomorphic to the sum of the trivial representation and the image $(1+\frac{1}{2}E)R(\rho_2,\beta_2)I(\beta_2)_f$, where $I(\beta_2)_f$ is the K_∞-invariant subspace of $I(\beta_2)$ and $E=\otimes_v E_v$. Here E_v is defined as follows: Let $R(\rho_5,\rho_2\Lambda)|_{S_1}$ be the restriction of $R(\rho_5,\rho_2\Lambda)$ to S_1. It is holomorphic at β_2. Let E_v be the value of $R(\rho_5,\rho_2\Lambda)|_{S_1}$ at β_2.

Then $R_v(\rho_2,\beta_2)I_v(\beta_2)=\pi_{1v}\oplus\pi_{2v}$, where π_{1v} is spherical and

$$E_v(f_v)=\begin{cases} f_v, & \text{if } f_v\in\pi_{1v}\\ -2f_v, & \text{if } f_v\in\pi_{2v}.\end{cases} \tag{3.3}$$

Let S be a finite set of finite places and $\pi^S=\otimes_{v\notin S}\pi_{1v}\otimes\otimes_{v\in S}\pi_{2v}$. Then

$$J(1)=\pi_0\oplus\bigoplus_{S,\mathrm{card}(S)\neq 1}\pi^S,$$

where π_0 is the trivial representation.

3.6.2. χ is non-trivial. From Propositions 3.2.2, 3.4.2 and 3.5.3, only the following characters contribute to the residual spectrum:

(1) $\chi\colon\chi_1=1,\chi_6^2=1,\chi_6\neq 1$
(2) $\chi'=\rho_6\chi\colon{\chi_1'}^2=1,\chi_1'\neq 1,\chi_6'=\chi_1'$
(3) $\chi''=\rho_1\rho_6\chi\colon{\chi_1''}^2=1,\chi_1''\neq 1,\chi_6''=1$
(4) $\tilde\chi\colon\tilde\chi^3=1,\tilde\chi_1\neq 1,\tilde\chi_6=1$

Under the identification $M_1\simeq GL_2$, where M_1 is the Levi subgroup of P_1, the above characters are given by $\chi=\chi(\mu,\nu)$, where μ,ν are grössencharacters of F:

(1) $\chi=\chi(\mu,\nu)$, $\mu=\nu$, $\mu^2=1,\mu\neq 1$
(2) $\rho_6\chi=\chi(\mu,\nu)$, $\mu=1,\nu^2=1,\nu\neq 1$
(3) $\rho_1\rho_6\chi=\chi(\mu,\nu)$, $\mu^2=1,\mu\neq 1,\nu=1$
(4) $\tilde\chi=\chi(\mu,\nu)$, $\mu^3=1,\mu\neq 1,\nu=\mu^2$

CASE 1. $\chi\colon\chi_1=1,\chi_6^2=1,\chi_6\neq 1$

From Proposition 3.2.2,

$$\mathrm{Res}_{\beta_2}\mathrm{Res}_{S_1}A(f,f';\Lambda)=c_1\big(f(\beta_2),R(\rho_2,\beta_2,\rho_2\chi)$$
$$\left(R(\rho_5,\text{“}\beta_2\text{''},\chi)f'(\beta_2)+c_2R\left(\sigma\left(\frac{\pi}{3}\right),\beta_4,\rho_3\chi\right)f'(\beta_4)+c_3R\left(\rho_6,\beta_4,\frac{2\pi}{3}\chi\right)f'(\beta_4)\right),$$

where c_1,c_2,c_3 are constants and $R(\rho_5,\text{“}\beta_2\text{''},\chi)=\otimes_v R(\rho_5,\text{“}\beta_2\text{''},\chi_v)$ and $R(\rho_5,\text{“}\beta_2\text{''},\chi_v)$ is the value at β_2 of the restriction $R(\rho_5,\Lambda,\chi_v)|_{S_1}$. It is an isomorphism from $I(\beta_2,\chi)$ to $I(\beta_2,\rho_5\chi)$.

Here we recall the inner product formula (3.1) and our short-hand notation. Note that $\sigma(\pi)\in W(\chi,\chi)$, $\rho_3\in W(\chi,\rho_6\chi)$ and $\sigma(\frac{2\pi}{3})\in W(\chi,\rho_1\rho_6\chi)$. Therefore, the above f''s are all in different spaces. Since $R\big(\sigma(\frac{\pi}{3}), \beta_4, \rho_3\chi\big)$ and $R(\rho_6, \beta_4, \frac{2\pi}{3}\chi)$ are intertwining operators and $R(\rho_5, \text{“}\beta_2\text{”}, \chi)$ is an isomorphism, $J(\chi)$ is isomorphic to the image

$$R(\rho_2, \beta_2, \rho_2\chi)I(\beta_2, \rho_2\chi)_f = \bigotimes_{v<\infty} R_v(\rho_2, \beta_2, \rho_2\chi_v)I_v(\beta_2, \rho_2\chi_v).$$

We already know from Moeglin-Waldspurger that if χ_v is trivial,

$$R_v(\rho_2, \beta_2, \rho_2\chi_v)I_v(\beta_2, \rho_2\chi_v) = \pi_{1v} \oplus \pi_{2v}.$$

Suppose χ_v is not trivial. Then $\chi_v \otimes \exp\big(\beta_2, H_B()\big)$ is a regular character of T. So we can apply Rodier's result as follows: By the cocycle relation,

$$R_v(\rho_2, \beta_2, \rho_2\chi_v) = R_v(\rho_5, -\beta_2, \rho_5\chi_v)R_v\big(\sigma(\pi), \beta_2, \rho_2\chi_v\big).$$

From Rodier [R, Cor 3 in p417], $R_v\big(\sigma(\pi), \beta_2, \rho_2\chi_v\big)I_v(\beta_2, \rho_2\chi_v)$ and the unique irreducible subrepresentation of $I_v(-\beta_2, \rho_2\chi_v)$ have the same Jacquet module. Therefore, $R_v\big(\sigma(\pi), \beta_2, \rho_2\chi_v\big)I_v(\beta_2, \rho_2\chi_v)$ is the unique irreducible subrepresentation of $I_v(-\beta_2, \rho_2\chi_v)$. Since $R_v(\rho_5, -\beta_2, \rho_5\chi_v)$ is an isomorphism, $R_v(\rho_2, \beta_2, \rho_2\chi_v)I_v(\beta_2, \rho_2\chi_v)$ is the unique irreducible subrepresentation of $I(-\beta_2, \chi_v)$.

Let $J_v = \{\pi_{1v}, \pi_{2v}\}$. If χ_v is not trivial, we take $\pi_{2v} = 0$. Let S be a finite set of finite places and $\pi^S = \otimes_{v\notin S} \pi_{1v} \otimes \otimes_{v\in S} \pi_{2v}$. Then

$$J(\chi) = \bigoplus_S \pi^S.$$

There is no condition on S.

CASE 2. $\chi' = \rho_6\chi$: ${\chi_1'}^2 = 1, \chi_1' \neq 1, \chi_6' = \chi_1'$

From Proposition 3.4.2 and the adjoint formula (3.2),

$$\begin{aligned}\operatorname{Res}_{\beta_4} &\operatorname{Res}_{S_5} A(f, f'; \Lambda)\\ &= c_1\big(f(\beta_4), R(\rho_3, \beta_2, \rho_3\chi')f'(\beta_2)\big)\\ &\quad + c_2\big(f(\beta_4), R(\rho_4, \beta_4, \rho_4\chi')\big)\big(f'(\beta_4) + R(\rho_1, \beta_4, \chi')f'(\beta_4)\big).\end{aligned}$$

Recall the inner product formula (3.1). We note that $\rho_4 \in W(\rho_6\chi, \rho_1\rho_6\chi)$ since $\rho_4\rho_6\chi = \rho_1\rho_6\chi$ and $\sigma(\pi) \in W(\rho_6\chi, \rho_6\chi)$, $\rho_3 \in W(\rho_6\chi, \chi)$. Therefore, the above f''s are all in different spaces. Here

$$R(\rho_4, \beta_4, \rho_4\chi')I(\beta_4, \rho_4\chi') = \bigotimes_v R_v(\rho_4, \beta_4, \rho_4\chi_v')I_v(\beta_4, \rho_v\chi_v').$$

Under the identification $M_1 \simeq \mathrm{GL}_2$, $\rho_6\chi = \chi(\mu, \nu)$, $\mu = 1$, $\nu^2 = 1$, $\nu \neq 1$. By inducing in stages, $I(\beta_4, \rho_4\chi_v') = \operatorname{Ind}_{P_1}^G \exp\big(\beta_4, H_{P_1}()\big) \otimes \operatorname{Ind}_{B_0}^{\mathrm{GL}_2} \rho_4\chi_v'$, where B_0 is a Borel subgroup of GL_2. $\pi_v = \operatorname{Ind}_{B_0}^{\mathrm{GL}_2} \rho_4\chi_v'$ is irreducible. Therefore, $R_v(\rho_4, \beta_4, \rho_4\chi_v')I_v(\beta_4, \rho_v\chi_v')$ is the Langlands' quotient of $\operatorname{Ind}_{P_1}^G \exp\big(\beta_4, H_{P_1}()\big) \otimes \pi_v$. In particular, it is irreducible.

We have $\rho_3 = \rho_1\rho_4\rho_6$ and $\rho_4\chi' = \chi$. So

$$R(\rho_3, \beta_2, \chi_v) = R(\rho_1, -\beta_4, \rho_4\rho_6\chi_v)R(\rho_4, \beta_4, \rho_6\chi_v)R(\rho_6, \beta_2, \chi_v).$$

If χ_v is not trivial, then $R(\rho_1, -\beta_4, \rho_4\rho_6\chi_v)$ and $R(\rho_6, \beta_2, \chi_v)$ are isomorphisms. Therefore, the image of $R(\rho_3, \beta_2, \chi_v)$ is irreducible.

If χ_v is trivial, $R(\rho_6, \beta_2, \chi_v)$ is not an isomorphism and we proceed as follows: Since $\rho_3 = \rho_1\sigma(\frac{2\pi}{3})$, $R(\rho_3, \beta_2, \chi_v) = R\big(\rho_1, -\beta_4, \sigma(\frac{2\pi}{3}), \chi_v\big)R\big(\sigma(\frac{2\pi}{3}), \beta_2, \chi_v\big)$. As in Lemma 3.2.4, we can show that $R\big(\rho_1, -\beta_4, \sigma(\frac{2\pi}{3})\chi_v\big)$ is the identity. Also from [M-W1, equation (15)], $R_v(\rho_6, -\beta_2)\big(R_v(\rho_2, \beta_2) - E_v R_v(\rho_2, \beta_2)\big) = 0$. From (3.3), $E_v f = -2f$ for $f \in \pi_{2v}$ and so $R_v(\rho_6, -\beta_2)f = 0$ for $f \in \pi_{2v}$. Therefore $R_v(\rho_6, -\beta_2)R_v(\beta_2, \beta_2)I(\beta_2)$ is irreducible. Since $\rho_6\rho_2 = \sigma(\frac{2\pi}{3})$, the image of $R_v\big(\sigma(\frac{2\pi}{3}), \beta_2\big)$ is irreducible. It is isomorphic to the image $R_v(\rho_4, \beta_4)I(\beta_4)$.

Therefore, $J(\chi')$ is isomorphic to the image $R(\rho_4, \beta_4, \rho_4\chi')I(\beta_4, \rho_4\chi')$. Since $\rho_6\beta_4 = \beta_2$, $J(\chi')$ is isomorphic to none other than $\otimes_v \pi_{1v}$ in Case 1.

CASE 3. $\chi'' = \rho_1\rho_6\chi\colon \chi_1''^2 = 1, \chi_1'' \neq 1, \chi_6'' = 1$
From Proposition 3.5.3 and the adjoint formula (3.2),

$$\begin{aligned}\operatorname{Res}_{\beta_4}\operatorname{Res}_{S_6} A(f,f';\Lambda) &= c_1(f(\beta_4), R(\rho_4, \beta_4, \rho_4\chi'')(f'(\beta_4) + R(\rho_1, \beta_4, \chi'')f'(\beta_4))\\ &\quad + c_2\left(f(\beta_4), R\left(\sigma\left(\frac{2\pi}{3}\right), \beta_2, \sigma\left(\frac{4\pi}{3}\right)\chi''\right)f'(\beta_2)\right),\end{aligned}$$

We proceed in the same way as Case 2. $J(\chi'')$ is isomorphic to the image $R(\rho_4, \beta_4, \rho_4\chi'')I(\beta_4, \rho_4\chi'')$. It is irreducible and it is isomorphic to the one in Case 2.

CASE 4. $\tilde{\chi}\colon \tilde{\chi}_1^3 = 1, \tilde{\chi}_1^2 \neq 1, \tilde{\chi}_6 = 1$
From Proposition 3.5.3,

$$\begin{aligned}\text{residue} &= \operatorname{Res}_{\beta_4}\operatorname{Res}_{S_6} A(f,f';\Lambda)\\ &= c\big(f(\beta_4), R(\rho_4, \beta_4, \rho_4\tilde{\chi})\big(f'(\beta_4) + R(\rho_1, \beta_4, \tilde{\chi})f'(\beta_4)\big).\end{aligned}$$

So $J(\tilde{\chi})$ is isomorphic to the image $R(\rho_4, \beta_4, \rho_4\tilde{\chi})I(\beta_4, \rho_4\tilde{\chi})$. By inducing in stages, $I(\beta_4, \rho_4\tilde{\chi}_v) = \operatorname{Ind}_{P_1}^{G}\exp\big(\beta_4, H_{P_1}()\big) \otimes \operatorname{Ind}_{B_0}^{GL_2}\rho_4\tilde{\chi}_v$, where B_0 is a Borel subgroup of GL_2. $\pi_v = \operatorname{Ind}_{B_0}^{GL_2}\rho_4\tilde{\chi}_v$ is irreducible. Therefore, $R_v(\rho_4, \beta_4, \rho_4\tilde{\chi}_v)I_v(\beta_4, \rho_v\tilde{\chi}_v)$ is the Langlands' quotient of $\operatorname{Ind}_{P_1}^{G}\exp\big(\beta_4, H_{P_1}()\big) \otimes \pi_v$. So $J(\tilde{\chi})$ is irreducible.

We have proved

THEOREM 3.6.1. *Let $J(\chi)$ be the subspace of $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$, which is K-finite, K_∞-invariant. Then the K-finite, K_∞-invariant subspace of $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$ is the direct sum of the following space:*

(1) $J(1) = \pi_0 \oplus \bigoplus_{S,\text{card}(S)\neq 1}\pi^S$, where $\pi^S = \otimes_{v\notin S}\pi_{1v} \otimes \otimes_{v\in S}\pi_{2v}$, π_{1v} is spherical.

(2) $J(\chi) = \oplus_S \pi^S$, where $\chi_1 = 1$, $\chi_6^2 = 1$, $\chi_6 \neq 1$. $\pi^S = \otimes_{v\notin S}\pi_{1v} \otimes \otimes_{v\in S}\pi_{2v}$. If χ_{6v} is not trivial, we set $\pi_{2v} = 0$.

(3) $J(\tilde{\chi})$ = the Langlands' quotient of $\operatorname{Ind}_{P_1}^{G}\exp\big(\beta_4, H_{P_1}()\big) \otimes (\operatorname{Ind}_{B_0}^{GL_2}\tilde{\chi})$, where $\tilde{\chi}^3 = 1$, $\tilde{\chi}_1 \neq 1$, $\tilde{\chi}_6 = 1$.

3.7. *Arthur Parameters.* In this section, we give the Arthur parameters for the residual spectrum of $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$. We say that a unipotent element u is distinguished if all maximal tori of Cent(u, G) are contained in the center of G°, the connected component of the identity. This is equivalent to the fact that the unipotent orbit O of u does not meet any proper Levi subgroup of G (Spaltenstein [Sp, p67]). (*i.e.*, if L is a Levi subgroup of a parabolic subgroup of G and $u \in L$ for a $u \in O$, then $L^\circ = G^\circ$.)

JACOBSON-MOROZOV THEOREM. *Suppose u is a unipotent element in a semi-simple algebraic group G. Then there exists a homomorphism $\phi\colon \mathrm{SL}_2 \mapsto G$ such that* $\phi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = u$.

LEMMA ([B-V, PROP. 2.4]). *Let u be a unipotent element and $\phi\colon \mathrm{SL}_2 \mapsto G$ be a homomorphism such that $\phi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = u$. Let $S_\phi = \mathrm{Cent}(\mathrm{im}\,\phi, G) \subset S_u = \mathrm{Cent}(u, G)$ and U^u be the unipotent radical of S_u. Then*

(1) $S_u = S_\phi \cdot U^u$, a semi-direct product. S_ϕ is reductive.

(2) The inclusion $S_\phi \subset S_u$ induces an isomorphism between $S_\phi / S_\phi^\circ Z_G$ and $S_u / S_u^\circ Z_G$.

Let F be a number field and let W_F be the global Weil group of F. For G a split group of type G_2, we can take the dual group $G^* = G_2(\mathbb{C})$. An Arthur parameter is a homomorphism

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \mapsto G^*,$$

defined modulo conjugacy, with the following properties: (The usual definition of Arthur parameter uses Langlands' hypothetical group L_F. But since we are only dealing with principal series, W_F is enough.)

(1) $\psi(W_F)$ is bounded and included in the set of semi-simple elements of G^*.

(2) The restriction of ψ to $\mathrm{SL}_2(\mathbb{C})$ is algebraic.

Let $S_\psi = \mathrm{Cent}(\mathrm{im}\,\psi, G^*)$ and

$$C_\psi = S_\psi / S_\psi^\circ Z_{G^*}.$$

Now we recall Moeglin's reformulation of Arthur's conjecture ([M3]): For each place v of F, we have local Arthur parameters $\psi_v = \psi | W_{F_v} \times \mathrm{SL}_2(\mathbb{C})$, as well as S_{ψ_v}, C_{ψ_v}. It is a part of local Arthur's conjecture that for each irreducible character η_v of C_{ψ_v}, there exists an irreducible representation $\pi(\psi_v, \eta_v)$. For each v, let Π_{ψ_v} be the set of $\pi(\psi_v, \eta_v)$.

We define the global Arthur packet $\Pi_\psi = L^2\big(G(F)\backslash G(\mathbb{A})\big)_\psi$ to be the set of irreducible representations $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ such that for each v, π_v belongs to Π_{ψ_v} and for almost all v, π_v is spherical.

ARTHUR'S CONJECTURE (GLOBAL). (1) The representations in the packet corresponding to ψ may occur in the discrete spectrum if and only if C_ψ is finite, *i.e.*, $S_\psi^\circ = 1$. We call such an Arthur parameter elliptic.

(2) For an elliptic Arthur parameter ψ, any $\pi \in \Pi_\psi$ occurs discretely in $L^2\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if

$$\sum_{x \in C_\psi} \prod_v \eta_v(x_v) \neq 0, \tag{3.4}$$

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where $\pi = \otimes_v \pi(\psi_v, \eta_v)$, $x = (x_v)$.

REMARK 3.7.1. If C_ψ is abelian, then the above condition is equivalent to: the character $\Pi_v \eta_v|_{C_\psi}$ of C_ψ is trivial. This is what happens in split classical groups [M1]. However, it is not true in our G_2 case since C_ψ is not abelian as we see below. It is S_3, the symmetric group on 3 letters.

Let Π_{res_v} be the subset of Π_{ψ_v}, parametrizing the local components of the residual spectrum. We will find Π_{res_v} and verify (3.4) for a representation $\pi = \otimes_v \pi_v$, $\pi_v \in \Pi_{\mathrm{res}_v}$ for all v, π_v spherical for almost all v.

REMARK 3.7.2. A representation in Π_{ψ_v} but not in Π_{res_v} will appear as a local component of a cuspidal automorphic representation. Suppose we know the local packet Π_{ψ_v} completely. Then it is a very difficult problem to determine when a representation $\pi \in \Pi_\psi$ is a cuspidal representation. Moeglin [M5] has a partial result on that in the case of split classical groups.

3.7.1. χ trivial.

The Arthur parameter is given by

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longmapsto G_2(\mathbb{C}),$$

where $\psi|_{W_F} \equiv 1$ and $\psi\colon \mathrm{SL}_2(\mathbb{C}) \longmapsto G_2(\mathbb{C})$ is given by a unipotent orbit of $G_2(\mathbb{C})$. In order that ψ be elliptic, the unipotent orbit has to be distinguished. There are two distinguished unipotent orbits of $G_2(\mathbb{C})$, namely, $G_2(\mathbb{C})$ and $G_2(a_1)$ ([Ca, p401]).

CASE 1. The unipotent orbit $G_2(\mathbb{C})$.

The unipotent orbit $G_2(\mathbb{C})$ gives the constant which corresponds to the residue

$$\mathrm{Res}_{\rho_B} \mathrm{Res}_{S_6} A(f, f'; \Lambda).$$

CASE 2. The unipotent orbit $G_2(a_1)$.

If ψ is determined by the unipotent orbit $G_2(a_1)$, then $C_\psi = C_{\psi_v} = S_3$, the symmetric group on 3 letters. There are 3 irreducible characters of S_3, namely, ψ_3, ψ_{21} and ψ_{111}. Here ψ_{111} is the sign character of S_3. They are class functions and the character table is given by

	ψ_3	ψ_{21}	ψ_{111}
C_1	1	2	1
C_2	1	0	-1
C_3	1	-1	1

Character table of S_3

Here C_1, C_2 and C_3 are the conjugacy classes in S_3, namely, $C_1 = \{1\}$, $C_2 = \{(1,2),(1,3),(2,3)\}$, $C_3 = \{(1,2,3),(1,3,2)\}$. From Section 3.6.1, we know that $R_v(\rho_2, \beta_2)I(\beta_2) = \pi_{1v} \oplus \pi_{2v}$, where π_{1v} is spherical. We attach π_{1v} to ψ_3 and π_{2v} to ψ_{21}. Therefore, in this case, $\Pi_{\mathrm{res}_v} = \{\pi_{1v}, \pi_{2v}\}$.

Then $\pi = \otimes_{v\notin S}\pi_{1v}\otimes\otimes_{v\in S}\pi_{2v}$ appears in $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if

$$\otimes_{v\notin S}\psi_3(1)\otimes\bigotimes_{v\in S}\psi_{21}(1)+2\bigotimes_{v\notin S}\psi_3\big((1,2,3)\big)\otimes\bigotimes_{v\in S}\psi_{21}\big((1,2,3)\big) = 2^s+2(-1)^s\neq 0, \tag{3.5}$$

i.e., $s\neq 1$, where $s=|S|$. This coincides with Moeglin-Waldspurger [M-W1, Appendix III]: In Moeglin-Waldspurger, there is an operator E_v which acts on π_{1v} and π_{2v} (See Section 3.6.1). Since we attached π_{1v} to ψ_3 and π_{2v} to ψ_{21}, E_v acts on the irreducible characters of S_3 as follows: $E_v(\psi_3)=1$ and $E_v(\psi_{21})=-2$. Then we can see that $E_v(\eta)=\eta(1)\eta\big((123)\big)$ for $\eta=\psi_3,\psi_{21}$. Therefore we can write (3.5) as follows:

$$2\bigotimes_{v\notin S}\psi_3\big((1,2,3)\big)\otimes\bigotimes_{v\in S}\psi_{21}\big((1,2,3)\big)\Big(1+\frac{1}{2}\bigotimes_{v\notin S}\bigotimes_{v\notin S}E_v(\psi_3)\otimes\bigotimes_{v\in S}E_v(\psi_{21})\Big)\neq 0$$

i.e., $\pi=\otimes_v\pi_v$ appears in $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if $(1+\frac{1}{2}E)\pi\neq 0$, where $E=\otimes E_v$.

REMARK 3.7.3. According to Arthur's local conjecture, the sign character ψ_{111} should give an irreducible representation which is a local component of a cuspidal automorphic representation. We do not know what it is.

REMARK 3.7.4. For O a distinguished unipotent orbit, let $A(u)=C(u)/C(u)^0$, where $u\in O$ and $C(u)$ is the centralizer of u. Let Springer(O) be the set of irreducible characters of $A(u)$ which are in the image of the Springer correspondence which is an injective map from the set of irreducible characters of W into the set of pairs (O,η), where O is a unipotent orbit and η is an irreducible character of $A(u)=C(u)/C(u)^0$, where $u\in O$. We note that by [Ca, p427], Springer$\big(G_2(a_1)\big)=\{\psi_3,\psi_{21}\}$ in $G_2(\mathbb{C})$. Therefore the local component Π_{res_v} of the residual spectrum attached to the trivial character of the torus is parametrized by Springer$\big(G_2(a_1)\big)$. Moeglin [M1] showed that for split classical groups, the residual spectrum attached to the trivial character of the torus is parametrized by distinguished unipotent orbits O and Springer(O). In other words, if the Arthur parameter ψ is given by the distinguished unipotent orbit O, then $\Pi_{\mathrm{res}_v}=$ Springer(O) and the multiplicity formula (3.4) holds.

Therefore we believe that the same thing would happen for all split groups. We state this as follows:

CONJECTURE. Let G be a split group over a number field F and T be a maximal torus of G. Then the residual spectrum attached to the trivial character of $T(\mathbb{A})/T(F)$ is parametrized by distinguished unipotent orbits O of $G^*(\mathbb{C})$, the L-group of G and Springer(O). More precisely, if the Arthur parameter $\psi\colon \mathrm{SL}_2(\mathbb{C})\longmapsto G^*(\mathbb{C})$ is given by the distinguished unipotent orbit O, then $\Pi_{\mathrm{res}_v}=$ Springer(O) and the multiplicity formula (3.4) holds.

We give an example of this conjecture in the case of split exceptional group F_4 and we hope to settle this example in the near future: Suppose the Arthur parameter $\psi\colon \mathrm{SL}_2(\mathbb{C})\longmapsto F_4(\mathbb{C})$ is given by the distinguished unipotent orbit $F_4(a_3)$. By [Ca, p401], $A(u)=S_4$, the symmetric group on 4 letters. There are 5 irreducible characters of S_4, namely, $\psi_4 = 1$, ψ_{31}, ψ_{22}, ψ_{211} and ψ_{1111}. By [Ca, p428], $\text{Springer}(F_4(a_3)) = \{\psi_4, \psi_{31}, \psi_{22}, \psi_{211}\}$. The character table of S_4 is given by

	C_1	C_2	C_3	C_4	C_5
ψ_4	1	1	1	1	1
ψ_{1111}	1	1	1	-1	-1
ψ_{211}	2	2	-1	0	0
ψ_{22}	3	-1	0	1	-1
ψ_{31}	3	-1	0	-1	1

Character table of S_4

Here C_i, $i = 1, \ldots, 5$, are the conjugacy classes in S_4, with representatives 1, (12)(34), (123), (12) and (1234), respectively: $|C_1| = 1$, $|C_2| = 3$, $|C_3| = 8$, $|C_4| = 6$ and $|C_5| = 6$. According to the conjecture, there will be 4 irreducible representations $\pi_{1v}, \ldots, \pi_{4v}$ attached to ψ_4, ψ_{211}, ψ_{22} and ψ_{31}, respectively. We divide $\text{Springer}(F_4(a_3))$ as follows: $\text{Springer}(F_4(a_3)) = \Pi_1 \cup \Pi_2 \cup \Pi_3$, where $\Pi_1 = \{\psi_4, \psi_{211}\}$, $\Pi_2 = \{\psi_4, \psi_{22}\}$ and $\Pi_3 = \{\psi_4, \psi_{31}\}$. The residual spectrum factors through Π_i, *i.e.*, it is the set of all $\pi = \otimes \pi_v$ such that there exists i, $\pi_v \in \Pi_i$ for all v. Let S be a finite set of finite places and $s = |S|$. If $\pi = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{2v}$, it appears in $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if $2^s + 3(2^s) + 8(-1)^s \neq 0$, *i.e.*, $s \neq 1$. If $\pi = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{3v}$, it appears in $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if $3^s + 3(-1)^s + 6 + 6(-1)^s \neq 0$, *i.e.*, $s \neq 1$. If $\pi = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{4v}$, it appears in $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if $3^s + 3(-1)^s + 6(-1)^s + 6 \neq 0$, *i.e.*, $s \neq 1$.

3.7.2. χ non-trivial. In order to find Arthur parameters for non-trivial characters, we have to look for endoscopic groups of $G_2(\mathbb{C})$, since Arthur parameters will factor through the endoscopic groups.

There are two equivalence classes of proper cuspidal endoscopic groups of $G_2(\mathbb{C})$, that is, $SL_3(\mathbb{C})$ and $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})/\{\pm 1\}$ ([A1, p30]). They are given as follows: Under the identification $M_1 \simeq GL_2$, by (2.1), $\beta_4^\vee(t) = \text{diag}(t, t)$. Then by [Ca, p93], $C\big(\beta_4^\vee(\omega)\big)$, the centralizer of $\beta_4^\vee(\omega)$ in $G_2(\mathbb{C})$, where $\omega^3 = 1$, $\omega \neq 1$, is reductive and its root system is $\Phi_1 = \{\pm\beta_1, \pm\beta_3, \pm\beta_5\}$, *i.e.*, $C\big(\beta_4^\vee(\omega)\big) \simeq SL_3$. The other one is $C\big(\beta_4^\vee(-1)\big)$. By [Ca, p93], its root system is $\Phi_1 = \{\pm\beta_1, \pm\beta_4\}$, *i.e.*, $C\big(\beta_4^\vee(-1)\big) \simeq SL_2(\mathbb{C}) \times SL_2(\mathbb{C})/\{\pm 1\}$.

The center of $SL_3(\mathbb{C})$ is $Z_3 = \{\omega I_3, \omega^3 = 1\} \simeq \mathbb{Z}_3$ and the center of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})/\{\pm 1\}$ is $Z_2 = \{\pm I_2\}$. Moreover, $S_3 = Z_3 \rtimes Z_2$.

CASE 1. The conjugacy class of χ: $\chi_1 = 1$, $\chi_6^2 = 1$, $\chi_6 \neq 1$

Under the identification $M_1 \simeq GL_2$, $\chi = \chi(\mu, \mu)$, $\mu^2 = 1$, $\mu \neq 1$, where μ is a grössencharacter of F. We have the embedding $SL_3(\mathbb{C}) \subset G_2(\mathbb{C})$.

The Arthur parameter factors through $SL_3(\mathbb{C})$:

$$\psi\colon W_F \times SL_2(\mathbb{C}) \longmapsto SL_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

$\psi|_{W_F}\colon w \longmapsto \begin{pmatrix} \mu(w) & & \\ & \mu(w) & \\ & & 1 \end{pmatrix}$ and $\psi\colon SL_2(\mathbb{C}) \longmapsto SL_3(\mathbb{C})$ is determined by the principal unipotent orbit of $SL_3(\mathbb{C})$. Here we note that under the embedding $SL_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$,

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the principal unipotent orbit of $SL_3(\mathbb{C})$ corresponds to the distinguished unipotent orbit $G_2(a_1)$ in $G_2(\mathbb{C})$ ([Ca, p401]). Then $S_\psi = Z_2$, $C_\psi = Z_2$ and $C_{\psi_v} = Z_2$ if μ_v is not trivial. $C_{\psi_v} = S_3$ if μ_v is trivial.

If μ_v is trivial, then $\Pi_{\mathrm{res}_v} = \{\pi_{1v}, \pi_{2v}\}$. If μ_v is not trivial, then $\Pi_{\mathrm{res}_v} = \{\pi_{1v}\}$. Then any $\pi = \otimes_{v\notin S}\pi_{1v} \otimes \otimes_{v\in S}\pi_{2v}$ appears in $L^2_d\big(G(F)\backslash G(\mathbb{A})\big)$ since

$$\bigotimes_{v\notin S}\psi_3(1)\otimes\bigotimes_{v\in S}\psi_{21}(1)+\bigotimes_{v\notin S}\psi_3(\tau)\otimes\bigotimes_{v\in S}\psi_{21}(\tau)\neq 0,$$

where τ is the non-trivial element in Z_2.

CASE 2. The conjugacy class of $\tilde{\chi}$: $\tilde{\chi}_1^3 = 1$, $\tilde{\chi}_1 \neq 1$, $\tilde{\chi}_6 = 1$

Under the identification, $M_1 \simeq GL_2$, $\tilde{\chi} = \chi(\mu,\nu)$, $\mu^3 = 1$, $\mu \neq 1$, $\nu = \mu^2$. The Arthur parameter factors through $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})/\{\pm 1\}$:

$$\psi\colon W_F \times SL_2(\mathbb{C}) \longmapsto SL_2(\mathbb{C}) \times SL_2(\mathbb{C})/\{\pm 1\} \hookrightarrow G_2(\mathbb{C}),$$

where $\psi|_{W_F}\colon w \longmapsto \begin{pmatrix}\mu(w) & \\ & \mu^{-1}(w)\end{pmatrix} \times \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\psi\colon SL_2(\mathbb{C}) \longmapsto SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ is determined by the principal unipotent orbits of $SL_2(\mathbb{C})$. We note that under the embedding $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})/\{\pm 1\} \hookrightarrow G_2(\mathbb{C})$, the principal unipotent orbit of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})/\{\pm 1\}$ corresponds to the unipotent orbit A_1 in $G_2(\mathbb{C})$ ([Ca, p401]). Then $S_\psi = Z_2$, $C_{\psi_v} = Z_2$ for μ_v non-trivial and $C_{\psi_v} = 1$ for μ_v trivial. In this case, Π_{res_v} consists of the Langlands' quotient which corresponds to the trivial character of $C_{\psi_v} = Z_2$. Therefore Π_{res} consists of one element.

REMARK 3.7.5. Arthur associated to an Arthur parameter ψ, an associated Langlands' parameter ϕ_ψ and conjectured that we could enlarge the L-packet $\Pi_{\phi_{\psi_v}}$ to Π_{ψ_v}. We note that in each of our cases, the associated Langlands' L-packet consists of only one element.

4. Decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1}$.

We have

$$\mathfrak{a}^*_{P_1} = X(M_1)\otimes\mathbb{R} = \mathbb{R}\beta_4, \mathfrak{a}_{P_1} = \mathbb{R}\beta_4^\vee$$

ρ_{P_1} is the half sum of roots generating N_1. Then $\rho_{P_1} = \frac{5}{2}\beta_4$.

Let $\tilde{\alpha} = \beta_4$ and identify $s \in \mathbb{C}$ with $s\tilde{\alpha} \in \mathfrak{a}^*_{\mathbb{C}}$. Let $\pi = \otimes\pi_v$ be a cusp form on $M_1 = GL_2$. Given a K-finite function φ in the space of π, we shall extend φ to a function $\tilde{\varphi}$ on G and set

$$\Phi_s(g) = \tilde{\varphi}(g)\exp\langle s+\rho_{P_1}, H_{P_1}(g)\rangle.$$

Define an Eisenstein series

$$E(s,\tilde{g},g,\rho_1) = \sum_{\gamma\in P_1(F)\backslash G(F)}\Phi_s(\gamma g).$$

It is known that $E(s,\tilde{\varphi},g,\rho_1)$ converges for $\mathrm{Re}(s) >> 0$ and extends to a meromorphic function of s in $\mathbb{C}$, with a finite number of poles in the plane $\mathrm{Re}(s) > 0$, all simple and on the real axis.

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It is also known that $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1}$ is spanned by the residues of the Eisenstein series for $\text{Re}(s) > 0$. We know that the poles of the Eisenstein series coincide with those of its constant terms. So it is enough to consider the constant term along P_1, which is

$$E_0(s,\tilde{\varphi},g,P_1) = \sum_{w\in\Omega} M(s,\pi,w)f(g)$$

where $\Omega = \{1, \rho_6\rho_1\rho_6\rho_1\rho_6\}$ and

$$M(s,\pi,w)f(g) = \int_{N_w^-} f(w^{-1}ng)\,dn$$

where

$$N_w^- = \prod_{\substack{\alpha>0\\ w^{-1}\alpha<0}} U_\alpha,$$

U_α is the one parameter unipotent subgroup and $f \in I(s,\pi) = \text{Ind}_{P_1}^G \pi \otimes \exp\big(s, H_{P_1}()\big)$.

We note that for each s, the representation of $G(\mathbb{A})$ on the space of Φ_s is equivalent to $I(s,\pi)$.

Then

$$M(s,\pi,w) = \bigotimes M(s,\pi_v,w), \quad M(s,\pi_v,w)f_v(g) = \int_{N_w(F_v)} f_v(w^{-1}ng)\,dn$$

where $f = \otimes f_v, f_v$ is the unique K_v-fixed function normalized by $f_v(e_v) = 1$ for almost all v.

Let ${}^LM_1 = \text{GL}_2(\mathbb{C})$ be the L-group of M_1. Denote by r the adjoint action of ${}^LM_1$ on the Lie algebra ${}^L\mathfrak{n}_1$ of ${}^LN_1$, the L-group of N_1.

Then

$$r = r_1 \oplus r_2$$
$$r_1 = r_3^0, \quad r_2 = \wedge^2\rho_2$$

where $r_3^0 = r_3 \otimes (\wedge^2\rho_2)^{-1}$ is the adjoint cube representation of $\text{GL}_2(\mathbb{C})$ (See [S4]). Here r_3 is the symmetric cube representation of $\text{GL}_2(\mathbb{C})$ and ρ_2 is the standard representation of $\text{GL}_2(\mathbb{C})$.

Then it is well-known ([S3]) that for $w = \rho_6\rho_1\rho_6\rho_1\rho_6$

$$M(s,\pi,w)f = \bigotimes_{v\in S} M(s,\pi_v,w)f_v \otimes \bigotimes_{v\notin S} \longrightarrow \tilde{f}_v \times \frac{L_S(s,\pi,r_1)L_S(2s,\pi,r_2)}{L_S(1+s,\pi,r_1)L_S(1+2s,\pi,r_2)}$$

where S is a finite set of places of F, including all the archimedean places such that for every $v \notin S$, π_v is a class 1 representation and if $f = \otimes_v f_v$, for $v \notin S, f_v$ is the unique K_v-fixed function normalized by $f_v(e_v) = 1$. $\tilde{f}_v$ is the K_v-fixed function in the space of $I\big(-s, w(\pi_v)\big)$.

Finally, $L_S(s,\pi,r_i) = \Pi_{v\notin S} L(s,\pi_v,r_i)$, where $L(s,\pi_v,r_i)$ is the local Langlands' L-function attached to π_v, r_i.

(1) Analysis of $L_S(s,\pi,r_1)$

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We know ([S4]) that $L_S(s,\pi,r_1)$ is absolutely convergent for $\mathrm{Re}(s)>1$ and hence has no zero there. It is expected ([S4, Bu-G-H, Ik]) that the completed L-function $L(s,\pi,r_1)$ has a pole for $\mathrm{Re}(s)>0$ if and only if $s=1$, $\omega_\pi^2=1$, $\omega_\pi\neq 1$ and π is the monomial representation corresponding to the quadratic character ω_π, where ω_π is the central character of π. We assume this fact.

REMARK. Ikeda [Ik] calculated the poles of the Rankin triple L-function $L(s,\pi\otimes\pi\otimes\pi)$ for π cuspidal representation of GL_2. It is related to the symmetric cube L-function of π as follows:

$$L(s,\pi\otimes\pi\otimes\pi)=L(s,\pi,r_3)\big(L(s,\pi\otimes\omega_\pi)\big)^2.$$

The symmetric cube L-function is given by

$$L(s,\pi,r_3)=L(s,\pi\otimes\omega_\pi,r_3^0).$$

$L(s,\pi,r_3)$ has a pole at $s=1$ when $\omega_\pi^6=1$ and $\omega_\pi^3\neq 1$ and so $L(s,\pi,r_3^0)$ has a pole when $\omega_\pi^2=1$ and $\omega_\pi\neq 1$.[1]

(2) Analysis of $L_S(s,\pi,r_2)$

For $v\notin S$,

$$L(s,\pi_v,r_2)=L(s,\omega_{\pi_v})=(1-\omega_{\pi_v}(\varpi)q_v^{-s})^{-1}$$

so $L_S(s,\pi,r_2)$ is the (partial) Hecke L-function. It has no zero for $\mathrm{Re}(s)>1$. The completed L-function $L(s,\pi,r_2)$ has a pole for $\mathrm{Re}(s)>0$ if and only if $s=1$, $\omega_\pi=1$.

(3) Analysis of $M(s,\pi_v,w)$ for $v\in S$.

For π_v tempered, the local factors $L(s,\pi_v,r_i)$ and $M(s,\pi_v,w)$ are holomorphic for $\mathrm{Re}(s)>0$. We show that for any $v\in S$,

$$L(s,\pi_v,r_1)^{-1}L(2s,\pi_v,r_2)^{-1}M(s,\pi_v,w)$$

is holomorphic. It is enough to show it for π_v complementary series. We follow [Ki]. Under the identification $M_1\simeq\mathrm{GL}_2$, by (2.1), for $\pi_v=\pi(\mu|\ |^r,\mu|\ |^{-r})$, $0\le r<\frac12$, complementary series of GL_2,

$$\mathrm{Ind}_{P_1}^G\,\pi_v\otimes\exp\big(\langle s\tilde\alpha,H_{P_1}()\rangle\big)=\mathrm{Ind}_B^G\,\chi(\mu,\mu)\otimes\exp\big(\langle\Lambda,H_B()\rangle\big),$$

where $\Lambda=(2r)\beta_3+(s-3r)\beta_4$. From this we have our assertion.

Now we assume that $r<\frac16$. Right now the best known result is $r<\frac15$ due to Shahidi [S3]. Then Λ is in the positive Weyl chamber for $s=\frac12$ and $s=1$ and we have

LEMMA 4.1. *For each v, the images of $M(\frac12,\pi_v,w)$ and $M(1,\pi_v,w)$ are irreducible.*

[1] Thanks to F. Shahidi who pointed this out.

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Conclusion. $E(s, \tilde{\varphi}, g, P_1)$ has a pole in the half plane $\mathrm{Re}(s) > 0$ if and only if
(1) $\omega_\pi = 1, s = \frac{1}{2}, L(\frac{1}{2}, \pi, r_3^0) \neq 0$,
(2) $\omega_\pi^2 = 1, \omega_\pi \neq 1, s = 1$, π monomial representation attached to ω_π.

Let $J_1(\pi_v)$ be the image of $M(\frac{1}{2}, \pi_v, w)$ and $J_2(\pi_v)$, the image of $M(1, \pi_v, w)$. They are the unique irreducible quotients of $I(\frac{1}{2}, \pi_v)$ and $I(1, \pi_v)$, respectively. Let $J_1(\pi) = \otimes_v J_1(\pi_v)$ and $J_2(\pi) = \otimes_v J_2(\pi_v)$. We have proved

THEOREM 4.2.

$$L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1} = \bigoplus_{\pi_1} J_1(\pi_1) \oplus \bigoplus_{\pi_2} J_2(\pi_2)$$

where π_1 runs over cuspidal representations of GL_2 with trivial central characters and $L(\frac{1}{2}, \pi, r_3^0) \neq 0$ and π_2 runs over monomial representations.

5. **Decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_2}$.**

In this case $\mathfrak{a}^*_{P_2} = X(M_2) \otimes \mathbb{R} = \mathbb{R}\beta_3$, $\mathfrak{a}_{P_2} = \mathbb{R}\beta_3^\vee$, $\rho_{P_2} = \frac{3}{2}\beta_3$.

Let $\tilde{\alpha} = \beta_3$ and identify $s \in \mathbb{C}$ with $s\tilde{\alpha} \in \mathfrak{a}^*_{\mathbb{C}}$. In this case, for π cuspidal representation of GL_2, the constant term of Eisenstein series is given by

$$E_0(s, \tilde{\varphi}, g, P_2) = \sum_{w \in \Omega} M(s, \pi, w) f(g)$$

where $\Omega = \{1, \rho_1\rho_6\rho_1\rho_6\rho_1\}$.

The adjoint action r of ${}^LM_2$ on ${}^L\mathfrak{n}_2$ is given as

$$r = r_1 \oplus r_2 \oplus r_3$$
$$r_1 = \rho_2, r_2 = \wedge^2\rho_2, r_3 = \rho_2 \otimes \wedge^2\rho_2.$$

Therefore for $w = \rho_1\rho_6\rho_1\rho_6\rho_1$,

$$M(s, \pi, w)f = \bigotimes_{v \in S} M(s, \pi_v, w) f_v \otimes \bigotimes_{v \in S} \tilde{f}_v \times \frac{L_S(s, \pi, r_1) L_S(2s, \pi, r_2) L_S(3s, \pi, r_3)}{L_S(s+1, \pi, r_1) L_S(2s+1, \pi, r_2) L_S(3s+1, \pi, r_3)}$$

where S is the same as in the case $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1}$.

Here

$$L(s, \pi_v, r_1) = L(s, \pi_v), \text{ the standard } L\text{-function for } \mathrm{GL}_2.$$
$$L(s, \pi_v, r_3) = L(s, \pi_v \otimes \omega_{\pi_v}), \text{ twisted by the central character.}$$
$$L(s, \pi_v, r_2) = L(s, \omega_{\pi_v}), \text{ Hecke } L\text{-function.}$$

We know that $L_S(s, \pi \otimes \theta)$ is absolutely convergent for $\mathrm{Re}\, s > 1$ for any grössencharacter θ. So it has no zero there. We know also that the completed L-function $L(s, \pi \otimes \theta)$ is entire for any θ. Under the identification $M_2 \simeq \mathrm{GL}_2$, by (2.2), for $\pi_v = \pi(\mu|\ |^r, \mu|\ |^{-r})$ complementary series of GL_2,

$$\mathrm{Ind}_{P_2}^G \pi_v \otimes \exp\big(\langle s\tilde{\alpha}, H_{P_2}()\rangle\big) = \mathrm{Ind}_B^G \chi(\mu, \mu) \otimes \exp\big(\langle \Lambda, H_B()\rangle\big),$$

where $\Lambda = (s-3r)\beta_3 + 6r\beta_4$. Therefore, for any $v \in S$,

$$\prod_{i=1}^{3} L(is, \pi_v, r_i)^{-1} M(s, \pi_v, w)$$

is holomorphic. Also if we assume $r < \frac{1}{6}$, Λ is in the positive Weyl chamber and the image of $M(\frac{1}{2}, \pi_v, w)$ is irreducible. Therefore, $E(s, \tilde{\varphi}, g, P_2)$ has a pole in the half plane $Res > 0$ if and only if $\omega_\pi = 1$, $s = \frac{1}{2}$ and $L(\frac{1}{2}, \pi, r_1) \neq 0$.

Let $J(\pi_v)$ be the image of $M(\frac{1}{2}, \pi_v, w)$ and $J(\pi) = \otimes_v J(\pi_v)$. Then we have

THEOREM 5.1.

$$L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_2} = \bigoplus_{\pi} J(\pi)$$

where π runs over cuspidal representations of GL_2 *with trivial central characters and* $L(\frac{1}{2}, \pi) \neq 0$.

REMARK 5.1. The referee suggested the problem of finding a connection between Arthur's conjecture and the non-vanishing of L-functions at $s = \frac{1}{2}$. Arthur [A1] did it for the group $P\,\mathrm{Sp}_4$. It would be interesting to do so in the above case.

REFERENCES

[A1] J. Arthur, *On some problems suggested by the trace formula*, In: Lie group representations, II, Lecture Notes in Math. **1041**, Springer-Verlag, 1984, 1–49.

[A2] ———, *Unipotent Automorphic Representations: Conjectures*, In: Orbites unipotentes et représentations II. Groupes p-adiques et réels, Astérisque **171–172**(1989), 13–71.

[A3] ———, *Eisenstein Series and the Trace Formula*, Proceedings of Symposia in Pure Mathematics, I, **33**(1979), 253–274.

[B-V] D. Barbasch and D. Vogan, Jr., *Unipotent representations of complex semisimple groups*, Ann. of Math. **121**(1985), 41–110.

[Bu-G-H] D. Bump, D. Ginzburg and J. Hoffstein, *The Symmetric Cube*, 1995, preprint.

[Ca] R. W. Carter, *Finite Groups of Lie Type, Conjugacy Classes and Complex Characters*, John Wiley and Sons, 1985.

[G-S] S. Gelbart and F. Shahidi, *Analytic Properties of Automorphic L-functions*, Perspectives in Mathematics **6**, Academic Press, 1988.

[Go] L. Goldstein, *Analytic Number Theory*, Prentice-Hall, 1971.

[Ik] T. Ikeda, *On the location of poles of the triple L-functions*, Comp. Math. **83**(1992), 187–237.

[Ja] H. Jacquet, *On the residual spectrum of* $\mathrm{GL}(n)$, In: Lie Group Representations II, Lecture Notes in Math. **1041**, Springer-Verlag, 1983, 185–208.

[K-S] C. D. Keys and F. Shahidi, *Artin L-functions and normalization of intertwining operators*, Ann. Scient. Ec. Norm. Sup. **21**(1988), 67–89.

[Ki] H. Kim, *The Residual Spectrum of* Sp_4, Comp. Math. (2)**99**(1995), 129–151.

[Ki-S] H. Kim and F. Shahidi, *Quadratic unipotent Arthur parameters and residual spectrum of* Sp_{2n}, Amer. J. Math. (2)**118**, 401–425.

[L1] R. P. Langlands, *Representations of abelian algebraic groups*, Notes, Yale University, 1968.

[L2] ———, *On the classification of irreducible representations of real algebraic groups*, In: Representation Theory and Harmonic Analysis on Semisimple Lie Groups, (eds. Paul J. Sally and David A. Vogan), AMS Math. Surveys and Monographs **31**(1989), 101–170.

[L3] ———, *Euler Products*, Yale University Press, 1971.

[L4] ———, *On the Functional Equations Satisfied by Eisenstein Series*, Lecture Notes in Math. **544**, Springer-Verlag, 1976.

[Li-Sc] J. S. Li and J. Schwermer, *Constructions of automorphic forms and related cohomology classes for arithmetic subgroups of* G_2, Comp. Math. **87**(1993), 45–78.

[M1] C. Moeglin, *Représentations unipotentes et formes automorphes de carréintégrable*, Forum mathematicum **6**(1994), 651–744.

[M2] ———, *Orbites unipotentes et spectre discret non ramifie, Le cas des groupes classiques déployé*, Comp. Math. **77**(1991), 1–54.

[M3] ———, *Une conjecture sur le spectre résiduel des groupes classiques*, 1994, preprint.

[M4] ———, *Letter*, 1994.

[M5] ———, *Cuspidal quadratic unipotent representations*, 1994, preprint.

[M-W1] C. Moeglin and J. L. Waldspurger, *Spectral decomposition and Eisenstein series, Une Paraphrase de l'Écriture*, Cambridge University Press, 1995.

[M-W2] ———, *Le Spectre Résiduel de* GL(n), Ann. Scient. Éc. Norm. Sup. **22**(1989), 605–674.

[R] F. Rodier, *Decomposition de la serie principale des groupes redutifs p-adiques*, In: Non-commutative harmonic analysis, Lecture Notes in Math. **880**, Springer-Verlag, 1981, 408–424.

[S1] F. Shahidi, *On certain L-functions*, Amer. J.Math. **103**(1981), 297–356.

[S2] ———, *Whittaker Models for Real Groups*, Duke Math. J. (1)**47**(1980), 99–125.

[S3] ———, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Ann. of Math. **127**(1988), 547–584.

[S4] ———, *Third symmetric power L-functions for* GL(2), Comp. Math. **70**(1989), 245–273.

[Sp] N. Spaltenstein, *Classes Unipotentes et Sous-groupes de Borel*, Lecture Notes in Mathematics **946**, Springer-Verlag, 1982.

[Wi] N. Winarsky, *Reducibility of principal series representations of p-adic Chevalley groups*, Amer. J. Math. (5)**100**(1978), 941–956.

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