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ON A CONJECTURE OF GRAHAM CONCERNING A SEQUENCE OF INTEGERS

BY
E. Z. CHEIN

Let $0<a_1<\cdots<a_n$ be integers and (a, b) denotes the greatest common divisor of a, b. R. L. Graham [1] has conjectured that

$$\frac{a_i}{(a_i, a_j)} \geq n$$

for some i and j. In a recent paper Weinstein [2] has improved Winterle's result [3] and has proven the following interesting theorem:

THEOREM. (Weinstein). *If A is the sequence $a_1<\cdots<a_n$, where $a_k=P$, a prime for some k and $P \nmid (a_i+a_j)/2$, $1 \leq i<j \leq n$, then*

$$\max_{i,j}\left\{\frac{a_i}{(a_i, a_j)}\right\} \geq n.$$

In this paper we prove that the condition $P \nmid (a_i+a_j)/2$ in Weinstein's Theorem is unnecessary by modifying Weinstein's argument. We use Weinstein's notation throughout the paper. Our principal result is the following

THEOREM. *If A is the sequence $0<a_1<\cdots<a_n$, where $a_k=P$, a prime for some k, then*

$$\frac{a_i}{(a_i, a_j)} \geq n$$

for some i and j.

Proof. Assume there exists a sequence A, say $0<a_1<\cdots<a_n$, where $a_k=P$ for some k and

$$\frac{a_i}{(a_i, a_j)} < n$$

for all i and j.

Let B be the subsequence $b_1<\cdots<b_g<\cdots<b_r$ of A consisting of all terms of A which are not divisible by P. By results of Winterle [3] and Vélez [4], the

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conjecture is true if a_1 is prime or $n-1$ is prime, so we can assume that neither a_1 nor $n-1$ is prime. Then we have

$$b_1+(g-1)\leq b_g<P<n-1, \qquad (g\geq 1)$$

and so

$$\frac{P}{b_1}\geq\frac{P}{p-g}>\frac{n-1}{n-g-1}.$$

This gives

$$\frac{(n-g-1)p}{(n-g-1)P}\geq(n-g-1)\frac{P}{b_1}\geq(n-g-1)\frac{P}{p-g}$$
$$>(n-g-1)\frac{n-1}{n-g-1}=n-1,$$

but $(n-g-1)P/((n-g-1)P, b_1)$ is an integer, so is greater than or equal to n. Hence

(1) $$a_i\leq(n-g-2)P$$

for all $a_i\in A\setminus B$. Also $P\nmid b_i$, so

$$b_r\leq n-1.$$

We now define a mapping $T(b_i)$ for all $b_i\in B$ by

(2) $$T(b_i)=\begin{cases} P^{h_i}(b_i-P) & \text{if} \quad g<i\leq r \quad \text{and} \quad P^{h_i}(b_i-P)\not\in A \\ nP & \text{if} \quad P^{h_i}(bi-P)\in A \\ (n+i)P & \text{if} \quad 1\leq i\leq g, \end{cases}$$

where h_i is the largest non-negative integer such that $P^{h_i}(b_i-P)\leq(n-g-2)P$.

We next show that T is $1-1$. If $1\leq i\leq g$, it is clear that the $T(b_i)$ are all distinct. In the case $g<i\leq r$, since $b_i\leq n-1$ and $b_1+g\leq P$, it follows that $b_i-P\leq n-g-2$. Then $h_i\geq 1$ so that $P\mid T(b_i)$. Also, since $(P^{h_i}(b_i-P), b_i)=1$ we must have $T(b_i)/(T(b_i), b_i)=T(b_i)$. Now if $T(b_i)\leq n-g-2$, then $T(b_i)P\leq (n-g-2)P$, which contradicts (2), the definition of $T(b_i)$. So

$$T(b_i)\not\in A$$

except possibly when $n-g-1\leq T(b_i)\leq n-1$.

Now $P\mid T(b_i)$ and $1+g<b_1+g\leq P$. Since there is at most one term of P consecutive integers which is divisible by P, we have

$$|\{T(b_i)|\,\gamma\geq r\geq 1\}\cap A|\leq 1.$$

Now if $T(b_i)=T(b_j)$, then $P^{h_i}(b_i-P)=P^{h_j}(b_j-P)$, so $b_i=b_j$. Hence the $T(b_i)$ are distinct for all i, so that T is 1–1.

We next define $F(a_i)$ for all $a_i \in A$ by

$$F(a_i) = \begin{cases} a_i & \text{if } \quad P \mid a_i \\ T(a_i) & \text{if } \quad P \nmid a_i \end{cases}.$$

Then $P \mid F(a_i)$ for all i. In view of (1) and (2), $F(a_i) \neq F(a_j)$ if $i \neq j$, so F is 1–1. From (1) and (2) we see that

$$|A| \leq (n-g-2)+g+1 = n-1,$$

which contradicts the fact $|A| = n$. This completes the proof of our theorem.

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Department of Mathematics
The Pennsylvania State University
University Park, Pennsylvania 16802

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