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FREE ACTIONS OF ABELIAN GROUPS ON A CARTESIAN POWER OF AN EVEN SPHERE

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ABSTRACT. We determine an algebraic condition necessary and sufficient for a group G to act freely on the nth Cartesian power of an even sphere, and characterize the abelian groups that satisfy this condition.

1. **Introduction**. Let X_n be the Cartesian product of n copies of S^{2k}, where k is any positive integer. Then X_n has Euler characteristic 2^n, so any group acting freely on X_n must have order 2^l, $l \leq n$. We consider the problem of which of these 2-groups can act freely on X_n, concentrating on abelian 2-groups. (This paper is 'orthogonal' to the ones by Carlsson [1] and Yogita [2], since they consider only actions trivial on integral homology. In the situation considered here, all free actions are nontrivial on homology.)

In §2 we show that deciding whether a given 2-group can act freely on X_n reduces to determining if an appropriate representation of the group on the cohomology algebra of X_n exists. Let S_n be the group of $n \times n$ signed permutation matrices, i.e. matrices with exactly one nonzero entry in each row and column and all of whose nonzero entries are ± 1. There is a canonical homomorphism $\psi: S_n \rightarrow \Sigma_n$, where Σ_n is the symmetric group on n letters. For $u \in S_n$, let $\sigma_1\sigma_2 \ldots \sigma_m$ be the decomposition of $\psi(u)$ into disjoint cycles. Thinking of u as a linear map $\boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$, let K_i be the subspace of $\boldsymbol{R}^n$ corresponding to σ_i, and define ϵ_i by $\det(u|K_i) = \epsilon_i \operatorname{sgn} \sigma_i$ (Clearly $\epsilon_i = \pm 1$). Set

$$\lambda(u) = \prod_{i=1}^{m} (1 + \epsilon_i).$$

Then we can characterize the 2-groups that act freely on X_n as follows.

THEOREM 1. *A 2-group G acts freely on X_n if and only if G admits a representation $\rho: G \rightarrow S_n$ such that, for any $g \in G$ with $g \neq 1$, $\lambda(\rho(g)) = 0$.*

REMARK. Note that $\lambda(\mathrm{id}) = 2^n$, so any representation of the type specified in the theorem is faithful.

In §3 we construct free actions of cyclic groups on spaces X_n. This gives a free action of G on some X_n for any finite abelian 2-group G. We also show that such a group

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cannot act freely on X_n for any n smaller than the 'obvious' value, and thus obtain the following result.

THEOREM 2. *Let G be an abelian 2-group, so that*

$$G \cong \mathbf{Z}_2^{n_1} \oplus \mathbf{Z}_4^{n_2} \oplus \cdots \oplus \mathbf{Z}_{2^l}^{n_l}$$

Then G acts freely on X_n if and only if

$$\sum_{i=1}^{l} n_i 2^{i-1} \leq n.$$

2. **Free actions and Lefschetz numbers**. The cohomology ring $H^*(X_n;\mathbf{Z})$ is the commutative algebra generated by n $2k$-dimensional elements $x_1, x_2, \ldots, x_n$ with relations $x_i^2 = 0$ for $1 \leq i \leq n$. Thus, for any self-map $f: X_n \to X_n$ the endomorphism $f^*: H^*(X_n; \mathbf{Z}) \to H^*(X_n; \mathbf{Z})$ is determined by the $n \times n$ matrix $\rho(f) = (a_{ij})$, where

$$f^*(x_i) = \sum_{j=1}^{n} a_{ij} x_j, \qquad 1 \leq i \leq n.$$

Then

$$0 = f^*(x_i^2) = 2 \sum_{j<k} a_{ij} a_{ik} x_j x_k,$$

so $\rho(f)$ has at most one nonzero entry in each row. Thus, there is a function $\sigma: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ so that $f^*(x_i) = a_{i\sigma(i)} x_{\sigma(i)}$ for $1 \leq i \leq n$. Since $H^{2kn}(X_n;\mathbf{Z})$ is generated by $x_1 x_2 \cdots x_n$, we have $\deg f = 0$ if σ is not a permutation and

$$\deg f = a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

if it is. In particular, if f is invertible, $\deg f = \pm 1$ and $\rho(f) \in S_n$.

Now suppose a group G acts on X_n. By the preceding paragraph, the action gives rise to a representation $\rho: G \to S_n$. If in addition the action is free, we have

$$L(g) = \sum_{i=0}^{2kn} (-1)^i \operatorname{Tr}(g_i^*: H^i(X_n;\mathbf{Z}) \to H^i(X_n;\mathbf{Z})) = 0$$

for every nonidentity element $g \in G$, by the Lefschetz fixed point theorem. To determine $L(g)$ directly from the matrix $\rho(g)$, we first assume without loss of generality that the decomposition of $\psi\rho(g)$ contains the cycle $(1\,2 \cdots l)$. Then g^* cyclically permutes the elements $x_1, x_2, \ldots, x_l$ in $H^2(X_n;\mathbf{Z})$, so g^* sends no monomial in the subalgebra of $H^*(X_n;\mathbf{Z})$ generated by $x_1, \ldots, x_l$ to a multiple of itself except $x_1 x_2 \cdots x_l$. In fact g^* sends this monomial to $a_{12} a_{23} \cdots a_{l1}$ times itself, and this number is $(-1)^{l-1} \det(\rho(g)|K)$, where K is the submodule of $H^2(X_n;\mathbf{Z})$ generated by $x_1, \ldots, x_l$. Thus, the trace of g^* on the subalgebra of $H^*(X_n;\mathbf{Z})$ generated by $x_1, \ldots, x_l$ is

$$1 + (-1)^{l-1} \det(\rho(g)|K) = 1 + \operatorname{sgn}(1\,2 \cdots l) \det(\rho(g)|K).$$

Now let $\sigma_1 \sigma_2 \cdots \sigma_m$ be the decomposition of $\psi\rho(g)$ into disjoint cycles, K_i be the submodule of $H^2(X_n; \boldsymbol{Z})$ generated by the x_j permuted by σ_i, and

$$\epsilon_i = \text{sign } \sigma_i \det (\rho(g)|K_i).$$

Then by the preceding analysis and the multiplicative property of trace on tensor products,

$$L(g) = \prod_{i=1}^{m} (1 + \epsilon_i) = \lambda(\rho(g)).$$

We have evidently proved the forward implication of Theorem 1.

REMARK. Henceforth we shall call those cycles σ_i with $\epsilon_i = -1$ essential. We have just proved that for any $g \neq 1$ in G, the matrix $\rho(g)$ has an essential cycle.

Now suppose $\rho: G \to S_n$ is a representation of a group G such that $\rho(g)$ has an essential cycle for all nonidentity $g \in G$. We define an action of G on X_n as follows. Represent an element of X_n as an n-tuple $(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$ of unit vectors $\boldsymbol{v}_i \in \boldsymbol{R}^{2k+1}$. Think of the n-tuple as a column vector and let the matrices $\rho(g)$ act on it. That is, put

$$g \cdot (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n) = (a_{1\sigma(1)}\boldsymbol{v}_{\sigma(1)}, a_{2\sigma(2)}\boldsymbol{v}_{\sigma(2)}, \ldots, a_{n\sigma(n)}\boldsymbol{v}_{\sigma(n)}),$$

where $\rho(g) = (a_{ij})$ and $\sigma = \psi\rho(g)$. To see that this action is free, suppose $g \neq 1$ fixes $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$. Now $\rho(g)$ has an essential cycle: without loss of generality we can assume the cycle is $(1\,2 \cdots l)$. Then we must have

$$\boldsymbol{v}_1 = a_{12}\boldsymbol{v}_2 = a_{12}a_{23}\boldsymbol{v}_3 = \cdots = a_{12}a_{23} \cdots a_{l1}\boldsymbol{v}_1 = -\boldsymbol{v}_1,$$

a contradiction. This completes the proof of Theorem 1.

3. **Abelian groups**. Let $\rho: G \to S_n$ be a representation of an abelian group G. We think of elements of S_n as acting linearly on the free $\boldsymbol{Z}$-module generated by $x_1, x_2, \ldots, x_n$. Each $g \in G$ gives rise to an element $\psi\rho(g) \in \Sigma_n$, so we can think of G as acting on $\{1, 2, \ldots, n\}$. We say that $g \in G$ fixes a G-orbit Ω if $\rho(g)x_i = x_i$ for all $i \in \Omega$, and that g negates Ω if $\rho(g)x_i = -x_i$ for all $i \in \Omega$. We have the following result.

LEMMA 1. *Let g be a member of G, Ω an orbit under the G-action on* $\{1, 2, \ldots, n\}$. *If $\psi\rho(g)$ fixes some $i \in \Omega$, then g either fixes or negates Ω.*

PROOF. Suppose $\rho(g)x_i = a_{ii}x_i$, where $i \in \Omega$, and let j be another member of Ω. Then there is some element h of G with $\rho(h)x_i = b_{ij}x_j$. Hence $\rho(hg)x_i = b_{ij}a_{ii}x_j$. But G is abelian, so $\rho(hg)x_i = \rho(gh)x_i = b_{ij}\rho(g)x_j$. Thus $\rho(g)x_j = a_{ii}x_j$, and the conclusion follows.

The next result gives a criterion for $\rho(g)$, $g \in G$, to have an essential cycle in a given orbit.

LEMMA 2. *For $g \in G$, an orbit Ω contains an essential cycle of $p(g)$ if and only if some power of g negates Ω. In this case* card $\Omega \geq p$, *where g^p is the lowest power of g that negates Ω.*

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PROOF. If $i \in \Omega$ is in an essential cycle of g, say of length l, then

$$\rho(g^l)x_i = -x_i.$$

Then g^l negates Ω, by Lemma 1. Conversely, suppose g^p negates Ω, and assume p minimal. Then no power of $\psi\rho(g)$ lower than the pth can fix anything in Ω (Lemma 1 again), so $\psi\rho(g)|\Omega$ must consist of cycles of length p, each an essential cycle of $\rho(g)$.

For any cyclic 2-group $\mathbf{Z}_{2^i}$, we can define an embedding in $S_{2^{i-1}}$ by sending a generator to the element u_i given by

$$u_i(x_j) = x_{j+1}, \qquad 1 \leq j \leq 2^{i-1} - 1, \qquad u_i(x_{2^{i-1}}) = -x_1.$$

Then evidently u_i has order 2^i and $\lambda(u_i) = 0$. In fact, $\lambda(u_i^r) = 0$ for all powers $r < 2^i$. This is immediate for $i = 1$, so assume $i \geq 2$. Then u_i has determinant 1: for odd powers r, $\psi(u_i^r)$ is a single cycle and $1 = \det(u_i^r) = \epsilon \operatorname{sgn} \psi(u_i^r) = (-1)^r\epsilon$, so $\epsilon = -1$. For even powers, note that $w = u_i^{2^{i-1}}$ negates the single orbit, and some power of u_i^r is w, for any even $r < 2^i$. Thus every power u_i^r, $r < 2^i$, has an essential cycle. By Theorem 1, this means that $\mathbf{Z}_{2^i}$, acts freely on $X_{2^{i-1}}$. Then any abelian 2-group

$$G \cong \mathbf{Z}_2^{n_1} \oplus \mathbf{Z}_4^{n_2} \oplus \cdots \oplus \mathbf{Z}_{2^l}^{n_l}$$

acts freely on

$$\prod_{j=1}^{n_1} X_1 \times \prod_{j=1}^{n_2} X_2 \times \cdots \times \prod_{j=1}^{n_l} X_{2^{l-1}} = X_{n_1 + 2n_2 + \cdots + 2^{l-1}n_l}.$$

Now suppose an abelian group G acts on X_n, so there is a representation $\rho: G \to S_n$ such that $\rho(g)$ has an essential cycle for every $g \neq 1$. Let G_0 be the set of elements of order ≤ 2 in G. Then G_0 is a vector space over $\mathbf{Z}_2$, and Lemma 2 implies that every nonidentity element of G_0 negates some orbit. (Though we shall think of G_0 as a vector space, we shall continue to use multiplicative notation.)

LEMMA 3. *Let $h_1, h_2, \ldots, h_s$ be a basis for a subspace $V \subset G_0$. Then there are orbits $\Omega_1, \Omega_2, \ldots, \Omega_s$ in $\{1, 2, \ldots, n\}$ and a basis $\{g_1, g_2, \ldots, g_s\}$ for V such that, for each i,*

1. *uh_i negates Ω_i, where $u \in$ span $\{h_1, \ldots, h_{i-1}\}$, and*
2. *g_i negates Ω_i, and no product of the g_j, $j \neq i$, does so.*

PROOF. We proceed by induction on s, the case $s = 1$ being immediate. By the induction hypothesis there exist orbits Ω_i, $1 \leq i \leq s - 1$, and a basis $\{k_1, \ldots, k_{s-1}\}$ for span $\{h_1, \ldots, h_{s-1}\}$ such that (1) and (2) (with g replaced by k) hold. Let N be the set of k_i such that something in span $\{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{s-1}, h_s\}$ negates Ω_i, and let u be the product of the elements of N. Now suppose something in span $\{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots k_{s-1}, uh_s\}$ negates Ω_ι, for some $1 \leq i \leq s - 1$. By the induction hypothesis, it must have form wuh_s, where $w \in W_i =$ span $\{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{s-1}\}$. If $k_i \in N$, then $u = k_iv$ for $v \in W_i$ and something of form yh_s, $y \in W_i$, negates Ω_i: but then wvh_s fixes Ω_i and yh_s negates it, so $wvy \in W_i$ negates Ω_i, contradicting the induction hypothesis. But if $k_i \notin N$, then $u \in W_i$ and having wuh_s negate Ω_i contradicts the definition of N. Let $g_s = uh_s$.

Now choose an orbit Ω_s that g_s negates (This evidently satisfies (1) for $i = s$). Let $W = \text{span}\,\{k_1, \ldots, k_{s-1}\}$. Then there is a homomorphism $f: W \to S_m$, where $m = \text{card}\,\Omega_s$, defined by $f(w) = \rho(w)|\hat{\Omega}_s$ (Here $\hat{\Omega}_s$ is the $\mathbf{Z}$-module generated by $\{x_i | i \in \Omega_s\}$). We identify $\bar{W} = W/\ker f$ with the image of f in the usual way, and denote the class of $w \in W$ in $\bar{W}$ by $\{w\}$. Now if $\mu = \rho(g_s)|\hat{\Omega}_s$ is not in $\bar{W}$ we can set $g_i = k_i$, so assume otherwise. Choose a basis B for $\bar{W}$ that includes μ. Now define g_i, $1 \leq i \leq s - 1$, to be $k_i g_s$ if μ occurs in the representation of $\{k_i\}$ in terms of B, and k_i otherwise. Then $\{g_1, g_2, \ldots, g_s\}$ is the required basis for V.

Now we can finish the proof of Theorem 2. Since G is a 2-group,

$$G \cong \mathbf{Z}_2^{n_1} \oplus \mathbf{Z}_4^{n_2} \oplus \cdots \oplus \mathbf{Z}_{2^l}^{n_l}$$

for some $n_1, \ldots, n_l$. Choose generators $r_1, r_2, \ldots, r_k$ for the summands, arranged so that r_i has order greater than or equal to r_{i+1}. If we raise each generator r_i to half its order, we obtain a basis for G_0. Now apply Lemma 3 with $V = G_0$: we obtain distinct orbits $\Omega_1, \ldots, \Omega_k$, and (1) of the lemma implies that, for each i, there is an element w_i that when raised to half the order of r_i negates Ω_i. Hence, by Lemma 2, card Ω_i is at least half the order of r_i. Now we have n_1 generators of order 2, n_2 of order 4, etc., so

$$n \geq \sum_{i=1}^{k} \text{card}\,\Omega_i \geq \sum_{i=1}^{l} 2^{i-1} n_i.$$

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