BULL. AUSTRAL. MATH. SOC. 52A40, 52A10
VOL. 21 (1980), 7-12.

FURTHER INEQUALITIES FOR CONVEX SETS WITH LATTICE POINT CONSTRAINTS IN THE PLANE

P.R. Scott

Let K be a bounded closed convex set in the plane containing no points of the integral lattice in its interior and having width w, area A, perimeter p and circumradius R. The following best possible inequalities are established:

$$(w-1)A \le \tfrac{1}{2}w^2 ,$$

$$(w-1)p \le 3w ,$$

$$(w-1)R \le w/\sqrt{3} .$$

1. Introduction

Let K be a bounded, closed, convex set in the euclidean plane, containing no points of the integral lattice in its interior. We denote the diameter, width, perimeter, area, inradius and circumradius of K by d, w, p, A, r and R respectively.

It is known [3] that the width satisfies

(1) $$w \le \tfrac{1}{2}(2+\sqrt{3})$$

with equality when and only when K is an equilateral triangle E, of side length $(2+\sqrt{3})/\sqrt{3}$. It has also been recently established [5] that

$$(w-1)(d-1) \le 1 ,$$

or equivalently,

(2) $$(w-1)d \le w$$

Received 19 June 1979.

https://doi.org/10.1017/S0004972700011242 Published online by Cambridge University Press

with equality when and only when K is a triangle of diameter d and width $w = d/(d-1)$ (Figure 1).

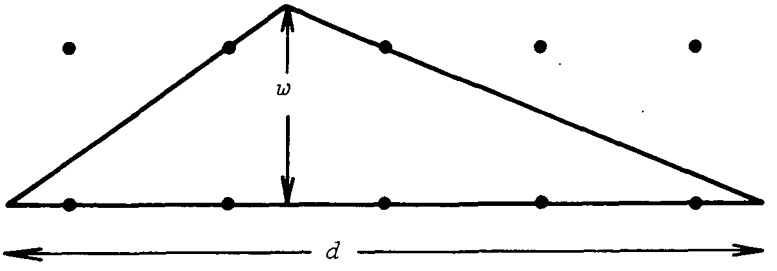

Figure 1

We shall prove several analogous results.

THEOREM 1. $(w-1)A \le \frac{1}{2}w^2$, *with equality when and only when* K *is a triangle of width* w *and diameter* $w/(w-1)$ (Figure 1).

THEOREM 2. $(w-1)p \le 3w$ *with equality when and only when* $K = E$.

THEOREM 3. $(w-1)R \le w/\sqrt{3}$ *with equality when and only when* $K = E$.

According to Blaschke's Theorem [*1*], every bounded convex figure of width w contains a circle of radius $w/3$. It follows that $w \le 3r$; equality holds here when and only when the figure is an equilateral triangle. Using this result and (1), we obtain the following corollaries.

COROLLARIES.

$$(w-1)A \le 3wr/2 \le 9r^2/2 ;$$

$$(w-1)A \le (7+4\sqrt{3})/8 \ (\approx 1.74) ;$$

$$(w-1)p \le 9r ;$$

$$(w-1)p \le (6+3\sqrt{3})/2 ;$$

$$(w-1)R \le \sqrt{3}.r ;$$

$$(w-1)R \le (3+2\sqrt{3})/6 .$$

In each case we have equality when and only when $K = E$.

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2. Proof of Theorems 2 and 3

To establish Theorem 3, we recall a theorem of Jung [2] which states that any set of diameter d is contained in a circular disc of radius $R \le d/\sqrt{3}$. Theorem 3 now follows immediately from (2), since

$$(w-1)R \le (w-1)d/\sqrt{3} \le w/\sqrt{3} .$$

For equality in Jung's result we require K to be an equilateral triangle; for equality in (2), K must be as in Figure 1. Hence equality occurs in Theorem 3 when and only when $K = E$.

We now show that Theorem 2 can be deduced from Theorem 1. If K is any convex polygon, we can partition K into triangles by joining each vertex to the (an) in-centre of K. Summing the areas of these triangles easily gives for K the inequality

$$A \ge \tfrac{1}{2}pr .$$

Since any convex set K in the plane can be approximated as closely as we please by a convex polygon, we conclude that this inequality is valid for any convex set K in the plane.

Assuming the validity of Theorem 1, we now have

$$(w-1)p \le 2(w-1)A/r \le w^2/r \le 3w$$

since $w \le 3r$ by Blaschke's Theorem. Hence $(w-1)p \le 3w$ as required. It is easily seen that equality occurs here when and only when $K = E$.

We notice that the inequality of Theorem 2 follows easily from (2) in the special case when K is a triangle, for then $p \le 3d$, and

$$(w-1)p \le (w-1)3d \le 3w .$$

3. Some preliminary results

We observe that the statement of Theorem 1 can be written as

$$\frac{1}{2A} - \frac{w-1}{w^2} \ge 0 .$$

We shall assume therefore that K is a set for which the left hand side of this inequality is as small as possible. Since $(w-1)/w^2$ is an increasing function of w, we choose K with A, w as large as possible.

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Let $\mathcal{D}$ be a largest circular disc contained in K, having radius r. It is known [4] that for any convex set K,

$$(w-2r)A \le w^2 r/\sqrt{3} .$$

Hence if $r \le \frac{1}{2}$,

$$(w-1)A \le (w-2r)A \le w^2 r/\sqrt{3} \le w^2/(2\sqrt{3}) < w^2/2 .$$

We may therefore assume that K contains a disc $\mathcal{D}$ of radius $r > \frac{1}{2}$.

By suitably translating K we may assume that the centre of $\mathcal{D}$ lies in the interior of the square with vertices $O(0, 0)$, $B(1, 0)$, $C(1, 1)$, $D(0, 1)$. Since K is convex, K is bounded by lines through the points O, B, C, D. If these lines form a convex quadrilateral Q, then Q contains no lattice points in its interior, and we may assume that K is Q. On the other hand, these lines may determine a triangular region T, as for example a degenerate quadrilateral, or when a line through D separates K from C. Such a region T may contain interior lattice points; nevertheless, it will be sufficient for us to establish the theorem for T.

4. Proof of Theorem 1

First let K be the convex quadrilateral Q. The following result is established in [3].

LEMMA. *The quadrilateral Q can be transformed into a kite Q' having the following properties:*

(a) $w(Q') \ge w(Q)$;

(b) *Q' contains no lattice point in its interior;*

(c) *Q' has its axis along the line $x = \frac{1}{2}$,*

(d) *the sides of Q' pass through O, B, C, D respectively;*

(e) $A(Q') \ge A(Q)$.

Property *(e)* is not stated explicitly in [3], but follows from the fact that Q' is obtained from Q by Steiner symmetrization and enlargement with scale factor $s \ge 1$.

Clearly we may take K to be the kite $Q' = XYZW$ (Figure 2). Let

$XZ = t$, $YW = u$. Then $2A = tu$. Also, computing the areas of the component parts of Q' gives

$$2A = 2 + (t-1).1 + (u-1).1$$
$$= t + u .$$

Hence

$$tu = t + u .$$

Suppose that $0 < t \le u$; then $t \le 2$. Now

$$A = \tfrac{1}{2}tu = \tfrac{1}{2}t^2/(t-1) < \tfrac{1}{2}w^2/(w-1) ,$$

since $w < t$, and $t^2/(t-1)$ is a decreasing function of t for $0 < t \le 2$. A similar argument holds for $0 < u \le t$.

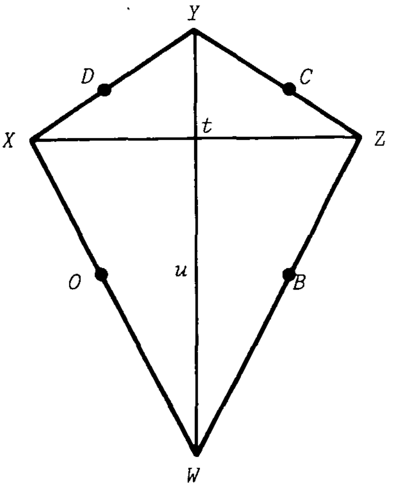

Figure 2

Hence if K is the quadrilateral Q ,

$$A(w-1) < \tfrac{1}{2}w^2 .$$

Now let K be the triangle T . In this case

$$A = \tfrac{1}{2}dw \le \tfrac{1}{2}w^2/(w-1)$$

using (2).

Thus for any K ,

$$(w-1)A \le \tfrac{1}{2}w^2 .$$

Equality occurs here when and only when K is a triangle as in Figure 1.

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Department of Pure Mathematics,
University of Adelaide,
Adelaide,
South Australia 5001,
Australia.