S. Friedberg
Nagoya Math. J.
Vol. 92 (1983), 1–20

ON THE IMAGINARY QUADRATIC DOI-NAGANUMA LIFTING OF MODULAR FORMS OF ARBITRARY LEVEL

SOLOMON FRIEDBERG*

In this paper, we use the theta function method [10] to give explicit Doi-Naganuma type maps associated to an imaginary quadratic field K, lifting cusp forms on any congruence subgroup of $SL(2, \boldsymbol{Z})$ to forms on $SL(2, \boldsymbol{C})$ automorphic with respect to an appropriate arithmetic discrete subgroup. The case of class number one, and form modular with respect to group $\Gamma_0(D)$ and character $\chi_0 = (-D/*)$, where $-D$ is the discriminant of K, has been treated by Asai [1]. In order to complete his discussion, we must first introduce a more general theta function associated to an indefinite quadratic form (here of type (3,1)), which we regard as a specialization of a symplectic theta function (see also [4]). Needed spherical harmonics appear naturally in such a context. While the definitions of the liftings are then straightforward, explicit computations are difficult. We develop the Fourier series [Theorems 3.1, 3.2, and 3.9] by expanding on an idea of Asai [1], who noticed that the theta function splits into simpler theta functions, but only on $A = \left\{a^{-1/2}\begin{pmatrix} a & \\ & 1\end{pmatrix} \middle| 0 < a \in \boldsymbol{R}\right\}$. (Note that if we think of $SL(2, \boldsymbol{C})/SU(2, \boldsymbol{C})$ as the quaternionic upper half space $\{z + a\boldsymbol{k} \mid z \in \boldsymbol{C},\ 0 < a \in \boldsymbol{R}\}$, then A is the 'imaginary' $\boldsymbol{k}$ axis). Namely, we develop an alternate expression for our theta function on *all* of $SL(2, \boldsymbol{C})$ (i.e. on the entire quaternionic upper half space), which gives this splitting on A [Proposition 3.7]. This alternate expression makes the direct computation of the Fourier series of the lift feasible, and even in Asai's situation appears to be a genuine simplification.

For k, N positive integers, and χ a character mod N, let $S_k(\Gamma(N))$ (resp. $S_k(\Gamma_0(N), \chi)$) denote the space of cusp forms of weight k on $\Gamma(N)$ (resp. on $\Gamma_0(N)$ with character χ). Then our liftings associate to each form $f \in S_k(\Gamma(N))$ a $2k-1$ vector of forms F on $SL(2, \boldsymbol{C})$, automorphic

Received July 22, 1981.
* Supported by a University of Chicago McCormick Fellowship.

https://doi.org/10.1017/S0027763000020535 Published online by Cambridge University Press

with respect to a certain subgroup conjugate to a congruence subgroup of $\Lambda = SL(2, \mathcal{O}_K)$, and as a vector transforming by the appropriate symmetric representation under $SU(2, \boldsymbol{C})$. As in Asai's paper, these forms are eigenfunctions of the Casimir operator, with eigenvalue $((k-1)^2-1)/2$. Further, at each cusp the constant term of our lifted form F vanishes if and only if f satisfies a certain orthogonality condition (orthogonality to a certain theta function associated to the field with respect to the Petersson inner product). In particular, a new eigenform has noncuspidal lifts if and only if it is constructed from the explicit list of theta series 'coming from the field' [Corollary 3.3]. A linear combination of our basic lifts gives a form with character $\chi\chi_0 \circ \mathbf{N}$, where $\mathbf{N} = \text{Norm}\,(\boldsymbol{C}/\boldsymbol{R})$, and for this combination we get a result analogous to those of Doi and Naganuma [3, 9]: a sum of the $h(K) =$ class number of K lifted Dirichlet series corresponding to $f = \sum a(n)e^{2\pi i n z}$, a new eigenform in *any* $S_k(\Gamma_0(N), \chi)$, is $\sum a(n)n^{-s} \cdot \sum a(n)\chi_0(n)n^{-s}$.

The theta function method has been used in the real quadratic case by Kudla [8] (although he did not calculate the lifted Fourier series there), and for the case of signature $(2, n-2)$ by Rallis and Schiffman [13] and Oda [11]. Other methods used to study the real quadratic case include those of Doi and Naganuma [3, 9] and Zagier [18]. The general question of when such a correspondence should exist has been considered by Howe [6]. Also, the results of Jacquet [7] and Saito [14] and Shintani can be used to speak of liftings. Unfortunately, these other points of view are difficult to use for explicit computations. We hope that our work will give a prototype of the explicit computation of liftings, and that the ideas involved will prove useful in these other contexts. For we feel that the issue of the explicit computation of liftings is of genuine arithmetic interest.

I would like to warmly thank Harold Stark for several helpful conversations.

Notation. $e(x) = e^{\pi i x}$.

§ 1. Theta functions and spherical harmonics

1.1. In this section we review some results of the author [4] which we will use to construct theta functions; the same method should prove useful for constructing theta functions for liftings in a variety of other contexts. For an alternative approach, see Shintani [16].

Let $\mathfrak{H}_n$ be the Siegel upper half space

$$\mathfrak{H}_n = \{Z \in M(n, \boldsymbol{C}) \,|\, {}^tZ = Z,\ \operatorname{Im}(Z) \text{ positive definite}\} .$$

We write $Z[v] = {}^tvZv$ for $v \in \boldsymbol{C}^n$ or $M(n, \boldsymbol{C})$, $Z \in M(n, \boldsymbol{C})$. Then one defines a symplectic theta function parametrized by $Z \in \mathfrak{H}_n$, $u, v, w \in \boldsymbol{C}^n$, and f a nonnegative integer, by:

$$\vartheta(Z, u, v, w, f) = \sum_{m \in \boldsymbol{Z}^n} ({}^twZ(m - v))^f e(Z[m - v] + 2{}^tmu - {}^tuv) .$$

Then $\vartheta(Z, u, v, w, f)$ has a transformation law with respect to a suitable subgroup of the integral symplectic group. Namely, call a symmetric matrix in $M(n, \boldsymbol{Z})$ *even* if it has even diagonal entries. Let

$$Sp(n, \boldsymbol{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(2n, \boldsymbol{R}) \middle| \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}[M] = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \right\} ,$$

where I is the $n \times n$ identity, and

$$\Gamma = \{M \in Sp(n, \boldsymbol{Z}) \,|\, {}^tCA,\ {}^tBD \text{ are even}\} .$$

Then $Sp(n, \boldsymbol{R})$ acts on $\mathfrak{H}_n$ in the usual way. We have

THEOREM 1.1. *For*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma , \qquad Z \in \mathfrak{H}_n ,$$

such that $Z(CZ + D)^{-1}CZ[w] = 0$ *if* $f > 1$,

$$\begin{aligned} &\vartheta(MZ, Au + Bv, Cu + Dv, {}^t(AZ + B)^{-1}Zw, f) \\ &\quad = x(M) \det(CZ + D)^{1/2} \vartheta(Z, u, v, w, f) . \end{aligned}$$

Here $x(M)$ *is an eighth root of unity.*

When a choice of the square root is made, the value of $x(M)$ can be explicitly determined. See Stark [17] or Friedberg [4]. In the next section, we shall give it precisely for the case of use here.

1.2. Let $\mathscr{L}$ be a discrete rank n $\boldsymbol{Z}$-module in $\boldsymbol{C}^n$, $Q \in M(n, \boldsymbol{C})$ give an integral quadratic form of type (p, q) with respect to $\mathscr{V} = \mathscr{L} \otimes \boldsymbol{R}$ (so if L is a matrix whose columns are basis vectors of $\mathscr{L}$, then ${}^tLQL$ is equivalent to $\begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$ over $GL(n, \boldsymbol{R})$). Say $p + q = n$. Associated to Q is a family of *majorants*: $R \in M(n, \boldsymbol{C})$, ${}^tR = R$, $RQ^{-1}R = Q$, and $R[\ell] > 0$ for all $\ell \in \mathscr{L} - \{0\}$. Write ${}^t xQy = (x, y)$. Then for any such R, and $z = x + iy \in \mathfrak{H}_1$, $u, v, w \in \boldsymbol{C}^n$, $f \in \boldsymbol{Z} \geq 0$, such that $Qw = Rw$ (similar results hold when $Qw = -Rw$), and $Q[w] = 0$ if $f > 1$, we define

$$\theta(z, u, v, w, f) = y^{q/2} \sum_{\ell \in \mathscr{L}} (w, \ell - v)^f e((xQ + iyR)[\ell - v] + 2(\ell, u) - (u, v)) .$$

Then we have

THEOREM 1.2. *Say* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Q})$ *is such that* $\begin{pmatrix} aI & b{}^tLQL \\ c({}^tLQL)^{-1} & dI \end{pmatrix} \in \Gamma$. *Then*

$$\theta(\sigma z,\ au + bv,\ cu + dv,\ w, f) = \hat{\chi}(\sigma)(cz + d)^{(p-q)/2+f}\theta(z, u, v, w, f) .$$

Here by $(cz + d)^{1/2}$ *we mean the principal value square root* ($-\pi/2 < \arg(z^{1/2}) \leq \pi/2$). *For* d *an odd prime,*

$$\hat{\chi}(\sigma) = \varepsilon_d^{-n}\left(\frac{2}{d}\right)^n\left(\frac{c^n (\det {}^tLQL)^{-1}}{d}\right),$$

where $\varepsilon_d = \begin{cases} 1 & d \equiv 1 \mod 4 \\ i & d \equiv 3 \mod 4 \end{cases}$, *and* $(-)$ *are Legendre symbols.*

Remarks. One can then use Dirichlet's theorem to determine $\hat{\chi}$ for any odd d; the case of odd c, but even d, is harder (cf. [4]). Also, the Γ condition can be rephrased in terms of $\mathscr{L}$ and its Q-dual $\mathscr{L}^*$ in an obvious way.

Proof. Note

$$y^{-q/2}\theta(z, u, v, w, f) = z^{-f}\vartheta(x{}^tLQL + iy{}^tLRL, {}^tLQu, L^{-1}v, L^{-1}w, f) .$$

Set $x{}^tLQL + iy{}^tLRL = Z$, ${}^tLQu = u'$, $L^{-1}v = v'$. Then the actions of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on z and $\begin{pmatrix} u \\ v \end{pmatrix}$ are the same as the actions of $\begin{pmatrix} aI & b{}^tLQL \\ c({}^tLQL)^{-1} & dI \end{pmatrix}$ on Z and $\begin{pmatrix} u' \\ v' \end{pmatrix}$ (in an obvious commutative diagram sense). Using Theorem 1.1 completes the proof (for details, see [4]).

The important thing about Theorem 1.2 for our purposes is that the transformation formula is independent of the choice of R. However, the possible majorants fill out the symmetric space attached to $SO(Q)$. Thus our theta function is really a function of z and R as well as the other variables, and by integrating with respect to z we lift a form to a function of R. First, though, we need to make an appropriate choice of quadratic form Q, and study precisely the associated spherical harmonics (specified by w above).

1.3. For the remainder of this paper we specialize the notation of Section 1.2 as follows. We use $M(2, C) \cong C^4$, and consider the real vector subspace $\mathscr{V} = \{X \in M(2, C) \mid {}^tX = \bar{X}\}$. Then $G = SL(2, C)$ acts on $\mathscr{V}$ by $X^g = {}^t\bar{g}Xg$. We give quadratic form Q by $Q[X] = -2\det X$, with majorant R: $R[X] = \operatorname{tr}(X^2)$. Then R^g is also a majorant for any $g \in G$, where $R^g[X] = R[X^g]$. Note Q is of type (3, 1) and $R^\kappa = R$ for all $\kappa \in SU(2, C) = K$. Also, $(X, Y) = (X^g, Y^g)$ for all $g \in G$. This guarantees that once one finds w satisfying the conditions of Section 1.2 with respect to R, $w^{g^{-1}}$, which satisfies them with respect to R^g, occurs in the theta functions as $(w^{g^{-1}}, X) = (w, X^g)$.

Asai [1] has described a natural basis for the $2k - 1$ dimensional space of spherical harmonic polynomials $p(X) = (w, X)^{k-1}$ where w is as in Section 1.2 for our choice of Q and R. Namely, put

$$A_1 = {}^t(a\ \ b)(a\ \ b)\begin{pmatrix} & -1 \\ 1 & \end{pmatrix},$$

and define the homogeneous polynomial $\eta_{k,\alpha}(X)$ of degree $k - 1$ as the coefficient of $a^{k-1-\alpha}b^{k-1+\alpha}$ in $(X, A_1)^{k-1}$ for each integer α such that $|\alpha| \leq k - 1$ (this would be $\eta_{k-1,\alpha}$ in Asai's notation). Further, put $\eta_{(k)}(X) = (\eta_{k,1-k}(X), \cdots, \eta_{k,k-1}(X))$. For example, $\eta_{(2)}\begin{pmatrix} m & r \\ \bar{r} & p \end{pmatrix} = (-\bar{r}, m - p, r)$. We have

LEMMA 1.3 [Asai]. *The polynomials $\eta_{k,\alpha}(X)$, $|\alpha| \leq k - 1$, are a basis for the space of spherical harmonic polynomials.*

Also, let $(a\ \ b)_n = {}^t(a^n, a^{n-1}b, \cdots, ab^{n-1}, b^n)$, and define the n-fold symmetric representation $\rho_n(g)$, for $g \in G$, by

$$((a\ \ b)^tg)_n = \rho_{n+2}(g)(a\ \ b)_n\,.$$

Note that we have skewed the indexing by 2 here. Then as in Asai [1], page 151, we have

LEMMA 1.4 [Asai]. $\eta_{(k)}(X^\kappa) = \eta_{(k)}(X)\rho_{2k}(\kappa)$ *for all* $\kappa \in K$.

We can also show

LEMMA 1.5. *For $y > 0$, $X = \begin{pmatrix} m & r \\ \bar{r} & p \end{pmatrix} \in \mathscr{V}$, we have*

$$\eta_{k,\alpha}(X) = (2\pi y)^{(\alpha+1-k)/2}(k-1)! \\ \times \sum_{\beta,\gamma} (\alpha + \beta)!^{-1}\gamma!^{-1}2^{-\beta}r^\alpha L_\beta^{(\alpha)}(4\pi r\bar{r}y)H_\gamma((2\pi y)^{1/2}(m - p))$$

where $L_\beta^{(\alpha)}$ and H_γ are Laguerre and Hermite polynomials, respectively, and *the sum is over* β *and* γ *nonnegative integers such that* $2\beta+\gamma=k-\alpha-1$ *and* $\alpha+\beta\geq 0$.

Indeed, this follows at once by combining the homogeneity of $\eta_{k,\alpha}$ of degree $k-1$ with Lemma 3 of Asai [1]. This formula will prove useful later on.

Finally, let us define the theta function to be used in our liftings. Let $\mathscr{L}$ be a lattice in $\mathscr{V}$, $V\in\mathscr{L}^*$, $z=x+iy\in\mathfrak{H}_1$, and define a theta function of the above type by

$$\theta_{k,\alpha}(z, V, g, \mathscr{L}) = y^{1/2}\sum_{X\in\mathscr{L}}\eta_{k,\alpha}((X-V)^g)e((xQ+iyR^g)[X-V])\,,$$

and denote the vector of theta functions by

$$\theta_{(k)}(z, V, g, \mathscr{L}) = (\theta_{k,1-k},\cdots,\theta_{k,k-1})$$

(for our purposes the variable u can be dispensed with). We often drop $\mathscr{L}$ from the notation, when there is no ambiguity.

Our principal interest is the lifting associated to lattice

$$\mathscr{L}_1=\left\{\begin{pmatrix} m & r\\ \bar{r} & p\end{pmatrix}\in\mathscr{V}\,|\,m\in MZ,\ p\in PZ,\ r\in\mathscr{I}\right\},$$

where $M,\ P\in Z-\{0\}$, and $\mathscr{I}$ is an ideal of $\mathscr{O}_K$. Slightly more generally, let $\mathscr{Q}_1=\mathscr{O}_K,\cdots,\mathscr{Q}_h$ be a set of wide ideal class representatives, with each $\mathscr{Q}_i$ of minimal norm in its class. Take $q_1=1,\cdots,q_h$ ideal numbers for these ideals in the sense of Hecke [5]; recall $\mathbf{N}(\mathscr{Q}_i)=\mathbf{N}(q_i)$. We set

$$\mathscr{I}_i=\mathscr{I}\mathscr{Q}_iq_i^{-1}\quad\text{and}\quad\mathscr{L}_i=\left\{\begin{pmatrix} m & r\\ \bar{r} & p\end{pmatrix}\in\mathscr{V}\,|\,m\in MZ,\ p\in PZ,\ r\in\mathscr{I}_i\right\}.$$

Observe that

$$\mathscr{L}_i^*=\left\{\begin{pmatrix} m & r\\ \bar{r} & p\end{pmatrix}\in\mathscr{V}\,|\,Pm\in Z,\ Mp\in Z,\ r\in\bar{\mathscr{I}}_i^{-1}\mathfrak{D}_K^{-1}\right\}$$

where $\mathfrak{D}_K$ denotes the different. The lifting associated to $\mathscr{L}_i$ and ideal $\mathscr{I}$ is of course closely related to that associated to $\mathscr{L}_1$ and ideal $\mathscr{I}\mathscr{Q}_i$. We can also use the variable V in the theta function to treat lattices of $\mathscr{V}$ having as a sublattice some $\mathscr{L}_i$, by summing over the various coset representatives for the quotient.

For a function $f(z)$ on $\mathfrak{H}_1$ and a matrix $\sigma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL^+(2,\boldsymbol{R})$, we define

https://doi.org/10.1017/S0027763000020535 Published online by Cambridge University Press

$$(f\,|\,[\sigma]_k)(z) = (\det\sigma)^{k/2}(cz+d)^{-k}f(\sigma z)$$

where $\sigma z = (az+b)(cz+d)^{-1}$ as usual. Also, write $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\iota} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and for lattice $\mathscr{L}$ as above, put $N_1 = \text{l.c.m.}\,(MP, \mathrm{N}(\mathscr{I})D)$, where as above $-D$ is the discriminant of K.

Proposition 1.6.

(1) *For all* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_1)$,

$$\theta_{k,\alpha}(z, dV, g)\,|\,[\sigma]_k = \chi_0(d)\theta_{k,\alpha}(z, V, g)\,.$$

In particular, for $\sigma \in \Gamma(N_1)$,

$$\theta_{k,\alpha}(z, V, g)\,|\,[\sigma]_k = \theta_{k,\alpha}(z, V, g)\,.$$

(2) *For all* $\kappa \in K$, $\gamma \in G(\mathscr{L}) = \{\gamma \in G\,|\,\mathscr{L}^{\gamma} = \mathscr{L}\}$,

$$\theta_{(k)}(z, V, \gamma g\kappa) = \theta_{(k)}(z, V^{\gamma}, g)\rho_{2k}(\kappa)\,.$$

(3) $\theta_{(k)}(z, V, g, \mathscr{L}) = (-1)^{k-1}\theta_{(k)}(z, \overline{V}, \overline{g}, \overline{\mathscr{L}})\rho_{2k}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$.

(4) $\theta_{(k)}(z, V, {}^t g^{-1}, \mathscr{L}) = (-1)^{k-1}\theta_{(k)}(z, V^{\iota}, \overline{g}, \mathscr{L}^{\iota})$.

(5) $\theta_{(k)}\left(z, V, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right) = (-1)^{k-1}\theta_{(k)}(z, V^{\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)}, g)\rho_{2k}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$.

Proof. The first part follows immediately from Theorem 1.2. To prove (2), combine Lemma 1.4 with the K invariance of R^g and the G invariance of (,). From the observation (of Asai) that $\eta_{k,\alpha}(\overline{X}) = (-1)^{\alpha}\eta_{k,-\alpha}(X)$ and the realness of Q and R, (3) follows. As for (4), note ${}^t g^{-1} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \overline{g} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, $V^{\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)} = \overline{V}^{\iota}$ for all $V \in \mathscr{V}$, and $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \in K$, and use (3). (5) is similar to (2), though $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \notin K$.

§ 2. The Doi-Naganuma lifting—Definition and basic results

2.1. Definition. For $f \in S_k(\Gamma(N))$, $|\alpha| \le k-1$, lattice $\mathscr{L}$ such that $N_1\,|\,N$, and $V \in \mathscr{L}^*$, the lift of f is given by

$$F_{\alpha}(g, V, \mathscr{L}) = \int_{\Gamma(N)\backslash\mathfrak{H}_1} \theta_{k,\alpha}(z, V, g, \mathscr{L})\overline{f(z)}y^{k-2}dxdy \tag{2.1.1}$$

and we write the associated vector lift as

$$A(f) = F(g, V, \mathscr{L}) = (F_{1-k}, \cdots, F_{k-1})\,.$$

This definition makes sense due to Proposition 1.6 (1) (independence of the choice of fundamental domain) and obvious estimates which give absolute convergence of the integral. Throughout this paper, F will denote the lift of a form $f \in S_k(\Gamma(N))$. As an immediate consequence of Proposition 1.6, we have

THEOREM 2.1.
(1) *For* $\kappa \in K$, $\gamma \in G(\mathscr{L})$, $F(\gamma g \kappa, V) = F(g, V^\gamma)\rho_{2k}(\kappa)$.
(2) $F(g, V, \mathscr{L}) = (-1)^{k-1} F(\bar{g}, \bar{V}, \bar{\mathscr{L}})\rho_{2k}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$.
(3) $F({}^t g^{-1}, V, \mathscr{L}) = (-1)^{k-1} F(\bar{g}, V^t, \mathscr{L}^t)$.
(4) $F\left(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, V\right) = (-1)^{k-1} F(g, V^{\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)})\rho_{2k}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$.

Note that

$$G(\mathscr{L}_i) \supseteq \left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G \,|\, \alpha, \delta \in \mathcal{O}_K,\ \beta \in \bar{q}_i P \mathscr{I}(\mathscr{I}, MP)^{-1},\ \gamma \in q_i M \bar{\mathscr{I}}(\bar{\mathscr{I}}, MP)^{-1}\right\}$$

(here $(a, b) = \text{g.c.d.}\,(a, b)$). By abuse of notation, we write this group as $\Lambda_0^0(q_i M \bar{\mathscr{I}}(\bar{\mathscr{I}}, MP)^{-1}, \bar{q}_i P \mathscr{I}(\mathscr{I}, MP)^{-1})$.

Thus in essence (2.1.1) lifts something of level N to something of level between $N\mathbf{N}(\mathscr{Q}_i)D^{-1}$ and $N\mathbf{N}(\mathscr{Q}_i)$. For $i \neq 1$, though $G(\mathscr{L}_i) \not\subseteq \Lambda$, $G(\mathscr{L}_i)$ has a subgroup conjugate to a congruence subgroup of Λ.

2.2. The extra variable V allows us to mix in characters. We give several examples.

(A) Let $V_1(v) = \begin{pmatrix} v/P & 0 \\ 0 & 0 \end{pmatrix}$; $v \in Z/MPZ$. This embeds Z/MPZ in $\mathscr{L}^*$. Then one sees $V_1(v)^{g'} \equiv V_1(\alpha\bar{\alpha}v) \bmod \mathscr{L}$ for

$$g' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Lambda^0(P\mathscr{I}_i) = \Lambda_0^0(1, P\mathscr{I}_i)\ .$$

Also, $(\alpha\bar{\alpha}, MP) = 1$ for $g' \in G(\mathscr{L}_i)$. Thus the set $S = \{v \in Z/MPZ \,|\, (v, MP) = r\}$ is fixed by $G(\mathscr{L}_i)$. Let ψ be a character on $(Z/(MPr^{-1})Z)^\times$, and set

$$\theta(z, g, \psi, r) = \sum_{v \in S} \psi(v/r)\theta_{(\kappa)}(z, V_1(v), g, \mathscr{L}_i)\ .$$

Then $\theta(z, g'g, \psi, r) = \psi(\delta\bar{\delta})\theta(z, g, \psi, r)$ for all $g' \in G(\mathscr{L}_i) \cap \Lambda^0(P\mathscr{I}_i)$, so the corresponding lift given by Petersson inner product with θ (a sum of the $A(f)$'s of 2.1.1) satisfies

$$F(g'g) = \psi(\delta\bar{\delta})F(g)$$

for all g' as indicated above. Notice also that for

https://doi.org/10.1017/S0027763000020535 Published online by Cambridge University Press

$$\sigma \in \Gamma_0(N_1)\,, \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \theta(z, g, \psi, r)\,|\,[\sigma]_k = \psi(d)\chi_0(d)\theta(z, g, \psi, r)\,,$$

so for $f \in S_k(\Gamma_0(N), \chi)$, the appropriate choice of ψ is $\psi = \chi\chi_0$. This is the linear combination of basic lifted forms mentioned in the introduction, and also the one whose properties are analogous to those discovered in the real quadratic case by Doi and Naganuma [3, 9].

(B) Similarly, we can mix in characters of $(Z/MPZ)^\times$ by using $V_4(v) = \begin{pmatrix} 0 & 0 \\ 0 & v/M \end{pmatrix}$, since $V_4(v)^{g'} \equiv V_4(\delta\bar{\delta}v) \bmod \mathscr{L}$ for

$$g' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Lambda_0(M\bar{\mathscr{J}}_i) = \Lambda_0^0(M\bar{\mathscr{J}}_i, 1)\,.$$

(C) Let $\tau\colon \bar{\mathscr{J}}_i^{-1}\mathfrak{D}_K^{-1}/\mathscr{J}_i \to \mathcal{O}_K/\mathscr{I}\bar{\mathscr{J}}\mathfrak{D}_K$ be an $\mathcal{O}_K$ module isomorphism. Define $V_2(r) = \begin{pmatrix} 0 & r \\ \bar{r} & 0 \end{pmatrix} \in \mathscr{L}^*$ for $r \in \bar{\mathscr{J}}_i^{-1}\mathfrak{D}_K^{-1}$. Again, for suitable g', say $g' \in \Lambda_0^0(\bar{\mathscr{J}}_i\mathfrak{D}_K M, \mathscr{J}_i P)$, $V_2(r)^{g'} \equiv V_2(\delta\bar{\alpha}r) \bmod \mathscr{L}_i$. Thus, mixing in characters ψ of $\mathcal{O}_K/\mathscr{I}\bar{\mathscr{J}}\mathfrak{D}_K$ as above, via $\sum \psi(\tau(r))\theta_{(k)}(z, V_2(r), g)$, one assigns to f a corresponding lifted form which satisfies $F(g'g) = \bar{\psi}(\delta\bar{\alpha})F(g)$ for all g' in the appropriate subgroup of $G(\mathscr{L}_i)$.

2.3. For $k > 2$, N as above, and $n \geq 1$, let

$$\varphi_n(z) = N^{-1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(N)} J(\gamma, z)^{-k} e(2n\gamma z/N)$$

be the n-th Poincaré series for $\Gamma(N)$; here as usual $J\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = cz + d$, and $\Gamma_\infty = \{\gamma \in \Gamma(N) \,|\, \gamma\infty = \infty\}$. Recall that the $\varphi_n(z)$ generate $S_k(\Gamma(N))$, since the Petersson inner product, which we denote throughout by $\langle\ ,\ \rangle$, of $\varphi_n(z)$ with $f = \sum_{m>0} a(m)e(2mz/N) \in S_k(\Gamma(N))$ is given by

$$\langle f, \varphi_n \rangle = \int_{\Gamma(N)\backslash \mathfrak{H}_1} f(z)\overline{\varphi_n(z)}y^{k-2}dxdy = (N/4\pi n)^{k-1}(k-2)!\,a(n)\,.$$

In order to lift the Poincaré series, we need to justify an interchange of summation and integration. Put $P[X] = Q[X] + R[X]$.

LEMMA 2.2. *For n a positive integer, $V \in \mathscr{L}^*$,*

$$\sum_{\substack{X \in \mathscr{L} \\ Q[X-V]=2n/N}} P[X - V]^{-s}$$

is absolutely convergent for $\mathrm{Re}\,(s) > 1$.

Proof. This is proven by an argument similar to Asai [1], Lemma 7.

LEMMA 2.3. *For $k > 2$, the (vector) series*

$$\sum_{\substack{X \in \mathscr{L} \\ Q[X-V]=2n/N}} \eta_{(k)}((X-V)^g)P[(X-V)^g]^{1/2-k} = \eta_{(k),n}(g, V)$$

converges absolutely and uniformly on compact subsets of G.

Proof. By combining Lemmas 4 and 8 of Asai [1], one can see that the absolute value of each component of $\eta_{(k),n}(g, V)$ is dominated by

$$c \sum_{\substack{X \in \mathscr{L} \\ Q[X-V]=2n/N}} P[X-V]^{-k/2}$$

for a constant c. The result then follows from Lemma 2.2 above.

PROPOSITION 2.4. *The Poincaré series $\varphi_n(z)$ lifts to*

$$\Gamma(k-1/2)\pi^{1/2-k}\eta_{(k),n}(g, V) .$$

Proof. This is a straightforward calculation, as in Kudla [8], Proposition 1; the lemmas above allow one to perform the interchange given by

$$\int_{\Gamma_\infty \backslash \mathfrak{H}_1} \theta_{(k)}(z, V, g)e(-2n\bar{z}/N)y^{k-2}dxdy$$
$$= \sum_{X \in \mathscr{L}} \eta_{(k)}((X-V)^g) \int_{\Gamma_\infty \backslash \mathfrak{H}_1} e((xQ+iyR)[(X-V)^g] - 2n\bar{z}/N)y^{k-3/2}dxdy .$$

COROLLARY 2.5. *For $k > 2$, the lifted forms are eigenfunctions of the Casimir operators on G, with eigenvalue $((k-1)^2-1)/2$.*

Proof. It suffices to prove this for the images of the Poincaré series, since they span $S_k(\Gamma(N))$. But then using Proposition 2.4 and Asai [1], Lemma 5, this is immediate.

While we have only considered the case $k > 2$ above, these results are expected to extend to $k = 2$ by using Hecke's trick. More directly, alternatively, Corollary 2.5 extends to all $k > 0$ as a result of Theorem 3.1 below.

§ 3. Fourier expansions and Dirichlet series

3.1. Throughout this section let $\mathscr{L}$ be one of the $\mathscr{L}_i$'s above, and $\mathscr{J}$ be the corresponding $\mathscr{J}_i$. Write $V \in \mathscr{L}^*$ as $V = \begin{pmatrix} v_1 & v_2 \\ \bar{v}_2 & v_4 \end{pmatrix}$. Set $S =$ Trace (C/R).

Take a set of coset representatives $\mathscr{R}$ for $\begin{pmatrix} 1 & Z \\ & 1 \end{pmatrix} \backslash SL(2, Z)/\Gamma(N)$; we write $\rho \in \mathscr{R}$ as $\rho = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}$. Without loss of generality, say $\rho_3 \neq 0$ for all $\rho \in \mathscr{R}$. Let

$$C_\alpha = \binom{2(k-1)}{k-\alpha-1} 2^{2-k} N P^{-1} i^{k-\alpha} (\mathrm{N}(\mathscr{I})^2 D)^{-1/2}$$

and for $r,\ v_2 \in \bar{\mathscr{I}}^{-1}\mathfrak{D}_K^{-1}$, put

$$\tau(\rho, r, v_2) = \rho_3^{-1} e(-2\rho_3^{-1}(\rho_1 \mathrm{N}(r) + \mathrm{S}(v_2 \bar{r}) + \rho_4 \mathrm{N}(v_2))) \times \sum_{j \in \mathscr{I}/\rho_3 \mathscr{I}} e(-2\rho_3^{-1}(\mathrm{S}(\bar{j}(r + \rho_4 v_2)) + \rho_4 \mathrm{N}(j))) .$$

Also, for $f \in S_k(\Gamma(N))$, $\rho \in SL(2, \boldsymbol{Z})$, write $f|[\rho^{-1}] = \sum a_\rho(n) e(2nz/N)$. Define a Grossencharacter by $\xi^\alpha(r) = (r/|r|)^\alpha$, and for $\mu \in \boldsymbol{C}$, $0 < \nu \in \boldsymbol{R}$, set $g(\mu, \nu) = \nu^{-1/2}\begin{pmatrix} \nu & \mu \\ 0 & 1 \end{pmatrix}$. Finally, for $v \in \bar{\mathscr{I}}^{-1}\mathfrak{D}_K^{-1}$, let $\phi_\alpha(z, v)$ be given by

$$\phi_\alpha(z, v) = \sum_{r \in \mathscr{I}} (r - v)^\alpha e(2\mathrm{N}(r - v) z) .$$

$\phi_\alpha(z, v)$ is a modular form on $\Gamma(N)$, by Theorem 1.2 (cf. Lemma 3.5 below).

THEOREM 3.1. *For $k > 0$ and $\alpha \geq 0$, $f \in S_k(\Gamma(N))$ has lift given by*

$$\begin{aligned} F_\alpha(g(\mu, \nu), V) = {} & \delta_{v_1 \bmod M, 0}\, \delta_{\alpha, k-1}\, \nu P^{-1} \langle \phi_\alpha(z, v_2), f \rangle \\ & + C_\alpha \nu^k \sum_{\rho \in \mathscr{R}} \sum_{\substack{t=1 \\ t\rho_3 \equiv -Pv_1 \bmod MP}}^{\infty} \sum_{\substack{r \in \bar{\mathscr{I}}^{-1}\mathfrak{D}_K^{-1} \\ r \neq 0}} (P^{-1}t)^{k-1} \overline{a_\rho(N \cdot \mathrm{N}(r))} \xi^\alpha(r) \\ & \times K_\alpha(4\pi P^{-1} t \nu |r|) e(2P^{-1} t(\mathrm{S}(r\bar{\mu}) - \rho_4 v_4)) \tau(\rho, r, v_2) . \end{aligned}$$

If $\alpha \leq 0$, $F_\alpha(g, V, \mathscr{L}) = (-1)^\alpha F_{-\alpha}(\bar{g}, \bar{V}, \bar{\mathscr{L}})$.

Here $\delta_{v_1 \bmod M, 0}$ is one if $v_1 \in M\boldsymbol{Z}$, and zero otherwise. Notice that in fact precisely those cosets such that $(\rho_3, MP)|Pv_1$ occur nontrivially in the sum.

In many cases we can simplify this expression. As an example, we shall show

THEOREM 3.2. *Let $f \in S_k(\Gamma_0(N), \chi)$, $f\left|\left[\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\right]_k\right. = \sum b(n) e(2nz)$. Then the Doi-Naganuma lift of f, (A) of 2.2, with $\psi = \chi\chi_0$ considered as a (not necessarily primitive) character* $\bmod N$, *$r = 1$, $P = 1$, $M = N$, is given by*

$$\begin{aligned} F_\alpha(g(\mu, \nu)) = {} & C'_\alpha \nu^k \sum_{t=1}^{\infty} \sum_{\substack{r \in \mathscr{I} \\ r \neq 0}} t^{k-1} \chi\chi_0(t) b(N \cdot \mathrm{N}(r) D^{-1} \mathrm{N}(\mathscr{I})^{-2}) \xi^\alpha(r) \\ & \times K_\alpha(4\pi t \nu |r| (D\mathrm{N}(\mathscr{I})^2)^{-1/2}) e(2\mathrm{S}(r \partial^{-1} \mathrm{N}(\mathscr{I})^{-1} t \bar{\mu})) . \end{aligned} \tag{3.1.1}$$

Here $\partial \in \mathfrak{H}_1$ is the purely imaginary generator of $\mathfrak{D}_K$, and

$$C'_\alpha = C_{|\alpha|} N^{1-k/2} i^{2-\alpha} \cdot \#\{\rho \in \mathscr{R} \mid (\rho_3, N) = 1\} .$$

These theorems will be proved, after some preliminary computations, in Sections 3.3 and 3.4 below. As an important Corollary of Theorem 3.2, we will show that the Dirichlet series at $\alpha = 0$ of an eigenform f when $h(K) = 1$ is given by $\sum \overline{b(n)} n^{-s} \sum \overline{b(n)} \chi_0(n) n^{-s}$.

Observe that though these theorems treat only the Fourier expansion at infinity, they can be used to compute the Fourier expansion at any cusp. For one gets the expansion at another cusp by computing $F(\gamma g(\mu, \nu), V)$ for $\gamma \in GL(2, \mathcal{O}_K)$ ($\gamma \in \Lambda$ gives only the principal cusps). But $F(\gamma g, V, \mathscr{L}_1) = F(g, V^\gamma, \mathscr{L}_1^\gamma)$, and while $\mathscr{L}_1^\gamma$ may not be of the form $\mathscr{L}_1$ (for any $M', P', \mathscr{I}'$), it is easy to see that $\mathscr{L}_1^\gamma$ must contain a sublattice (of finite index) of this type. If $i > 1$, the same is true for $\mathscr{L}_i$, though one must apply a γ' conjugate to $\gamma \in GL(2, \mathcal{O}_K)$ in the same way that a subgroup of $G(\mathscr{L}_i)$ is conjugate to a subgroup of Λ. So, since the conditions for constant term zero come out in terms of the ϕ_α's at all cusps, we have

COROLLARY 3.3. *All lifts* (2.1.1) *of a modular form f of weight k are cuspidal if and only if f is orthogonal to S_k', the space spanned by the $\phi_{k-1}(z, v, \mathscr{I}_i)$, for ideals $\mathscr{I}$ and v in $\bar{\mathscr{I}}_i^{-1}\mathfrak{D}_K^{-1}$.*

To determine whether a specific lift is cuspidal, observe that there are only a finite number of orthogonality conditions, which can, by the remarks above, be explicitly given in any instance. For example, while the lift (3.1.1) always has constant term zero at infinity, there is a nontrivial orthogonality condition at zero, by Theorem 2.1, (3). Also, one sees that the combination (C) is nontrivial for some nontrivial ψ.

3.2. The key step towards doing the integral (2.1.1) in general and so computing the Fourier expansion is the rewriting of $\theta_{(k)}$ in another way (Proposition 3.7) by using Theorem 1.2 on a sublattice of $\mathscr{L}$ to which there corresponds a lower dimensional theta function. Lemma 1.5 also plays an important role. In the following, $\sum_{\beta,\gamma}$ indicates a sum over nonnegative integers β and γ such that $2\beta + \gamma = k - \alpha - 1$, $\alpha + \beta \geq 0$ (the choice of α will be clear from the context). Also, for $X = \begin{pmatrix} m & r \\ \bar{r} & p \end{pmatrix} \in \mathscr{V}$, V as above, we write $X - V = \begin{pmatrix} m_1 & r_2 \\ \bar{r}_2 & p_4 \end{pmatrix}$.

Define the Maass differential operator δ_λ, $\lambda \in \boldsymbol{R}$, by

$$\delta_\lambda = (2\pi i)^{-1} y^{-\lambda} \frac{\partial}{\partial z} y^\lambda = (2\pi i)^{-1} \left\{ \lambda (2iy)^{-1} + \frac{\partial}{\partial z} \right\}, \qquad \text{and}$$

$$\delta_\lambda^\ell = \delta_{\lambda+2(\ell-1)} \cdots \delta_{\lambda+2}\delta_\lambda \qquad \text{for } 0 \le \ell \in \boldsymbol{Z}\,.$$

LEMMA 3.4.

(1) $\delta_\lambda^\ell(f\,|\,[\sigma]_\lambda) = (\delta_\lambda^\ell f)\,|\,[\sigma]_{\lambda+2\ell}$ *for all* $\sigma \in GL^+(2, \boldsymbol{R})$.

(2) *Set* $t_\gamma(z, m, p) = y^{1/2}(8\pi y)^{-\gamma/2}H_\gamma((2\pi y)^{1/2}(m-p))e(-2mpx + iy(m^2+p^2))$. *Then* $\delta_\gamma^\ell t_\gamma = t_{\gamma+2\ell}$ *for* $0 \le \gamma \in \boldsymbol{Z}$, $m, p \in \boldsymbol{R}$.

(3) $\delta_\lambda^\ell e(2az) = \ell!\,(-4\pi y)^{-\ell}L_\ell^{(\lambda-1)}(4\pi ay)e(2az)$ *for* $a \in \boldsymbol{R}$.

(4.1) $\delta_0^\ell(e(i\nu^2 y^{-1})) = (2y)^{-2\ell}\nu^{2\ell}(-1)^\ell e(i\nu^2 y^{-1})$.

(4.2) $\delta_1^\ell(y^{-1}e(i\nu^2 y^{-1})) = y^{-1}(2y)^{-2\ell}\nu^{2\ell}(-1)^\ell e(i\nu^2 y^{-1})$.

Here L and H are Laguerre and Hermite polynomials, respectively.

Proof. Parts (1), (2), and (3) may be found in Asai [1], Lemma 9. To prove part (4.1), induct on ℓ. Then (4.2) follows, as

$$\delta_1^\ell(y^{-1}e(i\nu^2 y^{-1})) = y^{-1}\delta_0^\ell(e(i\nu^2 y^{-1}))\,.$$

For $\alpha, \beta \ge 0$, $z \in \mathfrak{H}_1$, $u, v \in \boldsymbol{C}$, define

$$\begin{aligned}\phi_{\alpha,\beta}(z, u, v) = \sum_{r\in\mathscr{J}} & y^{-\beta}L_\beta^{(\alpha)}(4\pi y\mathrm{N}(r-v))(r-v)^\alpha \\ & \times e(2\mathrm{N}(r-v)z + 2\mathrm{S}(r\bar{u}) - \mathrm{S}(u\bar{v}))\,;\end{aligned}$$

the $\phi_\alpha(z, v)$ above is the same as $\phi_{\alpha,0}(z, 0, v)$.

LEMMA 3.5. *For* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D\mathrm{N}(\mathscr{J}))$,

$$\phi_{\alpha,\beta}(\sigma z, au+bv, cu+dv) = \chi_0(d)(cz+d)^{\alpha+2\beta+1}\phi_{\alpha,\beta}(z, u, v)\,.$$

Proof. The case $\beta = 0$ follows from Theorem 1.2, with two dimensional lattice given by $\mathscr{J}$, and $(u, v) = \mathrm{S}(u\bar{v})$ giving $Q = R$. Then by Lemma 3.4, (3), $\phi_{\alpha,\beta}(z, u, v) = (-4\pi)^\beta \beta!^{-1}\delta_{\alpha+1}^\beta\phi_{\alpha,0}$, so by part (1) we are done.

LEMMA 3.6. *For* $\alpha \ge 0$,

$$\begin{aligned}\theta_{k,\alpha}(z, V, g(\mu,\nu)) = (k-1)! & \sum_{\beta,\gamma}(\alpha+\beta)!^{-1}\gamma!^{-1}2^{-\beta}i^\gamma\nu^{\gamma+1}(2\pi)^{-\beta}P^{-1} \\ & \times \sum_{m\in M\boldsymbol{Z}, p\in P^{-1}\boldsymbol{Z}}\phi_{\alpha,\beta}(z, \mu p, v_2 - m_1\mu)y^{-\gamma}(m_1\bar{z}+p)^\gamma \\ & \times e(i\nu^2 y^{-1}|m_1 z + p|^2 - \mathrm{S}(\bar{\mu}pv_2) - 2pv_4)\,.\end{aligned} \tag{3.2.1}$$

Proof. First we establish for an integer $\gamma \ge 0$ the equation

$$\begin{aligned} y^{1/2}\sum_{p\in P\boldsymbol{Z}} & H_\gamma((2\pi y)^{1/2}(m_1\nu - (m_1\mu\bar{\mu} + \mathrm{S}(r_2\bar{\mu}) + p_4)\nu^{-1})) \\ & \times e(-2xm_1(m_1\mu\bar{\mu} + \mathrm{S}(r_2\bar{\mu}) + p_4) \\ & \qquad + iy(\nu^2 m_1^2 + \nu^{-2}(m_1\mu\bar{\mu} + \mathrm{S}(r_2\bar{\mu}) + p_4)^2))\end{aligned} \tag{3.2.2}$$

$$= P^{-1} i^{\gamma} \nu^{\gamma+1} (2\pi y^{-1})^{\gamma/2} \sum_{p \in P^{-1}\boldsymbol{Z}} (m_1 \bar{z} + p)^{\gamma}$$
$$\times\, e(iy^{-1}\nu^2 |m_1 z + p|^2 + 2p(m_1 \mu\bar{\mu} + \mathrm{S}(r_2\bar{\mu}) - v_4))\,.$$

For by applying the Poisson summation formula $\left(\text{Theorem 1.1 with } n = 1,\ Z = zP^2, \text{ and } M = \begin{pmatrix} & -I \\ I & \end{pmatrix}\right)$, one obtains (3.2.2) for $\gamma = 0$ (where $H_\gamma = 1$). For even γ, we then apply $\delta_0^{\gamma/2}$ to the resulting equation; using Lemma 3.4, parts (1), (2), and (4.1), and $y^{-\gamma} \left| \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right]_{\gamma} = y^{-\gamma}(c\bar{z} + d)^{\gamma}\right.$, one gets (3.2.2). For odd γ, we must first obtain (3.2.2) with $\gamma = 1$: apply $((2\pi)^{1/2} i m_1)^{-1} y^{1/2}(d/dx)$ to the case $\gamma = 0$ and add to this $m_1\nu(2\pi y)^{1/2}$ times the result for $\gamma = 0$. Then applying $\delta_1^{(\gamma-1)/2}$ establishes (3.2.2) for arbitrary γ (alternatively, we can avoid the use of the Maass operators by a more complicated application of the Poisson formula).

Next note

$$\begin{pmatrix} m & r \\ \bar{r} & p \end{pmatrix}^{g(\mu,\nu)} = \begin{pmatrix} m\nu & m\mu + r \\ m\bar{\mu} + \bar{r} & (m\mu\bar{\mu} + \mathrm{S}(r\bar{\mu}) + p)\nu^{-1} \end{pmatrix}.$$

When we sum over $m \in M\boldsymbol{Z}$, $r \in \mathscr{J}$, for each β and γ in $\sum_{\beta,\gamma}$,

$$(m_1\mu + r_2)^{\alpha} L_{\beta}^{(\alpha)}(4\pi y \mathrm{N}(m_1\mu + r_2)) e(2z\mathrm{N}(m_1\mu + r_2))$$

times equation (3.2.2) and combine this with Lemma 1.5 and the definition of ϕ, the result follows.

For $\mu = 0$, Lemma 3.6 gives the 'splitting' of the theta function as in [1]. As we see, however, the $\mu = 0$ case is only part of a more general phenomenon. In its most useful form, we express this splitting as

PROPOSITION 3.7. *For* $\alpha \geq 0$,

$$\theta_{k,\alpha}(z, V, g(\mu,\nu)) = (k-1)! \sum_{\beta,\gamma} (\alpha+\beta)!^{-1} \gamma!^{-1} 2^{-\beta} i^{\gamma} \nu^{\gamma+1} (2\pi)^{-\beta} P^{-1}$$
$$\times \Big\{ \delta_{v_1 \bmod M, 0}\, \delta_{\gamma,0}\, \phi_{\alpha,\beta}(z, 0, v_2) + \sum_{\rho \in \mathscr{R}} \sum_{\substack{t=1 \\ t\rho_3 \equiv -Pv_1 \bmod MP}}^{\infty} (P^{-1}t)^{\gamma} \sum_{\sigma \in \Gamma_\infty \backslash \Gamma(N)} \tag{3.2.3}$$
$$(\phi_{\alpha,\beta}(z, P^{-1}t\mu\rho_4, v_2 - P^{-1}t\mu\rho_3) \,|\, [\rho^{-1}]_{\alpha+2\beta+1} y^{-\gamma}$$
$$\times\, e(iP^{-2}t^2\nu^2 y^{-1} - P^{-1}t(\mathrm{S}(v_2\bar{\mu}\rho_4) + 2\rho_4 v_4))) \,|\, [\sigma\rho]_k \Big\}\,.$$

This result will allow us to use Rankin's method to compute the Doi-Naganuma lift, once we are able to compute the Fourier expansion of the $\phi_{\alpha,\beta}$ at all cusps (paragraph 3.3).

Proof. First consider the term in the sum of the right hand side of

(3.2.1) corresponding to m_1 and p such that $(m_1, p) = P^{-1}t(m', p')$, where $m', p' \in Z$ are relatively prime, $t \neq 0$, and $m' \equiv c \bmod N$, $p' \equiv d \bmod N$, $(c, d) = 1$, where c and d are fixed. But there is a unique $\sigma \in \Gamma_{-d/c}\backslash\Gamma(N)$ (where as usual $\Gamma_{-d/c}$ is the stabilizer of the cusp $-d/c$ in $\Gamma(N)$) such that $(m'\ p') = (c\ \ d)\sigma$. Using Lemma 3.5 and a lengthy but straightforward calculation, one sees

$$\begin{aligned}(3.2.4)\qquad &\phi_{\alpha,\beta}(z, P^{-1}t\mu p', v_2 - P^{-1}t\mu m')e(-P^{-1}t\mathrm{S}(\bar{\mu}p'v_2))\\ &\quad = \phi_{\alpha,\beta}(z, P^{-1}t\mu d, v_2 - P^{-1}t\mu c)e(-P^{-1}t\mathrm{S}(\bar{\mu}dv_2))\,|\,[\sigma]_{\alpha+2\beta+1}\,.\end{aligned}$$

Also, pulling out greatest common divisors in the pairs (m_1, p),

$$\begin{aligned}&\{(m_1, p)\,|\,m_1 \in MZ - v_1,\ p \in P^{-1}Z\}\\ &\quad = \{P^{-1}t(m', p')\,|\,t,\ m', p' \in Z,\ t \geq 1,\ \text{g.c.d. } (m', p') = 1,\\ &\qquad\qquad tm' \equiv -Pv_1 \bmod MP\}\,,\end{aligned}$$

so we can reduce the sum of (3.2.1) to the sum over relatively prime pairs (m', p') in congruence classes mod N specified by the coset representatives $\mathscr{R}$. Summing (3.2.4) over all m' and p' which are congruent mod N to c and d respectively fills out all $\sigma \in \Gamma_{-d/c}\backslash\Gamma(N)$. Then using Lemma 3.6 and the conjugacy of $\Gamma_{-d/c}$ to Γ_∞, one gets the second sum in (3.2.3). There is an additional term $(m_1, p) = (0, 0)$ which occurs if and only if $v_1 \equiv 0 \bmod M$, and by (3.2.1) gives a nonzero contribution only when $\gamma = 0$; this is responsible for the first term.

3.3. We get the Fourier expansion of the $\phi_{\alpha,\beta}$ at an arbitrary cusp and use this to complete the proof of Theorem 3.1.

PROPOSITION 3.8. *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$, $c \neq 0$, *and*

$$\begin{aligned}&\tau'(\sigma, u, v, v')\\ &\quad = (\mathrm{N}(\mathscr{I})^2 D)^{-1/2} i \sum_{v'' \in \mathscr{I}/c\mathscr{I}} e(2c^{-1}(a\mathrm{N}(v'') + d\mathrm{N}(v')) - \mathrm{S}(\bar{v}'(bv - du - 2v''c^{-1})))\,.\end{aligned}$$

Then

$$\begin{aligned}&\phi_{\alpha,\beta}(z, u, v)\,|\,[\sigma]_{\alpha+2\beta+1}\\ &\quad = (-1)^{\alpha+1}c^{\alpha+2\beta-1} \sum_{v' \in \bar{\mathscr{I}}^{-1}\mathfrak{D}_K^{-1}/c\mathscr{I}} \tau'(\sigma, u, v, v')\phi_{\alpha,\beta}(c^2 z, c(bv - du), u - c^{-1}(av - v'))\,.\end{aligned}$$

Proof. Note $\sigma = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} c & d \\ & 1/c \end{pmatrix}$. We compute the action of σ in the corresponding three parts. But for $\mathscr{I}'$ either $\mathscr{I}$ or $\bar{\mathscr{I}}^{-1}\mathfrak{D}_K^{-1}$, m and n relatively prime integers, one sees

$$\phi_{\alpha,\beta}(z, u, v, \mathscr{J}')\,|\left[\begin{pmatrix}1 & m/n\\ & 1\end{pmatrix}\right]_{\alpha+2\beta+1}$$
$$= \sum_{v'\in\mathscr{J}'/n\mathscr{J}} e(2m\mathbf{N}(v')/n - \mathbf{S}((u - mv/n)\bar{v}'))\phi_{\alpha,\beta}(z, u - mv/n, v + v', n\mathscr{J}) .$$

Combining this with Poisson summation $\left(\begin{pmatrix} & -I\\ I & \end{pmatrix}\text{ in Theorem 1.1}\right)$ gives the required expression.

In some cases one can get simpler expressions for this expansion, and thus a correspondingly simpler expression for the lift. We give one variation on this theme in 3.4; another is to use the results of Asai [2] when $\mathbf{N}(\mathscr{J})D$ is square free. Also, for cosets in the theta group, one may use Lemma 3.5 directly.

To get the expression of Theorem 3.1, one combines Propositions 3.7 and 3.8 with the formulas

$$\int_0^\infty \exp\{-at - b/t\}L_\beta^{(\alpha)}(at)t^{\alpha+\beta-1}dt = (-1)^\beta 2(\beta!)^{-1}a^{-\alpha/2}b^{\alpha/2+\beta}K_\alpha(2(ab)^{1/2})$$

for $a, b > 0$, and

$$(k-1)!\sum_{\beta,\gamma} 2^\gamma(\beta!\,\gamma!\,(\alpha+\beta)!)^{-1} = \binom{2(k-1)}{k-\alpha-1}$$

given in Asai [1], page 161, after performing the interchange of summation and integration as indicated above. As for the 'constant' (ν) term, note that for all $f\in S_k(\Gamma(N))$, v_2 as above,

$$\langle f, \phi_{\alpha,\beta}(z, 0, v_2)\rangle = 0 \qquad \text{for } \beta\neq 0 .$$

Indeed, this follows from the straightforward generalization of Lemma 6 of Shimura [15] to $\Gamma(N)$. Finally, the relationship between $\alpha\geq 0$ and $\alpha\leq 0$ is simply a restatement of Theorem 2.1, (2).

3.4. In this section we state and prove a more general version of Theorem 3.2. Let $P = 1$, $M = N$. We compute the lift associated to $V\in\mathscr{L}^*$ such that $(v_1, N) = 1$ in a simpler way, avoiding the sum of Proposition 3.8. Namely, for $\rho\in\mathscr{R}$, we may assume $(\rho_3, N) = 1$, or else ρ does not occur in the sum (3.2.3). Then there are integers w and w' such that $w\rho_3 + Nw'\rho_4 = 1$, so

$$\rho^{-1} = \tau_1\tau_2\,,\quad \tau_1 = \begin{pmatrix} & -1\\ N & \end{pmatrix}\begin{pmatrix}-\rho_3 & w'\\ -N\rho_4 & -w\end{pmatrix},\quad \tau_2 = \begin{pmatrix}N^{-1} & -N^{-1}\rho_3^{-1}\rho_1 + w'\rho_3^{-1}\\ & 1\end{pmatrix}.$$

One sees that for $f\in S_k(\Gamma_0(N), \chi)$, $\alpha\geq 0$,

$$\langle \sum_{\sigma \in \Gamma_\infty \backslash \Gamma(N)} \{\phi_{\alpha,\beta}(z, t\mu\rho_4, v_2 - t\mu\rho_3) | [\rho^{-1}] y^{-\gamma} e(it^2\nu^2 y^{-1})\} | [\sigma\rho], f \rangle$$
$$= \bar{\chi}(-\rho_3) N^{-\gamma/2} \int_0^\infty \int_0^1 \phi_{\alpha,\beta}(z, t\mu\rho_4, v_2 - t\mu\rho_3) | [\tau_1] y^{-\gamma} e(it^2\nu^2(Ny)^{-1}) \overline{f_N(z)} y^{k-2} dxdy\,,$$

where

$$f_N(z) = f | \left[\begin{pmatrix} & -1 \\ N & \end{pmatrix}\right]_k \in S_k(\Gamma_0(N), \bar{\chi})\,.$$

Also, we have

$$\phi_{\alpha,\beta}(z, t\mu\rho_4, v_2 - t\mu\rho_3) | [\tau_1]_{\alpha+2\beta+1}$$
$$= C_1 \chi_0(-\rho_3) \phi_{\alpha,\beta}(Nz/D\mathbf{N}(\mathscr{I})^2, \bar{\partial}^{-1}\mathbf{N}(\mathscr{I})^{-1}(wv_2 - t\mu), -\partial\mathbf{N}(\mathscr{I})\rho_4 v_2)$$

where

$$C_1 = i^{\alpha-1}(N/D\mathbf{N}(\mathscr{I})^2)^{(\alpha+2\beta+1)/2}\,.$$

This is proved by using Theorem 1.2 (notice we need an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Q})$ which may not be in $SL(2, \boldsymbol{Z})$). Combining these last two formulas with (3.2.3) and a lengthy computation similar to that sketched above, one gets the following result from which Theorem 3.2 immediately follows.

THEOREM 3.9. *Let* $P = 1$, $M = N$, $(v_1, N) = 1$, *and* $f \in S_k(\Gamma_0(N), \chi)$. *For each* $m \in \boldsymbol{Z}$, $(m, N) = 1$, *let* $m\hat{m} \equiv 1 \bmod N$ *for some* $\hat{m} \in \boldsymbol{Z}$. *Then the lifting* (2.1.1) *is given by*

$$F_\alpha(g(\mu, \nu), V) = C''_\alpha \nu^k \sum_{\rho \in \mathscr{R}} \overline{\chi\chi_0(-\rho_3)} \sum_{\substack{t=1 \\ t\rho_3 \equiv -v_1 \bmod N}}^{\infty} t^{k-1} \sum_{\substack{r \in \mathscr{I} \\ r \neq 0}} \overline{b(N \cdot \mathbf{N}(r) D^{-1} \mathbf{N}(\mathscr{I})^{-2})} \xi^\alpha(r)$$
$$\times K_\alpha(4\pi t\nu |r| / (D\mathbf{N}(\mathscr{I})^2)^{1/2}) e(2\mathbf{S}(r\partial^{-1}\mathbf{N}(\mathscr{I})^{-1}(\overline{t\mu - \hat{\rho}_3 v_2})) - 2\hat{\rho}_3\rho_4\mathbf{N}(v_2) - 2t\rho_4 v_4)\,,$$

where $C''_\alpha = C'_\alpha(\#\{\rho \in \mathscr{R} | (\rho_3, N) = 1\})^{-1}$, *and* $b(n)$ *is as in Theorem* 3.2.

3.5. To discuss Dirichlet series, for convenience we write

$$F_\alpha(g(\mu, \nu), V, \mathscr{L})$$
$$= \delta_{k-1,|\alpha|} C_\alpha(0, V, \mathscr{L})\nu + c_\alpha \sum \nu^k C(r, V, \mathscr{L}) \xi^\alpha(r) K_\alpha(4\pi\nu |r|) e(2\mathbf{S}(r\bar{\mu}))$$

where c_α depends only on α and k, and we understand here and in (3.5.1) below that $\sum$ means a sum over r in $\bar{\mathscr{J}}^{-1}\mathfrak{D}_K^{-1} - 0$. ($C(r, V, \mathscr{L})$ and c_α can of course be read off directly from Theorems 3.1, 3.2, or 3.9). Notice

$$C_{k-1}(0, V, \mathscr{L}) = \delta_{v_1 \bmod M, 0} P^{-1} \langle \phi_{k-1}(z, v_2), f \rangle = (-1)^{k-1} C_{1-k}(0, \bar{V}, \bar{\mathscr{L}})\,.$$

A standard argument using the Mellin transform

$$\int_0^\infty (F_\alpha(g(0,\nu), V, \mathscr{L}) - \delta_{k-1,|\alpha|}C_\alpha(0, V, \mathscr{L})\nu)\nu^{2s-k-1}d\nu$$

along with Theorem 2.1, (3), gives

PROPOSITION 3.10.

(3.5.1) $\Phi_\alpha(s, V, \mathscr{L}) = c_\alpha(2\pi)^{-2s}\Gamma(s+\alpha/2)\Gamma(s-\alpha/2)\sum C(r, V, \mathscr{L})\xi^\alpha(r)\mathrm{N}(r)^{-s}$

has a meromorphic continuation to the entire s plane, which is holomorphic for $|\alpha| < k-1$ *and for* $|\alpha| = k-1$ *except for simple poles of residue* $-C_\alpha(0, V, \mathscr{L})\cdot 2$ *and* $(-1)^{k-1}C_\alpha(0, V^\iota, \mathscr{L}^\iota)\cdot 2$ *at* $s = (k-1)/2$ *and* $(k+1)/2$ *respectively when these quantities are nonzero.* Φ_α *has the functional equation* $\Phi_\alpha(s, V, \mathscr{L}) = (-1)^{k-1}\Phi_\alpha(k-s, V^\iota, \mathscr{L}^\iota)$.

Consider the case given by Theorem 3.2. There

$$C(r) = \sum_{t|r} t^{k-1}\chi\chi_0(t)\overline{b(N\mathrm{N}(rt^{-1}))}\ ,$$

where the sum is over positive integers t such that $(t)|(r)$ in $\mathscr{J}^{-1}\mathfrak{D}_K^{-1}$. In the case of normalized new (eigen) form f, so $b(n) = (\text{constant})\ \overline{a(n)}$, and class number one, the relevant Dirichlet series becomes

$$D_\alpha(s) = U^{-1}\sum_{r\in\mathcal{O}_K} C_1(r)\xi^\alpha(r)\mathrm{N}(r)^{-s}$$

where

$$C_1(r) = \sum_{t|r \text{ in } \mathcal{O}_K} t^{k-1}\chi\chi_0(t)a(\mathrm{N}(rt^{-1}))\ ,$$

and U is the number of units of K (this changes Φ_α by constants and $(D\mathrm{N}(\mathscr{J}))^{-s}$). Note C_1 is multiplicative, and for $\mathfrak{p}|p$, $p\nmid N$, $\chi_0(p) = 1$, resp. -1, we have

$$C_1(\mathfrak{p}^n) = a(p^n)\ , \qquad \text{resp. } \sum_{f=0}^{n}(-1)^f\chi(p)^f p^{f(k-1)}a(p^{2n-2f})\ .$$

Thus, for α even ($4|\alpha$, $6|\alpha$, in cases $D = 4, 3$, resp.), $\mathrm{Re}(s)$ sufficiently large, a calculation shows that

$$D_\alpha(s) = \prod_{\mathfrak{p}}(1 - \xi^\alpha(\mathfrak{p})C_1(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s} + \xi^{2\alpha}(\mathfrak{p})\chi\circ\mathrm{N}(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{k-1-2s})^{-1}\ ,$$

and in particular

https://doi.org/10.1017/S0027763000020535 Published online by Cambridge University Press

$$D_0(s) = \sum a(n)n^{-s} \sum a(n)\chi_0(n)n^{-s} .$$

In the case of $h(K) > 1$, adding the Dirichlet series corresponding to the lattices $\mathscr{L}_i$, we similarly can obtain

$$\sum a(n)\sigma(n)n^{-s} \sum a(n)\bar{\sigma}(n)\chi_0(n)n^{-s} \tag{3.5.2}$$

where σ' is any character of the ideal class group and for each rational prime p we fix a prime $\mathfrak{p}'|p$ in K, and then set

$$\sigma(p) = \sigma'(\mathfrak{p}')$$

and extend σ multiplicatively (because (3.5.2) has an Euler product, it is independent of the choice of $\mathfrak{p}'$).

As a closing remark, note that the method of Section 4 of Asai [1] extends to other situations (for example, using the results of Ogg [12]), and leads to some interesting characterizations of forms in S_k'. However, the general situation will require Rankin's method for arbitrary $\Delta(N)$, with N not necessarily square free, and there are many details which need to be checked, hence we shall not deal with this here.

References

[1] T. Asai, On the Doi-Naganuma lifting associated with imaginary quadratic fields, Nagoya Math. J., **71** (1978), 149–167.

[2] T. Asai, On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution, J. Math. Soc. Japan, **28** (1976), 48–61.

[3] K. Doi and H. Naganuma, On the functional equation of certain Dirichlet series, Invent. Math., **9** (1969), 1–14.

[4] S. Friedberg, On theta functions associated to indefinite quadratic forms, preprint.

[5] E. Hecke, Mathematische Werke, Vandenhoeck & Ruprecht, Gottingen, 1970, no. 14.

[6] R. Howe, θ-series and invariant theory, in automorphic forms, representations, and L-functions, part 1, Proc. Symp. Pure Math., **33** (1979), 275–285.

[7] H. Jacquet, Automorphic forms on $GL(2)$, II, Lecture Notes in Math., **278**, Springer, 1972.

[8] S. Kudla, Theta-functions and Hilbert modular forms, Nagoya Math. J., **69** (1978), 97–106.

[9] H. Naganuma, On the coincidence of two Dirichlet series associated with cusp forms of Hecke's "Neben"-type and Hilbert modular forms over a real quadratic field, J. Math. Soc. Japan, **25** (1973), 547–555.

[10] S. Niwa, Modular forms of half integral weight and the integral of certain theta-functions, Nagoya Math. J., **56** (1974), 147–161.

[11] T. Oda, On modular forms associated with indefinite quadratic forms of signature $(2,n-2)$, Math. Ann., **231** (1977), 97–144.

[12] A. Ogg, On a convolution of L-series, Invent. Math., **7** (1969), 297–312.

[13] S. Rallis, On a relation between $\widetilde{SL}_2$ cusp forms and automorphic forms on orthogonal groups, in Automorphic forms, representations, and L-functions, part 1, Proc. Symp. Pure Math., **33** (1979), 297–314.

[14] H. Saito, Automorphic forms and algebraic extensions of number fields, Lecture in Math., no. 8, Kinokuniya Book Store Co. Ltd., Tokyo Japan 1975.
[15] G. Shimura, The special values of the zeta functions associated with cusp forms, Comm. Pure Appl. Math., **29** (1976), 783–804.
[16] T. Shintani, On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J., **58** (1975), 83–126.
[17] H. Stark, On the transformation formula for the symplectic theta function and applications, J. Fac. Sci. Univ. Tokyo, **29** (1982), 1–12.
[18] D. Zagier, Modular forms associated to real quadratic fields, Invent. Math., **30** (1975), 1–46.

Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge, Massachusetts 02138
USA