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LACUNARY SETS FOR GROUPS AND HYPERGROUPS

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Abstract

In this paper, we generalize the classical F. and M. Riesz theorem to compact groups and compact commutative hypergroups. The group $SU(2)$ of unitary matrices is also studied.

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1. Introduction

F. and M. Riesz proved the following result ([21], p. 335).

(CLASSICAL) F. AND M. RIESZ THEOREM. *Let μ be a measure on the unit circle $\mathbb{T} = \{e^{i\theta}, 0 < \theta \leq 2\pi\}$ whose Fourier coefficients*

$$\hat{\mu}(n) = \int_0^{2\pi} e^{-in\theta}\, d\mu(\theta)$$

with negative index are equals to zero. Then μ is absolutely continuous with respect to Lebesgue measure.

J. H. Shapiro gave a new proof of this theorem based on a study of duals of subspaces of $L^p(\mathbb{T})$ for $0 < p < 1$ [22]. His ideas were used by G. Godefroy for the study of Riesz subsets of commutative discrete groups [10]. In another direction R. G. M. Brummelhuis generalized Shapiro's methods to compact metrizable groups whose center contains a circle group [3].

In this paper, we give extensions of the classical F. and M. Riesz theorem to compact groups and compact commutative hypergroups. See [15, 23, 24] for the definition and the fundamental properties of hypergroups. A hypergroup is a locally compact space which has enough structure so that a convolution on the space of finite regular Borel measures can be defined. Classical examples are the space of conjugacy classes of a compact group, spaces of orbits in a locally compact group of automorphisms and double-cosets of certain non-normal closed subgroups of a compact group. The class of hypergroups includes the class of locally compact topological groups.

We briefly describe the contents of this paper. In Section 2 we recall some basic properties of hypergroups. In Section 3 we construct approximate units of the space $L^1(K)$ for K a compact hypergroup. In Section 4, we study lacunary sets in the dual of a compact hypergroup. We first extend a result of R. E. Dressler and L. Pigno [7]: the union of a Riesz set and a Rosenthal set is a Riesz set. We also investigate the class of Riesz sets, nicely placed sets and Shapiro sets in the dual object of a compact commutative hypergroup. Shapiro sets are Riesz sets (Theorem 4.6), for which the Mooney-Havin theorem extends. Following [10, 17], we use the localization technique to construct nicely placed sets. The stability by union is studied and extensions to the Mooney-Havin theorem are given. Section 5 is devoted to some examples, we consider in particular the hypergroup K of the conjugacy classes of the compact group $SU(2)$ and construct nicely placed sets in the dual $\widehat{K}$. In Section 6, we use techniques of infinite dimensional Banach space theory to study $\Lambda(1)$-sets in the dual object of compact hypergroups and give non-commutative extensions of G. F. Bachelis and S. E. Ebenstein's result [1] and of F. Lust-Piquard's result [16]. We also answer a question of R. G. M. Brummelhuis [5]. In Section 7, we generalize a result of G. Godefroy [10] to compact groups whose center contains a copy of the circle group; we improve a result of R. G. M. Brummelhuis [4, 5]. We also give some applications to the unit sphere of $\mathbb{C}^n (n \geq 2)$ and to the Bergman-Shilov boundary of a bounded symmetric domain.

NOTATION. Let K be a compact group or a compact hypergroup. We denote by $\mathscr{C}(K)$ the space of continuous functions on K and by $\mathscr{M}(K)$ the dual of $\mathscr{C}(K)$, the space of finite Borel measures on K. We denote by $\mathscr{M}^+(K)$ the subset of positive measures in $\mathscr{M}(K)$ and $D(K)$ the algebra generated by the discrete measures. We denote by δ_x the Dirac measure at the point x. We denote by μ_a (resp. μ_s) the absolute continuous (resp. singular) part of a measure μ in $\mathscr{M}(K)$ with respect to the Haar measure. The $L(1, \infty)$ ("weak L^1") quasi-norm is defined by

$$\|f\|_{1,\infty} = \sup_{\lambda \geq 0} \{\lambda P(|f| \geq \lambda)\}.$$

We denote by $B(X)$ the closed unit ball of a Banach space X and by I_E the characteristic function of the set E.

2. Basic properties of hypergroups

Our main reference for hypergroups is [15]. Let us recall the definition of a hypergroup.

DEFINITION 2.1. Let K be a locally compact Hausdorff space. The space K is a hypergroup if there exists a binary mapping $(x, y) \to \delta_x \star \delta_y$ of $K \times K$ into $\mathscr{M}^+(K)$ satisfying the following conditions.

(1) The mapping $(\delta_x \star \delta_y) \to \delta_x \star \delta_y$ extends to a bilinear associative operator $\star$ from $\mathscr{M}(K) \times \mathscr{M}(K)$ into $\mathscr{M}(K)$ such that

$$\int_K f d(\mu \star \nu) = \int_K \int_K \int_K f d(\delta_x \star \delta_y)\, d\mu(x) d\nu(y)$$

for all continuous functions f on K vanishing at infinity.

(2) For each $x, y \in K$, the measure $\delta_e \star \delta_y$ is a probability measure with compact support.

(3) The mapping $(\mu, \nu) \to \mu \star \nu$ is continuous from $\mathscr{M}^+(K) \times \mathscr{M}^+(K)$ into $\mathscr{M}^+(K)$; the topology on $\mathscr{M}^+(K)$ being the cone topology.

(4) There exists $e \in K$ such that $\delta_e \star \delta_y = \delta_x = \delta_x \star \delta_e$ for all $x \in K$.

(5) There exists a homeomorphism involution $x \to x^-$ of K onto K such that, for all $x, y \in K$, we have

$$(\delta_x \star \delta_y)^- = \delta_{y^-} \star \delta_{x^-}, \text{ where } \delta_x^- \text{ is defined by}$$
$$\int_K f(k) d\delta_x^-(k) = \int_K f(k^-) d\delta_x(k),$$

and also,

$$e \in \operatorname{supp}(\delta_x \star \delta_y) \text{ if and only if } y = x^-$$

where $\operatorname{supp}(\delta_x \star \delta_y)$ is the support of the measure $\delta_x \star \delta_y$.

(6) The mapping $(x, y) \to \operatorname{supp}(\delta_x \star \delta_y)$ is continuous from $K \times K$ into the space $\mathscr{C}(K)$ of compact subsets of K, where $\mathscr{C}(K)$ is given the topology whose subbasis is given by all

$$\mathscr{C}_{U,V} = \{A \in \mathscr{C}(K) : A \cap U \neq \varnothing \text{ and } A \subset V\}$$

where U, V are open subsets of K.

Note that in general $\delta_x \star \delta_y$ is not necessarily a discrete measure. A hypergroup K is *commutative* if $\delta_x \star \delta_y = \delta_y \star \delta_x$ for all x, y in K. Let us first

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recall some properties of *compact commutative hypergroups*. Such a hypergroup K carries a Haar measure m such that $m(K) = 1$ and $\delta_x \star m = m$ for all x in K. If f is a Borel function on K and $x, y \in K$ then $x \star y$ is defined by $f(x \star y) = \int_K f\, d(\delta_x \star \delta_y)$. A complex-valued function χ on K is said to be *multiplicative* if $\chi(x \star y) = \chi(x)\chi(y)$ for all x and y in K. The dual $\widehat{K}$ of K is the space of characters; that is the space of multiplicative continuous functions χ on K such that $\chi(x^-) = \overline{\chi(x)}$ for all x in K. The space $\widehat{K}$ is an orthogonal basis for $L^2(K)$. If K is compact then $\widehat{K}$ is discrete. Let us note that $\widehat{K}$ is not necessarily a hypergroup. For $\mu \in \mathscr{M}(K)$, the *Fourier-Stieltjes transform* $\hat{\mu}$ of μ is defined on $\widehat{K}$ by

$$\hat{\mu}(\chi) = \int_K \overline{\chi}\, d\mu, \quad \text{for all } \chi \in \widehat{K}.$$

The mapping $\mu \to \hat{\mu}$ is a norm-decreasing $\star$-algebra isomorphism from $\mathscr{M}(K)$ into the space of bounded functions on $\widehat{K}$. We are also concerned with compact, not necessarily commutative, hypergroups K. Such hypergroups also carry a Haar measure and are unimodular. The dual object Σ is then the set of equivalence classes of continuous irreducible representations of K. If K is commutative, then Σ is to be identified with $\widehat{K}$, the space of characters. In the non-commutative case complications arise because not all continuous irreducible representations of K have representation space of dimension 1. However all the representation spaces have finite dimension when K is compact [24]. Let us recall a few definitions and properties [24]. The *Fourier-Stieltjes transform* of a measure μ in $\mathscr{M}(K)$ is then defined by

$$\hat{\mu}(\tau) = \int_K \overline{\tau}\, d\mu, \quad \text{for each } \tau \in \Sigma.$$

It is an operator-valued function on Σ. The spectrum of a measure μ in $\mathscr{M}(K)$ is defined by

$$\operatorname{spec} \mu = \{\alpha \in \Sigma,\ \hat{\mu}(\alpha) \neq 0\} = \operatorname{supp} \hat{\mu}.$$

For any subset Λ of Σ, we let

$$\mathscr{M}_\Lambda(K) = \{\mu \in \mathscr{M}(K),\ \operatorname{spec} \mu \subset \Lambda\}.$$

Let $\mathscr{T}(K)$ denote the space of trigonometric polynomials on K : $\mathscr{T}(K) = \{f \in L^1(K) : \operatorname{spec} f$ is a finite set$\}$. If $\Lambda \subset \Sigma$, let $\mathscr{T}_\Lambda(K) = \{f \in \mathscr{T}(K) : \operatorname{spec} f \subset \Lambda\}$.

3. Approximate units in $L^1(K)$

In this section, we construct approximate units in $L^1(K)$ for K a compact hypergroup. We get a generalization of a result of J. Boclé [2, Theorem II,

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page 17], see also [22, Lemma 1.1]. In the following K will denote a compact not necessarily commutative hypergroup, m its Haar measure. We assume $m(K) = 1$. The main result of this section is the following

THEOREM 3.1. *Let K be a compact hypergroup. There exists a net of functions $\{h_\alpha\}$ in $L^1(K)$ such that for all α:*

(1) $h_\alpha \in \mathscr{T}(K)$.
(2) $\|h_\alpha\|_1$ *is bounded.*
(3) *If $\mu \in \mathscr{M}_s(K)$ then the net $\{\mu \star h_\alpha\}$ converges in Haar measure to zero.*
(4) *If $f \in L^1(K)$ then the net $\{f \star h_\alpha\}$ converges in L^1-norm to f.*

PROOF. Consider $\mathscr{U}$ a basis of neighborhoods at e consisting of symmetric sets. We direct the net in the usual way: $U \geq V$ if $U \subset V$. For $V \in \mathscr{U}$, let $f_V = m(V)^{-1} I_V$. We now prove that the net $\{\mu \star f_V\}_{V \in \mathscr{U}}$ converges in Haar measure to 0 when μ belongs to $\mathscr{M}_s(K)$. Since $|\mu \star f_V| \leq |\mu| \star f_V$ [15, 6.1.B] we may without loss of generality suppose μ to be a positive measure. Let $\varepsilon > 0$ and $a > 0$. Since μ is a regular and singular measure on K there exist a compact set H and an open set U such that $H \subset U \subset K$ and $\mu(U) = \mu(K) = \|\mu\|$, $\mu(U \setminus H) < \varepsilon a/2$, $m(U) < \varepsilon/2$. Define λ in $\mathscr{M}(K)$ as follows:

$$\lambda(B) = \mu(B \cap H) \text{ for } B \text{ a Borel subset of } K .$$

Then $\mu = \lambda + \theta$ where $\theta(K) < \varepsilon a/2$. By [15, 3.2.D], there is a neighborhood W in $\mathscr{U}$ such that $W \star H \subseteq U$. By definition, one has:

$$(\lambda \star I_W)(t) = \int_K I_W(y^- \star t) d\lambda(y) = \int_K (I_W \star \delta_t)(y) d\lambda(y)$$

and,

$$(\lambda \star I_W)(t) = \int_{(W \star t) \cap H} (I_W \star \delta_t)(y) d\mu(y) .$$

By [15, 4.1.B], the set $(W \star t) \cap H$ is empty if and only if the set $\{t\} \cap (W^- \star H)$ is empty; that is t does not belong to the set $W^- \star H$ and by the symmetry of W, t does not belong to $W \star H$. It follows that if t does not belong to U then the set $(V \star t) \cap H$ is empty for any $V \subset W$. Now we proceed as in the group case [22]. For $V \subset W$, we have $\mu \star f_V = \theta \star f_V$ off U. Hence,

$$\int_{K \setminus U} (\mu \star f_V)(t)\, dm(t) = \int_{K \setminus U} (\theta \star f_V)(t) dm(t) \leq \|\theta\| \, \|f_V\|_1 \leq \varepsilon a/2 .$$

By Chebyshev's inequality:

$$m\{\{\mu \star f_V > a\} \cap (K \setminus U)\} \leq \varepsilon/2$$

and

$$m\{\mu \star f_V > a\} \leq \varepsilon/2 + m(U) < \varepsilon .$$

By [15, 5.1.B], we also get that if $g \in L^1(K)$, $\{g \star f_V\}_{V \in \mathscr{U}}$ converges in L^1-norm to g. Since each f_V is an element of $L^2(K)$ and $\mathscr{T}(K)$ is dense in $L^2(K)$ [24], we get the theorem.

Let us note that R. C. Vrem constructed in $L^1(K)$, with K a compact hypergroup, approximate units satisfying the assertions (1), (2) and (4) of the Theorem 3.1. See [24].

4. Lacunary sets for compact hypergroups

Following [10, 16], we now define Riesz, nicely placed, Shapiro and Rosenthal sets. In the sequel, K will be a *compact hypergroup*, Σ its dual and m the Haar measure on K.

DEFINITION 4.1. A subset Λ of Σ is a *Riesz* set if every measure μ in $\mathscr{M}_\Lambda(K)$ is absolutely continuous with respect to the Haar measure of K.

DEFINITION 4.2. Let X be a closed subspace of L^1. The space X is *nicely placed* if $B(X)$ is closed in $L(1, \infty)$. A subset Λ of Σ is *nicely placed* if $L^1_\Lambda(K)$ is nicely placed in $L^1(K)$.

We denote by $[\Lambda]$ the smallest nicely placed subset of Σ containing Λ.

DEFINITION 4.3. A subset Λ of Σ is a *Shapiro set* if every subset of Λ is nicely placed.

DEFINITION 4.4. A subset Λ of Σ is a *Rosenthal set* if $L^\infty_\Lambda(K) = \mathscr{C}_\Lambda(K)$.

It is known that every Rosenthal subset of a commutative discrete group is a Riesz set [16]. More generally, R. E. Dressler and L. Pigno have shown that the union of a Riesz set and of a Rosenthal set is a Riesz set [7]. G. Godefroy extended this result [11]; we generalize Godefroy's result to compact hypergroups.

PROPOSITION 4.5. *If K is compact hypergroup and Σ its dual, if $\Lambda \subset \Sigma$ is such that $\mathscr{M}_\Lambda = L^1_\Lambda \oplus (\mathscr{M}_s)_\Lambda$ and if Λ_0 is a Rosenthal set, then*

$$\mathscr{M}_{\Lambda \cup \Lambda_0} = L^1_{\Lambda_1 \cup \Lambda_2} \oplus (\mathscr{M}_s)_\Lambda .$$

Let us remark that if Λ is a Riesz set then $(\mathscr{M}_s)_\Lambda = \{0\}$ and $\Lambda \cup \Lambda_0$ is a Riesz set.

PROOF. Let μ be in $\mathscr{M}_{\Lambda \cup \Lambda_0}(K)$ and consider $g_n = k_n \star \mu$ (where (k_n) is an approximation of the identity in $L^1(K)$). We have,

$$\begin{aligned}\int_K (k_n \star \mu) f\,dm &= \iint_k k_n(x \star y^-) f(x) d\mu(y) dm(x)\\ &= \iint_K k_n^-(y \star x^-) f^-(x^-) d\mu(y) dm(x^-)\\ &= \iint_K k_n^-(x) f^-(y^- \star x) d\mu(y) dm(x)\\ &= \int_K k_n^-(\mu \star f^-) dm.\end{aligned}$$

We let $\overline{\Lambda} = \{\overline{\alpha}, \alpha \in \Lambda\}$ and $\Lambda' \in \Sigma \setminus \overline{\Lambda}$. If $f \in L^\infty_{\Lambda'}$ then $f^- \in L^\infty_{\Lambda}$ and $\mu \star f^- \in L^\infty_{\Lambda_0} = \mathscr{C}_{\Lambda_0}$. Therefore $\lim_{n\to+\infty}(\int g_n f\,dm)$ exists for every $f \in L^\infty_{\Lambda'}$.

And now the proof proceeds exactly as in the commutative compact group case, see [11].

In the sequel, K will be a commutative compact hypergroup and $\widehat{K}$ its dual. The second part of this section is devoted to the proof of the following result.

THEOREM 4.6. *Let K be a commutative compact hypergroup. Then every Shapiro subset of $\widehat{K}$ is a Riesz set.*

Let $\mathscr{C} \subset \mathscr{P}(\widehat{K})$ be a family of subsets of $\widehat{K}$. We consider the class $\mathring{\mathscr{C}} = \{A \in \mathscr{P}(\widehat{K}):$ for all $B \subset A$, $B \in \mathscr{C}\}$. We have the following lemmas.

LEMMA 4.7. *Let K be a commutative compact hypergroup and $\mathscr{C}$ be a family of subsets of $\widehat{K}$. If every $\Lambda \in \mathscr{C}$ satisfies:*

$$\mu \in \mathscr{M}_\Lambda(K) \text{ implies } \mu_s \in \mathscr{M}_\Lambda(K),$$

then every $\Lambda \in \mathring{\mathscr{C}}$ is a Riesz set.

PROOF. The proof here is similar to the group case [10, Lemma 1.1].

LEMMA 4.8. *Let K be a commutative compact hypergroup, Λ be a subset of $\widehat{K}$ and $\mu \in \mathscr{M}_\Lambda(K)$. Then $\mu_s \in \mathscr{M}_{[\Lambda]}(K)$.*

PROOF. The proof proceeds as in the group case [10, Lemma 1.5]. Note that this proof does require Theorem 3.1.

Theorem 4.6 follows from these lemmas.

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Remark. With the same arguments it can be shown that if G is a compact group and Σ its dual then every Shapiro subset of Σ is a Riesz set.

A way to construct Riesz, nicely placed and Shapiro sets is the *localization technique* [10, 17]. Let us introduce the following topology on $\widehat{K}$. For $\alpha \in \widehat{K}$, we say that V_α is a τ-neighborhood of α if there exists a measure $\nu_\alpha \in D(K)$ such that

$$\hat{\nu}_\alpha(\alpha) \neq 0, \; V_\alpha \supset \operatorname{spec} \nu_\alpha . \tag{1}$$

This defines a topology. Let us just mention that if U_α and V_α are two τ-neighborhoods of α in $\widehat{K}$ then there exist two measures ν_α and μ_α in $D(K)$ such that

$$\alpha \in U_\alpha \supset \operatorname{spec} \nu_\alpha , \; \alpha \in V_\alpha \supset \operatorname{spec} \mu_\alpha .$$

We have $\nu_\alpha \star \mu_\alpha \in D(K)$ and $\alpha \in U_\alpha \cap V_\alpha \supset \operatorname{spec}(\nu_\alpha \star \mu_\alpha)$. Therefore $U_\alpha \cap V_\alpha$ is a τ-neighborhood of α. This topology corresponds to the Bohr topology in the commutative group case.

Theorem 4.9. *If Λ is a subset of $\widehat{K}$ and if for every $\alpha \in \widehat{K}$ there exists a τ-neighborhood V_α of α such that*

$$\mu \in \mathscr{M}_{\Lambda \cap V_\alpha}(K) \text{ implies } \mu_s \in \mathscr{M}_{\Lambda \cap V_\alpha}(K) \tag{2}$$

then we also have

$$\mu \in \mathscr{M}_{\Lambda}(K) \text{ implies } \mu_s \in \mathscr{M}_{\Lambda}(K) .$$

Proof. Let $\Lambda \subset \widehat{K}$, $\alpha \notin \Lambda$ and V_α be a τ-neighborhood of α satisfying (1) and (2). Let $\mu \in M_\Lambda(K)$. Then $\nu_\alpha \star \mu \in \mathscr{M}_{\Lambda \cap \operatorname{spec} \nu_\alpha}(K)$ and $\nu_\alpha \star \mu \in \mathscr{M}_{\Lambda \cap V_\alpha}(K)$. Then $(\nu \star \mu)_s \in \mathscr{M}_{\Lambda \cap V_\alpha}(K)$. Since $\alpha \notin \Lambda$, $\alpha \notin \Lambda \cap V_\alpha$ and $(\widehat{\nu_\alpha \star \mu})_s(\alpha) = 0$. Since $L^1(K)$ and $\mathscr{M}_s(K)$ are closed and invariant under convolutions by the elements of $D(K)$ [15], we have $(\nu_\alpha \star \mu)_s = \nu_\alpha \star \mu_s$ [17]. Thus $\widehat{\mu}_s(\alpha) = 0$ and $\mu_s \in \mathscr{M}_\Lambda(K)$.

Let us notice that if Λ is a subset of $\widehat{K}$ and if for every $\alpha \in \widehat{K}$ there exists a τ-neighborhood V_α of α such that $\Lambda \cap V_\alpha$ is a Riesz set then Λ is also a Riesz set.

Theorem 4.10. *If Λ is a subset of $\widehat{K}$ and if for every $\alpha \in \widehat{K}$ there exists a τ-neighborhood V_α of α such that $\Lambda \cap V_\alpha$ is a nicely placed set then Λ is a nicely placed set.*

Proof. We need two lemmas.

Lemma 4.11. *Let $\nu \in D(K)$ and $(f_n)_{n\geq 1}$ be a bounded sequence in $L^1(K)$ which converges to 0 in $L(1, \infty)$. Then $\nu \star f_n$ belongs to $L^1(K)$ and there exists a subsequence $(\tilde{f}_k)_{k\geq 1}$ of (f_n) such that the sequence*

$$\left(\nu \star \frac{1}{n}\sum_{k=1}^{n} \tilde{f}_n\right)_{n\geq 1}$$

converges to 0 in $L(1, \infty)$.

Proof. By [15, 6.2.B], we have $\nu \star f_n \in L^1(K)$ and

$$\|\nu \star f_n\|_1 \leq \|\nu\| \, \|f_n\|_1 .$$

By [10, Lemma 1.2], there exists a subsequence $(\tilde{f}_k)$ of (f_n) such that the sequence

$$\left(\nu \star \frac{1}{n}\sum_{k=1}^{n} \tilde{f}_k\right)_{n\geq 1}$$

converges in $L(1, \infty)$. It is easy to see that the assumption that the sequence (f_n) converges to 0 in $L(1, \infty)$ implies that the sequence $(\nu \star \frac{1}{n}\sum_{k=1}^{n} \tilde{f}_k)_{n\geq 1}$ also converges to 0 in $L(1, \infty)$.

Lemma 4.12. *Let Λ be a subset of $\widehat{K}$ and $\alpha \notin \Lambda$. If there exists a τ-neighborhood V_α of α such that $\alpha \notin [V_\alpha \cap V]$ then $\alpha \notin [\Lambda]$.*

Proof. Let $\nu_\alpha \in D(K)$ which satisfies (1). Let $f \in L^1_\Lambda(K)$. By Lemma 4.11, $\nu_\alpha \star f \in L^1(K)$. In fact, $\nu_\alpha \star f \in L^1_{V_\alpha \cap \Lambda}(K)$. Thus $\nu_\alpha \star L^1_\Lambda(K) \subset L^1_{V_\alpha \cap \Lambda}(K)$. Lemma 4.11 proves that if X is nicely placed then $C = \{f : \nu_\alpha \star f \in X\}$ is also nicely placed: consider a sequence (f_n) in C which converges to f in $L(1, \infty)$, then $\nu_\alpha \star f_n$ belongs to X and $\frac{1}{n}\nu_\alpha \star \sum_{k=1}^{n} f_k$ also belongs to X. It follows by Lemma 4.11 that f belongs to C. Then we have:

$$\nu_\alpha \star L^1_{[\Lambda]}(K) \subset L^1_{[V_\alpha \cap \Lambda]}(K). \tag{3}$$

Since $\nu_\alpha \star \alpha = \hat{\nu}_\alpha(\alpha).\alpha$ and $\hat{\nu}_\alpha(\alpha) \neq 0$, (3) shows that $\alpha \notin [\Lambda]$.

Let us now prove Theorem 4.10. Let $\alpha \notin \Lambda$ and V_α a τ-neighborhood satisfying (1), then $\alpha \notin \Lambda \cap V_\alpha = [\Lambda \cap V_\alpha]$. And by Lemma 4.11, $\alpha \notin [\Lambda]$.

Corollary 4.13. *If Λ is a subset of $\widehat{K}$ and if for every $\alpha \in \widehat{K}$ there exists a τ-neighborhood V_α of α such that $\Lambda \cap V_\alpha$ is a Shapiro set then Λ is a Shapiro set.*

Corollary 4.14. *Let* Λ_1 *be a nicely placed subset of* $\widehat{K}$ *and* Λ_2 *a* τ-*closed subset of* $\widehat{K}$. *Then the union* $\Lambda_1 \cup \Lambda_2$ *is nicely placed. In particular, every* τ-*closed subset is nicely placed.*

This extends a result of Y. Meyer [17]. We also get an extension of the Mooney-Havin theorem [13, 18].

Corollary 4.15. *Let* Λ_1 *be a nicely placed subset of* $\widehat{K}$ *and* Λ_2 *be a* τ-*closed subset of* $\widehat{K}$. *Then the space* $L^1/L^1_{\Lambda_1\cup\Lambda_2}(K)$ *is weakly sequentially complete.*

Proof. The result follows from Corollary 4.14 and from the fact that if Λ is a nicely placed subset of $\widehat{K}$ then the space $L^1/L^1_\Lambda(K)$ is weakly sequentially complete [10].

5. Examples

Conjugacy classes of compact non-commutative groups.

Let G be a compact non-commutative group, with normalized Haar measure σ and Σ be its dual object. We say that a subset Λ of Σ is *central Riesz* if every central measure μ in $\mathscr{M}_\Lambda(G)$ is absolutely continuous with respect to the Haar measure of G. Central nicely placed and central Shapiro subsets of Σ are defined in the same way. For $x \in G$, let $x^G = \{t^{-1}xt\,,\, t \in G\}$ the conjugacy class of x. Let $K = \{x^G\,,\, x \in G\}$ have the quotient topology. The space K, with the operation

$$\delta_{x^G} \star \delta_{y^G} = \int_G \delta_{(t^{-1}xty)^G}\, d\sigma(t)$$

is a compact commutative hypergroup [15]. Each $\tau \in \Sigma$ has a representation space of finite dimension d_τ and trace χ_τ. The functions χ_τ are called characters but the *hypergroup characters* are normalized by dividing χ_τ by d_τ. More precisely, if $\pi : G \to K$ is the natural mapping then ψ_τ on K is defined by: $\psi_\tau \circ \pi = d_\tau^{-1}\chi_\tau$ and $\widehat{K} = \{\psi_\tau\,,\, \tau \in \Sigma\}$. The Haar measure m on K is induced from the Haar measure on G. $\widehat{K}$ is a commutative discrete hypergroup. The functions defined on K (respectively the measures of K) correspond to the central functions defined on G (respectively the central measures of G). It follows that Riesz (respectively nicely placed) subsets of $\widehat{K}$ correspond to central Riesz (respectively central nicely placed) subsets of Σ. Let us now consider two examples:

a) The special unitary group $SU(2)$ consists of all 2×2 matrices $\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$ where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Following [14] we now recall the construction of the dual Σ of $SU(2)$. Let l be a number in the set $\{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$. We construct a representation of $SU(2)$ of dimension $2l+1$ as follows.

Let H_l be the linear space of all complex one variable polynomials of degree not exceeding $2l$. Let $u = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$. For all $f \in H_l$, we define:

$$(T_u^{(l)} f)(z) = (\beta z + \overline{\alpha})^{2l} f\left(\frac{\alpha z - \overline{\beta}}{\beta z + \overline{\alpha}}\right).$$

The mapping $T^{(l)} : u \to T_u^{(l)}$ is a $(2l+1)$-dimensional representation of $SU(2)$ and the set $\{T^{(0)}, T^{(\frac{1}{2})}, T^{(1)}, \dots\}$ is a complete set of continuous unitary irreducible representations of $SU(2)$. That is, for each $n = 1, 2, 3, \dots, \Sigma$ contains exactly one element of dimension n. We write its character by χ_n. We now describe the hypergroup K of conjugacy classes of $SU(2)$, see [15] for more details. We identify K with $[0, \pi]$ where θ in $[0, \pi]$ corresponds to the conjugacy class containing the matrix

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

The hypergroup character ψ_n on K corresponding to χ_n is equal to

$$\psi_n(\theta) = \frac{\sin n\theta}{n \sin\theta}, \ \theta \in]0, \pi[; \ \ \psi_n(0) = \psi_n(\pi) = 1.$$

The Haar measure m on K is given by

$$\int_K f\, dm = \frac{2}{\pi} \int_0^\pi f(\theta) \sin^2\theta\, d\theta.$$

We now give some examples of τ-closed subsets of $\widehat{K}$. A subset F of $\widehat{K}$ is τ-closed if there exists a measure ν in $D(K)$ such that

$$F = \{\psi_n,\ n \in \mathbb{N}^\star,\ \hat{\nu}(\psi_n) = 0\}.$$

(1) Let $r \in \mathbb{N}^\star \setminus \{1\}$. Then the set $\{\psi_{kr}, k \in \mathbb{N}^\star\}$ is τ-closed. Indeed, take $\nu = \delta_{\pi/r}$. Then $\hat{\nu}(\psi_n) = (\sin n\pi/r)/(n \sin \pi/r) = 0$ if and only if n is a multiple of r in $\mathbb{Z}$.

(2) Let $r \in \mathbb{N}^\star$. Then the set $\{\psi_{(2k+1)r}, k \in \mathbb{N}\}$ is τ-closed. Consider the discrete measure $\nu = \sin\theta_1 \delta_{\theta_1} - \sin\theta_2 \delta_{\theta_2}$, with $\theta_1 = \pi/r - \pi/\sqrt{2}$ and $\theta_2 = \pi/\sqrt{2}$.

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Then,

$$\begin{aligned}\hat{\nu}(\psi_n) &= \tfrac{1}{n}(\sin n(\pi/r - \pi/\sqrt{2}) - \sin n\pi/\sqrt{2}) \\ &= \tfrac{2}{n}\sin n(\pi/2r - \pi/\sqrt{2})\cos n\pi/2r\end{aligned}$$

and

$$\begin{aligned}(\hat{\nu}(\psi_n) = 0) \quad &\text{if and only if } (\cos n\pi/2r = 0) \\ &\text{if and only if } (n \in \{(2k+1)r,\, k \in \mathbb{Z}\}).\end{aligned}$$

b) C. F. Dunkl and D. E. Ramirez constructed in [8] an interesting family of compact countable hypergroups H_a, for $0 < a \le \frac{1}{2}$. H_a has the topological structure of the one-point compactification of the nonnegative integers. The case $H_{1/p}$, p prime, is the set of equivalence classes of the p-adic integers modulo the group of units (under multiplication). The hypergroup H_a is identified with $\{0, 1, 2, \dots\infty\}$. The invariant measure m is given by

$$m(\{k\}) = a^k(1-a) \text{ for } k < \infty, \quad m(\{\infty\}) = 0.$$

$\widehat{H}_a = \{\psi_0, \psi_1, \psi_2, \dots\}$ where $\psi_0 \equiv 1$ and

$$\psi_n(k) = \begin{cases} 1 & \text{for } k \ge n \\ a/(a-1) & \text{for } k = n-1, \quad n \ge 1 \\ 0 & \text{for } k < n-1. \end{cases}$$

The following sets are τ-clopen:

(1) $\{\psi_p, \psi_{p+1}, \psi_{p+2}, \dots\}$ for all $p \ge 2$.
(Take $\nu = \delta_{p-2}$)

(2) $\{\psi_p\}$ for all $p \in \mathbb{N}$. (Take $\nu = ((a+1)/a\delta_{p-1} - \delta_p) + \sum_{j=0}^{\infty} 1/2^j(\delta_{p+2j+1} - \delta_{p+2j+2})$ for $p \ge 1$ and $\nu = \sum_{j=0}^{\infty} 1/2^j(\delta_{2j} - \delta_{2j+1})$ for $p = 1$).

(3) $\{\psi_0, \psi_1, \dots, \psi_p\}$ for all $p \in \mathbb{N}$.
(Take $\nu = \sum_{j=0}^{\infty} 1/2^j(\delta_{p+2j} - \delta_{p+2j+1})$).

6. $\Lambda(1)$-sets in compact hypergroups

It is known that if G is a compact commutative group and Λ a subset of its dual then we have the following equivalences.

(1) Λ is a $\Lambda(1)$-set if and only if $L^1_\Lambda(G)$ is reflexive [1]

(2) Λ is a Riesz set if and only if $L^1_\Lambda(G)$ has Radon-Nikodym property [16].

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Let us recall that a Banach space X has the *Radon-Nikodym property* (RNP) if and only if every linear continuous operator

$$T : L^1(\Omega, \mathscr{A}, \mu) \to X$$

(where μ is a probability measure) is representable by a X-valued strongly μ-measurable and bounded function F that is for all $\varphi \in L^1(\Omega, \mathscr{A}, \mu)$,

$$T(\varphi) = \int \varphi(\omega) F(\omega) d\mu(\omega).$$

If X is a reflexive Banach space then X has RNP [6]. If follows that a $\Lambda(1)$-set is a Riesz set. Our aim is to extend the results (1) and (2) to compact hypergroups. In the following K will denote a compact not necessarily commutative hypergroup, m its Haar measure and Σ its dual.

DEFINITION 6.1. Let $0 < p < \infty$. A subset Λ of Σ is called $\Lambda(p)$-set if for some $0 < q < p$ there exists a constant C such that

$$\|f\|_p \leq C\|f\|_q, \text{ for all } f \in \mathscr{T}_\Lambda(K). \tag{4}$$

Λ is called central $\Lambda(p)$ if (4) holds for all f in $\mathscr{T}_\Lambda(K)$ which are central.

PROPOSITION 6.2. *Let Λ be a subset of Σ. Then Λ is $\Lambda(1)$-set if and only if $L^1_\Lambda(K)$ is reflexive.*

PROOF. Suppose that Λ is a $\Lambda(1)$-set of Σ. Then for some $0 < q < 1$ there exists a constant C such that

$$\|f\|_1 \leq C\|f\|_q, \text{ for all } f \in \mathscr{T}_\Lambda(K).$$

Let (F_α) be an approximate unit satisfying Theorem 3.1. Let $f \in L^1_\Lambda(K)$ then $F_\alpha \star f \in \mathscr{T}_\Lambda(K)$ and, for $0 < q < 1$, we have

$$\|F_\alpha \star f\|_1 \leq C\|F_\alpha \star f\|_q.$$

Let $\varepsilon > 0$ and α be such that

$$\|F_\alpha \star f - f\|_1 \leq \varepsilon.$$

Then,

$$\begin{aligned}\|f\|_1 &\leq \|F_\alpha \star f\|_1 + \|f - F_\alpha \star f\|_1 \leq C\|F_\alpha \star f\|_q + \varepsilon \\ &\leq C[\|F_\alpha \star f - f\|_q^q + \|f\|_q^q]^{1/q} + \varepsilon \leq C[\|F_\alpha \star f - f\|_1^q + \|f\|_q^q]^{1/q} + \varepsilon \\ &\leq C(\varepsilon^q + \|f\|_q^q)^{1/q} + \varepsilon.\end{aligned}$$

We have proved that there exists a constant C depending only on q such that for all $f \in L^1_\Lambda(K)$, we have

$$\|f\|_1 \leq C\|f\|_q;$$

that is, the L^1 and L^q topologies coincide on $L^1_\Lambda(K)$ and $L^1_\Lambda(K)$ is reflexive. For the converse, see [1].

PROPOSITION 6.3. *Let Λ be a subset of Σ. Then Λ is Riesz if and only if $L^1_\Lambda(K)$ has RNP.*

PROOF. Suppose first that Λ is Riesz. The space $L^1_\Lambda(K)$ has RNP if and only if its separable subspaces have RNP [6]. Let S be a separable subspace of $L^1_\Lambda(K)$, then $S \subset L^1_{\Lambda'}(K)$ with $\Lambda' \subset \Lambda$ and Λ' countable. Since $L^1_\Lambda(K) = \mathscr{M}_\Lambda(K)$, $L^1_{\Lambda'}(K)$ is a separable dual. On the other hand, suppose that $L^1_\Lambda(K)$ has RNP. Let $\mu \in \mathscr{M}_\Lambda(K)$, $f \in L^1(K)$, $\theta \in \mathscr{C}(K)$ then

$$\begin{aligned}\langle f \star \mu, \theta\rangle &= \int_K (f\star\mu)\theta\, dm = \int_K f(\theta\star\mu^-)dm \quad [15, 6.2D] \\ &= \int_K f(g)\int_K \theta(g\star y^-)d\mu^-(y)dm(g) \\ &= \int_K\int_K\int_K f(g)\theta(s)d(\delta_g\star\delta_{y^-})(s)d\mu^-(y)dm(g).\end{aligned}$$

Also,

$$\begin{aligned}\int_K f(g)\langle\delta_g\star\mu, \theta\rangle\, dm(g) &= \int_K f(g)\int_K \theta(x)d(\delta_g\star\mu)(x) \\ &= \int_K\int_K f(g)\theta_g(y)d\mu(y) \quad [15, 3.1.F] \\ &= \int_K\int_K\int_K f(g)\theta(s)d(\delta_g\star\delta_y)(s)d\mu(y)dm(g)\end{aligned}$$

and

$$\langle f\star\mu, \theta\rangle = \int_K f(g)\langle\delta_g\star\mu, \theta\rangle dm(g).$$

Let T_μ be the operator from $L^1(K)$ into $L^1_\Lambda(K)$ defined by $T_\mu(f) = f\star\mu$. $L^1_\Lambda(K)$ has RNP then the function $g \to \delta_g\star\mu$ is almost everywhere $L^1_\Lambda(K)$-valued. Thus μ is in $L^1_\Lambda(K)$.

Let G be a compact group, Σ its dual and Λ a subset of Σ. Let us denote by $L^{1C}_\Lambda(G)$ the subspace of central functions of $L^1_\Lambda(G)$. RNP and reflexivity are isomorphic properties therefore by Proposition 6.2 and Proposition 6.3, we get

COROLLARY 6.4. (1) *Λ is a central $\Lambda(1)$-set if and only if $L^{1C}_\Lambda(G)$ is reflexive.*

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(2) Λ *is a central Riesz set if and only if* $L^{1C}_{\Lambda}(G)$ *has RNP.*

COROLLARY 6.5. *If* Λ *is a central* $\Lambda(1)$*-set then* Λ *is a central Riesz set.*

This answers a question of R. G. M. Brummelhuis [5].

REMARK. In the non-commutative group case, central $\Lambda(p)$-sets are more abundant than $\Lambda(p)$-sets, see [5]. For example for all $n \geq 2$, the dual object of $SU(n)$ does not contain an infinite $\Delta(p)$-set for any $p > 0$ ([19, 20]).

7. Shapiro sets for compact groups whose center contains the circle

G. Godefroy proved the following result [10]. Let Γ be a totally ordered discrete group and Λ be a subset of Γ such that

$$\Lambda \cap \{\alpha' \leq \alpha\} \text{ is a } \Lambda(1)\text{-set for every } \alpha \in \Gamma.$$

Then Λ is a Shapiro set of Γ.

We generalize this result to compact groups whose center contains a circle group and precise a result of R. G. M. Brummelhuis [5]. Examples of such groups are the unitary group $U(n)$, isotropy groups of bounded symmetric domains. In this section G will denote a compact metrizable group whose center contains a circle group. We denote by Σ its dual and by m its Haar measure. For τ in Σ, let $H(\tau)$ denote the representation space of τ and d_τ the dimension of $H(\tau)$. If χ_τ denotes the character of τ then for $F \in \mathscr{T}(G)$ and $g \in G$

$$F(g) = \sum_{\tau \in \Sigma} d_\tau (\chi_\tau \star F)(g) = \sum_{\tau\in\Sigma} d_\tau \operatorname{tr}\{\widehat{F}(\tau)\tau(g)\}$$

where tr means trace.

Fix an injective homomorphism $\mathbb{T} = \{e^{i\theta},\ \theta \in\ ]-\pi, \pi]\} \hookrightarrow Z(G)$, the center of G; $e^{i\theta}$ will denote an element of $\mathbb{T}$ as well as an element of $Z(G)$. By Schur's lemma there exists for each τ in Σ a unique $n(\tau) \in \mathbb{Z}$ such that

$$\tau(e^{i\theta}) = e^{in(\tau)\theta} I d_{H(\tau)}, \text{ for all } e^{i\theta} \in \mathbb{T}. \tag{5}$$

If f is a function on G and g in G, the "slice" function f_g on $\mathbb{T}$ is defined by:

$$f_g(e^{i\theta}) = f(e^{i\theta} g).$$

For f in $\mathscr{T}(G)$, $f_g \in \mathscr{T}(\mathbb{T})$ and

$$f_g(e^{i\theta}) = \sum_{m\in\mathbf{Z}} \pi_m f(g) e^{im\theta}$$

where the projections π_m are defined by

$$\pi_m f(g) = \sum_{\substack{\tau\in\Sigma \\ n(\tau)=m}} d_\tau \operatorname{tr}[\hat{f}(\tau)\tau(g)].$$

Define the projection P_N on $\mathcal{T}(G)$ by

$$P_N(f) = \sum_{m\le N} \pi_m f. \tag{6}$$

The following lemma follows from [5].

LEMMA 7.1. *For all p, $0<p<1$, there exists a constant C such that for all f in $\mathcal{T}(G)$,*

$$\|P_N f\|_p \le C\|f\|_1.$$

We need the following lemma

LEMMA 7.2. *Let $\Lambda\subset\Sigma$, $f\in L^1(G)$ and $n(\Lambda)=\{n(\tau),\tau\in\Lambda\}$. If f belongs to $L^1_\Lambda(G)$ then for almost every $g\in G$, f_g belongs to $L^1_{n(\Lambda)}(\mathbb{T})$.*

PROOF. There exists a sequence $(f^{(n)})_{n\ge1}$ of trigonometric polynomials in $\mathcal{T}_\Lambda(G)$ such that $(\|f^{(n)}\|_1)_{n\ge1}$ is bounded and $(f^{(n)})_{n\ge1}$ converges to f in L^1-norm.

Up to a subsequence, we may assume that $M=\int_G\sum_n|f^{(n)}(g)-f(g)|dm(g)$ is finite; by invariance we get that, for all $e^{i\theta}$ in $\mathbb{T}$,

$$M=\int_G\sum_n|f^{(n)}(ge^{i\theta})-f(ge^{i\theta})|dm(g).$$

So

$$\int_{\mathbb{T}}\int_G\sum_n|f^{(n)}(ge^{i\theta})-f(ge^{i\theta})|dm(g)\frac{d\theta}{2\pi}\ \text{ is finite}.$$

Hence, for almost every g in G, $\int_{\mathbb{T}}\sum_n|f^{(n)}(ge^{i\theta})-f(ge^{i\theta})|d\theta/2\pi$ is finite. So that, for almost every g in G, $(f_g^{(n)})_{n\ge1}$ converges to f_g in $L^1(\mathbb{T})$. Since $f_g^{(n)}\in L^1_{n(\Lambda)}(\mathbb{T})$, it follows that $f_g\in L^1_{n(\Lambda)}(\mathbb{T})$.

Let us remark that if, for almost every $g\in G$, the slice function f_g belongs to $L^1_{n(\Lambda)}(\mathbb{T})$ then the function f belongs to $L^1_{\widetilde{\Lambda}}(G)$ where $\widetilde{\Lambda}=\{\tau\in\Sigma, n(\tau)=n(\beta)$, for some $\beta\in\Lambda\}$. We are now ready to prove the main result of this section.

THEOREM 7.3. *Let G be a compact group whose center contains a copy of the circle group. Let Λ be a subset of Σ such that*

$$\{\tau \in \Lambda,\ n(\tau) \leq N\} \text{ is a } \Lambda(1)\text{-set for every } N \in \mathbb{Z}. \tag{7}$$

Then Λ is a Shapiro subset of Σ.

PROOF. Since (7) is hereditary, it is enough to prove that Λ is nicely placed. Let $(f_n)_{n\geq 1}$ be a sequence in $B(L^1_\Lambda)$ which converges to f in $L(1, \infty)$, and let τ be in $\Sigma \setminus \Lambda$; we have to prove that $\hat{f}(\tau) = 0$.

Let $\alpha \in \Sigma$ be such that $n(\alpha) \geq n(\tau)$. Consider the projection $P_{n(\alpha)}$ defined by (6). We let $\Gamma_{n(\alpha)} = \{\delta \in \Sigma,\ n(\delta) > n(\alpha)\}$. If $\operatorname{supp} \hat{f} \subset \Lambda$ then $P_{n(\alpha)} f \in L^1_{\Lambda\setminus\Gamma_{n(\alpha)}}$. Since $\Lambda \setminus \Gamma_{n(\alpha)}$ is a $\Lambda(1)$-set, there exists $K > 0$ such that for every f in L^1_Λ,

$$\|P_{n(\alpha)} f\|_1 \leq K \|P_{n(\alpha)} f\|_{1/2}.$$

Also by Lemma 7.1, $\|P_{n(\alpha)} f\|_{1/2} \leq C\|f\|_1$. Thus $P_{n(\alpha)}$ is $\|\cdot\|_1$-continuous from L^1_Λ into $L^1_{\Lambda\setminus\Gamma_{n(\alpha)}}$ and $L^1_\Lambda = L^1_{\Lambda\setminus\Gamma_{n(\alpha)}} \oplus L^1_{\Lambda\cap\Gamma_{n(\alpha)}}$. Let (f'_n) be a subsequence of (f_n) which converges almost everywhere. Let $g'_n = P_{n(\alpha)}(f'_n)$ and $h'_n = f'_n - g'_n$. The sequences (g'_n) and (h'_n) are $\|\cdot\|_1$-bounded and thus by Lemma 1.2 [10] there exist subsequences (g''_n) and (h''_n), indexed by the same set, which converge in Cesaro mean almost everywhere and in $L(1, \infty)$ to g and h, respectively. We have $f = g + h$.

Since $\Lambda \setminus \Gamma_{n(\alpha)}$ is a $\Lambda(1)$-set, the space $L^1_{\Lambda\setminus\Gamma_{n(\alpha)}}$ is $L(1, \infty)$ closed and thus $g \in L^1_{\Lambda\setminus\Gamma_{n(\alpha)}}$ and $\hat{g}(\tau) = 0$. It remains to show that $\hat{h}(\tau) = 0$. The sequence $k_n = \frac{1}{n}\sum_{j=1}^n h''_j$ converges to h in $L(1, \infty)$ and is bounded in $L^1_{\Lambda\cap\Gamma_{n(\alpha)}}$. In particular k_n is in $L^1_{\Gamma_{n(\alpha)}}$. Thus we have to show that $L^1_{\Gamma_{n(\alpha)}}$ is nicely placed.

In fact, it suffices to prove that $L^1_{\Gamma^+}$ is nicely placed where $\Gamma^+ = \{\alpha \in \Sigma, n(\alpha) > 0\}$. Let $(f^{(n)})$ be a sequence in $B(L^1_{\Gamma^+})$ which converges to f in $L(1, \infty)$. We have to show that $\hat{f}(\alpha) = 0$ for $\alpha \notin \Gamma^+$.

Up to a subsequence we may suppose that $(f^{(n)})$ converges to f a.e. And for almost every $g \in G$, $e^{i\theta} \in \mathbb{T}$, $f_g^{(n)}(e^{i\theta})$ tends to $f_g(e^{i\theta})$. By Lemma 7.2, we have that $f_g^{(n)} \in L^1_{n(\Gamma^+)}$ and $(\|f_g^{(n)}\|_1)$ is bounded. Since $n(\Gamma^+)$ is nicely placed, $f_g \in L^1_{n(\Gamma^+)}$ for almost every g and by the remark after Lemma 7.2 $f \in L^1_{\Gamma^+}$. This proves the theorem.

Our result improves Theorem 2.1 of [5].

We now give some applications of this theorem.

First, we give a non commutative extension of the Mooney-Havin theorem ([13, 18]).

COROLLARY 7.4. *Let G be a compact group whose center contains a copy of circle group. Let Λ be a subset of Σ satisfying* (7) *then the space $L^1/L^1_\Lambda(G)$ is weakly sequentially complete.*

PROOF. The result follows from Theorem 7.3 and Lemma 1.8 of [10].

We are now interested in the "central version" of Theorem 7.3. Let us note that if (f_n) is a bounded sequence in $L^{1C}_\Lambda(G)$ which converges to f in $L(1, \infty)$ then f is also a central function. And if f is a central trigonometric polynomial then the projection $P_N f$ is also a central function.

THEOREM 7.5. *Let G be a compact group whose center contains a copy of the circle group. Let Λ be a subset of Σ such that*

$$\{\tau \in \Lambda, n(\tau) \leq N \text{ is a central } \Lambda(1)\text{-set for every } N \in \mathbb{Z}.$$

Then Λ is a central Shapiro subset of Σ.

Let us note that under these assumptions Λ is central Riesz. Another easy consequence of the theorem is the following corollary.

COROLLARY 7.6. *Let G be a compact group whose center contains a copy of the circle group. Let Λ be a subset of Σ such that*

(1) *For each $m \in \mathbb{Z}$ the set $\{\tau \in \Lambda : n(\tau) = m\}$ is a $\Lambda(1)$-set.*

(2) *The set $\{n(\tau) : \tau \in \Lambda\}$ is bounded from below.*

Then Λ is a Shapiro subset of Σ.

Let us mention that under these assumptions R. G. M. Brummelhuis proved that Λ is a Riesz subset of Σ [3].

REMARK. Let G be a compact group and Λ be a subset of its dual. We denote by $\tilde{\Lambda}$ the set of α's in Λ such that the restriction of $F_\alpha : f \to \hat{f}(\alpha)$ to $B(^{1C}_\Lambda(G))$ is L^p-continuous $(0 < p < 1)$. Then we can show that for every central Shapiro set Λ, one has $\Lambda = \tilde{\Lambda}$. We can also give another "central" extension of Theorem 3.2 of [3].

PROPOSITION 7.7. *Let G be a compact group whose center contains a copy of the circle group. Let Λ be a subset of Σ such that*

(1) *For each $m \in \mathbb{Z}$ the set $\{\tau \in \Lambda, n(\tau) = m\}$ is central Shapiro.*

(2) *The set $\{n(\tau), \tau \in \Lambda\}$ is bounded from below.*

Then Λ is a central Riesz subset of Σ.

PROOF. Let μ be in $\mathscr{M}_\Lambda^C(G)$ and (F_n) be a sequence of central trigonometric polynomials satisfying Theorem 3.1. Then $F_n \star \mu$ belongs to $\mathscr{T}_\Lambda^C(G)$ and μ_a belongs to the set C which is the closure of the set $\|\mu\| B(\mathscr{T}_\Lambda^C)$ in $\|\cdot\|_p$ $(0 < p < 1)$. Then for all $F \in B(\mathscr{T}^C(G))$, $F \star \mu_a$ also belongs to C. Following [3] we consider for $m \in \mathbb{Z}$ the projection $\pi_m : \mathscr{T}(G) \to \mathscr{T}(G)$ defined by

$$(\pi_m f)(k) = \int_{-\pi}^{\pi} f(e^{i\theta}k)e^{-im\theta}\, d\theta/2\pi = \sum_{n(\tau)=m} d_\tau(\chi_\tau \star f)(k).$$

Since the set $\{n(\tau), \tau \in \Lambda\}$ is bounded from below, π_m is L^p-continuous on $\mathscr{T}_\Lambda(G)(0 < p < 1)$ [3]. Let (f_n) be a sequence in $\mathscr{T}_\Lambda^C(G)$ with $(\|f_n\|_1)$ bounded and let f be in $\mathscr{T}(G)$ such that (f_n) converges to f in $\|\cdot\|_p$ $(0 < p < 1)$ then $(\pi_m(f_n))$ converges to $\pi_m(f)$ in $\|\cdot\|_p$. Moreover we have that spec $\pi_m(f) \subset \{\tau \in \Lambda, n(\tau) = m\}$ since the set $\{\tau \in \Lambda, n(\tau) = m\}$ is central Shapiro. Therefore f belongs to $\mathscr{T}_\Lambda^C(G)$. It follows that for all $F \in \mathscr{T}^C(G)$, $F \star \mu_a$ belongs to $\mathscr{T}_\Lambda^C(G)$ and,

$$\text{for all } F \in \mathscr{T}^C(G),\ F \star \mu_s \in \mathscr{T}_\Lambda^C(G). \tag{8}$$

For each $\sigma \in \Lambda$ and each $k \in G$, consider the linear functional:

$$f \to d_\sigma(\chi_\sigma \star f)(k). \tag{9}$$

Let us denote by Y the space $T^C_{\{\tau \in \Lambda, n(\tau)=n(\sigma)\}}$. Let (f_n) and f be in Y such that $(\|f_n\|_1)$ is bounded and (f_n) converges to f in $\|\cdot\|_p$ $(0 < p < 1)$. We have that $(\hat{f}_n(\tau))$ converges to $\hat{f}(\tau)$ for $\tau \in \{\tau \in \Lambda, n(\tau) = n(\sigma)\}$ since this set is central Shapiro. And $((\chi_\sigma \star f_n)(k) = \mathrm{tr}(\hat{f}_n(\sigma)\sigma(k))$ converges to $(\chi_\sigma \star f)(k)$. Hence the functionals (9) are L^p-continuous on the bounded sets of Y. From the L^p-continuity of the projection $\pi_{n(\sigma)}$ it follows that

(10) functionals (9) are L^p-continuous on bounded subsets of $\mathscr{T}_\Lambda^C(G)$.

(8) and (10) imply that $\mu_s = 0$ (see [3] for more details).

We give some applications to the unit sphere in $\mathbb{C}^n$, homogeneous spaces and bounded symmetric domains.

Let $S = S_{2n-1} = \{z \in \mathbb{C}^n, |z| = 1\}$ be the unit sphere in $\mathbb{C}^n$. S_1 is just the unit circle $\mathbb{T}$. S_3 can be identified with $SU(2)$ and we can then apply the results of Section 5. For $n > 2$, S_{2n-1} does not have the structure of a compact group. We will extend the results of Section 7 to S_{2n-1} and more generally to homogeneous spaces.

Let us recall some basic facts about S_{2n-1} [3]. For non-negative integers p and q, we denote by $H(p, q)$ the vector space of all harmonic homogeneous polynomials on $\mathbb{C}^n$ that have degree p in z and degree q in $\overline{z}$. We let σ be the rotation-invariant positive Borel measure on S for which $\sigma(S) = 1$. The space $L^2(S, \sigma)$ is the direct sum of the pairwise orthogonal spaces $H(p, q)$. Let π_{pq} be the orthogonal projection of $L^2(S, \sigma)$ onto $H(p, q)$. Fix p and q. To every $z \in S$ corresponds a unique K_z in $H(p, q)$ that satisfies

$$(\pi_{pq}f)(z) = \int_S f\overline{K}_z \, d\sigma \quad (f \in L^2(S, \sigma)).$$

We can define $\pi_{pq}\mu$ when $\mu \in \mathscr{M}(S)$. Consider now

$$\operatorname{spec} \mu = \{(p, q) \in \mathbb{N} \times \mathbb{N}, \ \pi_{pq}\mu \neq 0\}.$$

Let τ_{pq} be the restriction of the left regular representation of $U(n)$ on $L^2(S, \sigma)$ to $H(p, q)$, that is

$$(\tau_{pq}(U)f)(l) = f(U^{-1}l); \ f \in H(p, q), \ U \in U(n), \ l \in S.$$

The τ_{pq} are pairwise non-equivalent and they represent all irreducible representations of $U(n)$ which occur in $L^2(S, \sigma)$, and $n(\tau_{pq}) = q - p$, where $n(\tau_{pq})$ is defined by the relation (5).

From Theorem 7.3 we then get,

Corollary 7.8. *Let $\Delta \subset \mathbb{N} \times \mathbb{N}$ be such that for all $N \in \mathbb{Z}$*

$$\{(p, q) \in \Delta, q - p \leq N\} \text{ is a } \Lambda(1)\text{-set}.$$

Then Δ is a Shapiro set.

We can improve Theorem 1.1 of [3]:

Corollary 7.9. *Let $\Delta \subset \mathbb{N} \times \mathbb{N}$ be such that*

(1) *For each $N \in \mathbb{Z}$, $\{(p, q) \in \Delta, q - p = N\}$ is finite,*

(2) *$\{q - p, (p, q) \in \Delta\}$ is bounded from below (or above).*

Then Δ is a Shapiro set.

Let K be a compact group whose center contains a copy of the unit circle. Let H be a closed subgroup of K. We will extend Corollary 7.8 to the homogeneous space K/H. Functions (resp. measures) on K/H can be identified with functions (resp. measures) on K which are right H-invariant. Let σ be the K-invariant measure on K/H for which $\sigma(K/H) = 1$. If $\mu \in \mathscr{M}(K/H)$

is a right H-invariant measure on K then $\pi_\tau \mu = d_\tau \chi_\tau \star \mu$ ($\tau \in \Sigma$ the dual of K) is again H-invariant. The map $\pi_\tau : f \to d_\tau \chi_\tau \star f (\tau \in \Sigma)$ is an orthogonal projection of $L^2(K/H, \sigma)$ which is different from 0 if and only if τ occurs in the left regular representation of K on $L^2(K/H, \sigma)$. For $f \in L^1(K/H, \sigma)$, consider

$$\operatorname{spec} f = \{\tau \in \Sigma, \ \pi_\tau f \neq 0\}.$$

The " homogeneous version" of Theorem 7.3 (see also Theorem 3.7 [3]) is

COROLLARY 7.10. *Let* $\Delta \subset \Sigma$ *be such that for all* $N \in \mathbb{Z}$,

$$\{\tau \in \Delta, \ n(\tau) \leq N\} \text{ is a } \Lambda(1)\text{-set}.$$

Then for every $\Gamma \subset \Delta$, $B(L^1_\Gamma(K/H), \sigma)$ *is* L^p*-closed* $(0 < p < 1)$.

Note that for $K = U(n)$, $H = U(n-1)$ then $K/H = S_{2n-1}$.

Following [3] we can apply these results to bounded symmetric domains. Let $\Omega \in \mathbb{C}^n$ be a bounded symmetric domain, we may assume Ω to be convex and circular. Let K be the stabilizer of 0 in the component of the identity of the group of holomorphic automorphisms of Ω. The center of K contains a copy of the circle. We can apply Corollary 7.10 to the Bergman-Shilov boundary S of Ω. Let $H^2(S)$ be the closure in $L^2(S, \sigma)$ of the holomorphic polynomials restricted to S. $H^2(S)$ is K-invariant under the left regular representation of K in $L^2(S, \sigma)$. Let $\widehat{K}_{Hol}$ be the set of irreducible representations of K which occur in $H^2(S)$. Let $H(p)$ be the space of holomorphic polynomials which are homogeneous of degree p $(p \in \mathbb{N})$ restricted to S. $H(p)$ is K-invariant and decomposes as a finite sum of representations in $\widehat{K}_{hol}$. If $\tau \in \widehat{K}_{Hol}$ then $n(\tau) \leq 0$ and $n(\tau) = -p$ if τ occurs in $H(p)$ (see (5) for the definition of $n(\tau)$). Therefore $\widehat{K}_{Hol}$ satisfies conditions (1) and (2) of Corollary 7.9 and $\widehat{K}_{Hol}$ is a Shapiro set.

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References

[1] G. F. Bachelis and S. E. Ebenstein, 'On $\Lambda(p)$-sets', *Pacific J. Math.* **54** (1974), 35–38.
[2] J. Boclé, 'Sur la théorie ergodique', *Ann. Inst. Fourier* (*Grenoble*) **10** (1960), 1–45.

[3] R. G. M. Brummelhuis, 'An F. and M. Riesz theorem for bounded symmetric domains', *Ann. Inst. Fourier* (*Grenoble*) **37** (1987), 139–150.

[4] ——, 'Variations on a theme of Frederic and Marcel Riesz', Thesis.

[5] ——, 'A note on Riesz sets and lacunary sets', *J. Austral. Math. Soc.* (*Series A*) **48** (1990), 57-65.

[6] J. Diestel, *Geometry of Banach spaces. Selected topics*, Lecture Notes in Math. **485** (Springer, Berlin, 1975).

[7] R. E. Dressler and L. Pigno, 'Une remarque sur les ensembles de Rosenthal et Riesz', Note aux C.R.A.S. **280** (1975), 1281–1282.

[8] C. F. Dunkl and D. E. Ramirez, 'A family of countable compact P_*- hypergroups', *Trans. Amer. Math. Soc.* **202** (1975), 338–356.

[9] J. J. F. Fournier and K. A. Ross, 'Random Fourier series on compact abelian hypergroups', *J. Austral. Math. Soc.* (*Series A*) **37** (1984), 45–81.

[10] G. Godefroy, 'On Riesz subsets of abelian discrete groups', *Israel J. Math.* **61** (1988), 301–331.

[11] ——, 'On coanalytic families of sets in harmonic analysis', Illinois J. Math. **35** (1991), 241–249.

[12] P. Halmos, *Measure theory* (Van Nostrand, Princeton, 1950).

[13] V. P. Havin, 'Weak completeness of the space L^1/H_0^1', *Vestnik Leningard Univ.* **13** (1973), 73–81 (in Russian).

[14] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis* II (Springer, Berlin, 1970).

[15] R. I. Jewett, 'Spaces with an abstract convolution of measures', *Adv. Math.* **18** (1975), 1–101.

[16] F. Lust-Piquard, 'Ensembles de Rosenthal et ensembles de Riesz', *C.R. Acad. Sci. Paris* **282** (1976), 833–835.

[17] Y. Meyer, 'Spectres des mesures et mesures absolument continues', *Studia Math.* **30** (1968), 87–99.

[18] M. C. Mooney, ' A theorem on bounded analytic functions', *Pacific J. Math.* **43** (1972), 457–463.

[19] J. F. Price, 'Non a sono inoemi infiniti di tipo $\Lambda(p)$ per $SU(2)$', *Boll. Un. Mat. Ital.* **4(4)** (1971), 879–881.

[20] D. Rider, '$SU(n)$ has no infinite local Λ_p sets', *Boll. Un. Mat. Ital.* **12(4)** (1975), 155–160.

[21] W. Rudin, *Real and complex analysis* (Mc Graw-Hill, New York, 1966).

[22] J. Shapiro, 'Subspaces of $L^p(G)$ spanned by characters, $0 < p < 1$', *Israel J. Math.* **29** (1978), 248–264.

[23] R. Spector, 'Aperçu de la théorie des hypergroupes', in: *Analyse harmonique sur les groupes de Lie. sém. Nancy-Strasbourg* 1973-1975, Lecture Notes in Math. **497** (Springer, Berlin, 1976) pp. 643–673.

[24] R. C. Vrem, 'Harmonic analysis on compact hypergroups', *Pacific J. Math.* **85** (1979), 239–251.

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