ON SQUARE ROOTS AND LOGARITHMS OF SELF-ADJOINT OPERATORS

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All operators considered in this paper are bounded and linear (everywhere defined) on a Hilbert space. An operator A will be called a square root of an operator B if

$$A^2 = B. \quad \text{.........(1)}$$

A simple sufficient condition guaranteeing that any solution A of (1) be normal whenever B is normal was obtained in [**1**], namely : *If B is normal and if there exists some real angle θ for which* $\operatorname{Re}(Ae^{i\theta}) \geqslant 0$, *then* (1) *implies that A is normal.* Here, $\operatorname{Re}(C)$ denotes the real part $\frac{1}{2}(C+C^*)$ of an operator C.

The object of the present note is to use the above result to generalize the well-known fact that a (self-adjoint) non-negative operator has a unique non-negative square root (cf. [**3**], p. 256, also [**2**], p. 725) and to obtain a certain uniqueness theorem for logarithms of positive self-adjoint operators. The following will be proved :

(I) *If B is a non-negative self-adjoint operator, and if A is any solution of* (1) (*A not assumed to be self-adjoint or even normal*) *satisfying* $\operatorname{Re}(A) \geqslant 0$, *then necessarily A is the* (*unique*) *non-negative self-adjoint square root of B.*

(II) *If A is a logarithm of a positive self-adjoint operator $B = \int \lambda \, dE$, so that $e^A = B$ (>0), and if*

$$\| A \| \leqslant 2 \log 2, \quad \text{.........(2)}$$

then necessarily A is the self-adjoint operator

$$A = \int \log \lambda \, dE \quad (\log \lambda \text{ real}). \quad \text{.........(3)}$$

The proof of (I) follows from an application of the italicized assertion in the first paragraph. For, since B is self-adjoint and hence normal, A is normal. Since the square of any number in the spectrum of A is in the spectrum of B, it follows that the spectrum of A is real. Therefore A is self-adjoint and, in view of the assumption $\operatorname{Re}(A) \geqslant 0$, is non-negative (hence uniquely determined). This completes the proof of (I).

In order to prove (II), it will first be shown that

$$e^{A/2} = B^{1/2}, \quad \text{.........(4)}$$

where $B^{1/2}$ denotes the (unique) positive square root of B. To this end, note that

$$\| e^{A/2} - I \| \leqslant e^{\|A\|/2} - 1 \leqslant 1,$$

the second inequality following from (2). Hence $\operatorname{Re}(e^{A/2}) \geqslant 0$ and (4) now follows from (I).

Since the inequality (2) holds also if A is replaced by $A/2^n$ for $n = 1, 2, \dots$, it follows that $e^{A/2^n} = B^{1/2^n}$ for $n = 0, 1, 2, \dots$. Consequently, $e^{rA} = H^r$ for any rational number of the form $r = m/2^n$ $(n = 0, 1, 2, \dots;\ m = 0, \pm 1, \pm 2, \dots)$ and hence, by continuity,

$$e^{tA} = B^t = \int \lambda^t \, dE$$

for every real t. A differentiation with respect to t of this (operator) identity yields

$$Ae^{tA} = \left(\int \log \lambda \, dE\right) B^{t};$$

hence for $t = 0$, the relation (3), at least for some determination of $\log \lambda$. But $\| A \| \geqq | \log \lambda |$ for every λ in the (real) spectrum of B and so relation (2) implies that $\log \lambda$ is real. This completes the proof of (II).

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