Canad. Math. Bull. Vol. **26** (3), 1983

A LEMMA ON PROJECTIVE GEOMETRIES AS MODULAR AND/OR ARGUESIAN LATTICES

BY
ALAN DAY(*)

ABSTRACT. A projective geometry of dimension $(n-1)$ can be defined as modular lattice with a spanning n-diamond *of atoms* (i.e.: $n+1$ atoms in general position whose join is the unit of the lattice). The lemma we show is that one could equivalently define a projective geometry as a modular lattice with a spanning n-diamond that is (a) is generated (qua lattice) by this n-diamond and a coordinatizing diagonal and (b) every non-zero member of this coordinatizing diagonal is invertible. The lemma is applied to describe certain freely generated modular and Arguesian lattices.

§1. **Introduction.** A projective geometry of dimension $(n-1)$ can be defined as a modular lattice with a spanning n-diamond *of atoms* (see Crawley and Dilworth [2] or Day [4]). In this note we provide another necessary and sufficient condition for a modular lattice with a spanning n-diamond to be a projective geometry of dimension $(n-1)$ and apply it to prove that projective geometries of prime order and dimension ≥ 3 (respectively $=2$) are projective modular (resp. Arguesian) lattices. The first aforementioned result is due to Freese [7].

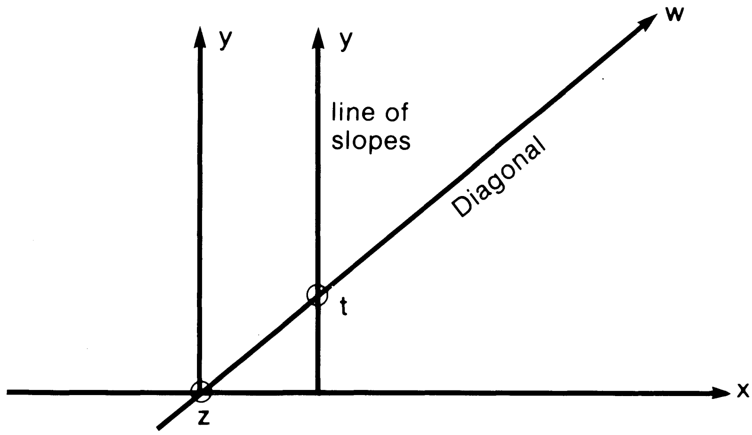
Let M be a bounded modular lattice; a spanning n-diamond in M is a sequence $\mathbf{d}=(d_1,\ldots,d_{n+1})$ in M satisfying for all $i\neq j=1,\ldots,n+1$, $(nD1)$ $\bigvee(d_k: k\neq i)=1$ and $(nD2)$ $d_i\wedge\bigvee(d_k: k\neq i,j)=0$. Although there is complete symmetry in the definition of a spanning n-diamond, we will write $\mathbf{d}=(x_1,\ldots,x_{n-1},z,t)$, $h=\vee(x_i; i=1,\ldots,n-1)$, the "hyperplane at infinity", $w=h\wedge(z\vee t)$, the infinity point on the line $z\vee t$; $A=\{p\in M: p\vee h=1$ and $p\wedge h=0\}$, the affine plane; and $D=\{a\in A: a\leq z\vee t\}=\{a\in L: a\vee w=z\vee t$ and $a\wedge w=0\}$, the coordinatizing diagonal which will become the (planar ternary) ring. The affine plane A can now be coordinatized by D in that there are inverse bijections between A and D^{n-1} viz: $p\mapsto((z\vee t)\wedge(\bar{x}_i\vee p))$ and $(a_i)\mapsto\bigwedge(\bar{x}_i\vee a_i)$ where $\bar{x}_i=\bigvee(x_j: j\neq i)$.

We will need to examine the case where $n=2$ (i.e. the projective plane) more closely, so let (x, y, z, t) be a spanning 3-diamond in a modular lattice M.

Received by the editors September 29, 1981 and, in revised form, August 14, 1982.
(*) This research was supported by an NSERC Operating Grant, A-8190.
AMS Subject Codes: 06C05, 51A05
Received by the editors September 29, 1981 and, in revised form, August 14, 1982.
© 1983 Canadian Mathematical Society

https://doi.org/10.4153/CMB-1983-046-1 Published online by Cambridge University Press

We can visualize this as the affine plane A with $h = x \vee y$, the line at infinity as:



The projective isomorphism $[0, z \vee w] \overset{x}{\underset{\wedge}{=}} [0, z \vee y]$ defines for each $b \in D$ a y-intercept point $b_0 = (z \vee y) \wedge (x \vee b)$ satisfying for $p \leq z \vee y$, $p = b_0$ for some $b \in D$ if and only if $p \vee y = z \vee y$ and $p \wedge y = 0$. Similarly the projective isomorphisms $[0, z \vee w] \overset{x}{\underset{\wedge}{=}} [0, y \vee t] \overset{z}{\underset{\wedge}{=}} [0, x \vee y]$ provide a "slope point at infinity" $b \overset{x}{\underset{\wedge}{=}} b_1 \overset{z}{\underset{\wedge}{=}} b_\infty$ for each $b \in D$. Note that $z_\infty = x$ and $t_\infty = w$. Furthermore $q \leq x \vee y$ is such a slope point if and only if it is a complement of y (in $[0, x \vee y]$). We now can define the ternary operator on D by:

$$T(a, m, b) = (z \vee t) \wedge \{x \vee [(y \vee a) \wedge (m_\infty \vee b_0)]\}, \quad a, b, c \in D.$$

Easy (modular) calculations show that T is indeed a function from D into D. We now can define multiplication and addition on D by:

$$a \otimes b = T(a, b, z)$$

$$a \oplus b = T(a, t, b).$$

Left and right differences can now be defined by

$$a \,\Delta_l c = (z \vee t) \wedge \{x \vee [(y \vee z) \wedge \{w \vee [(y \vee a) \wedge (x \vee c)]\}]\}$$

and

$$c \,\Delta_r b = (z \vee t) \wedge \{y \vee [(x \vee c) \wedge (w \vee b_0)]\}.$$

These make $(D; \oplus, z)$ into a loop since $c = a \oplus b$ iff $a = c \,\Delta_r b$ iff $b = a \,\Delta_l c$.

In general multiplication does *not* have left and right division as operations in D. One need only consider $\mathscr{L}({}_R R^3)$, the lattice of left submodules of a given "bad" ring R. If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of R, $x = R\mathbf{e}_1$, $y = R\mathbf{e}_2$, $z = R\mathbf{e}_3$ and $t = R(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$, we obtain $D = \{\bar{a} = R(a\mathbf{e}_1 + a\mathbf{e}_2 + \mathbf{e}_3) : a \in R\}$ with $T(\bar{a}, \bar{m}, \bar{b}) = \overline{am + b} \cdot a \in R$ is then invertible if and only if $z \wedge \bar{a} = 0$ and

$z \vee \bar{a} = z \vee t$. In general we define $\mathrm{Inv}(D) = \{a \in D : z \vee a = z \vee t \text{ and } z \wedge a = 0\}$ and can show $a \in \mathrm{Ind}(D)$ if and only if there exists $b, c \in D$ with $b \otimes a = t = a \otimes c$ (t is the unit of $\otimes$!).

Now if M is a projective plane then $\mathrm{Inv}(D) = D \setminus \{z\}$ since the meet of any two distinct points is 0. Furthermore one can obtain every point of the geometry by lattice operations from the points $D \cup \{x_1, \ldots, x_{n-1}\}$. Our lemma is the converse.

LEMMA. *Let M be a modular lattice with spanning n-diamond $\langle x_1, \ldots, x_{n-1}, z, t\rangle$ and suppose $M = \langle D \cup \{x_1, \ldots, x_{n-1}\}\rangle$ the lattice generated by $D \cup \{x_1, \ldots, x_{n-1}\}$; then M is a projective geometry of dimension $(n-1)$ if and only if* $\mathrm{Inv}(D) = D \setminus \{z\}$.

§2. **The case** $n = 3$

CLAIM 1. For $a, b \in D$, the following are equivalent:

$$(1) \quad a \wedge b = 0 \quad \text{and} \quad a \vee b = z \vee t$$

$$(2) \quad a\ \Delta_l b \in \mathrm{Inv}(D)$$

$$(3) \quad b\ \Delta_r a \in \mathrm{Inv}(D).$$

Proof. Modular lattice calculations give $z \wedge (a\ \Delta_l b) = z \wedge (b\ \Delta_r a) = z \wedge (w \vee (a \wedge b))$ and $z \vee (a\ \Delta_l b) = z \vee (b\ \Delta_r a) = z \vee (w \wedge (a \vee b))$. Therefore $a\ \Delta_l b$ (resp. $b\ \Delta_r a$) is in $\mathrm{Inv}(D)$ if and only if $w \wedge (a \vee b) \leq w \leq w \vee (a \wedge b)$ are complements of z in $[0, z \vee t]$ if and only if $w \wedge (a \vee b) = w = w \vee (a \wedge b)$ by modularity if and only if $a \vee b = z \vee t$ and $a \wedge b = 0$.

COROLLARY 1. *If $D = \mathrm{Inv}(D) \cup \{z\}$, then $\{0, z \vee t, w\} \cup D$ is a sublattice of M isomorphic to M_α where $\alpha = 1 + |D|$.*

COROLLARY 2. *If $D = \mathrm{Inv}(D) \cup \{z\}$, then $\{0, y \vee z, y\} \cup D_0$ is a sublattice of M isomorphic to M_α where $\alpha = 1 + |D|$ and $D_0 = \{a_0 : a \in D\}$.*

COROLLARY 3. *If $D = \mathrm{Inv}(D) \cup \{z\}$, then $\{0, x \vee y, y\} \cup D_\infty$ is a sublattice of M isomorphic to M_α where $\alpha = 1 + |D|$ and $D_\infty = \{a_\infty : a \in D\}$.*

We now need to represent M as a projective plane by defining points, lines and incidences. We let

$$P = A \cup \{y\} \cup D_\infty$$

$$L = \{h = x \vee y\} \cup \{y \vee a : a \in D\} \cup \{m_\infty \vee b_0 : m, b \in D\}$$

and pIl iff $p \leq l$. To complete the proof for $n = 3$ we must show that $(P, L, \leq)$ is a projective geometry and that $M = \{0, 1\} \cup P \cup L$.

CLAIM 2. $p \leq h$ iff $p \in \{y\} \cup D_\infty$.

Proof. Trivial as $p \wedge h = 0$ for all $p \in A$.

https://doi.org/10.4153/CMB-1983-046-1 Published online by Cambridge University Press

Claim 3. $p \leq y \vee a$ iff $p \in \{y\} \cup \{(y \vee a) \wedge (x \vee b) : b \in D\}$.

Proof. Easy.

Claim 4. $p \leq m_\infty \vee b_0$ iff $p \in \{m_\infty\} \cup \{(y \vee a) \wedge (x \vee T(a, m, b)) : a \in D\}$.

Proof. Clearly any point on $m_\infty \vee b_0$ besides m_∞ must come from A, and for such a point

$$\begin{aligned}
(y \vee a) \wedge (x \vee c) \leq m_\infty \vee b_0 \quad & \text{iff} \quad (y \vee a) \wedge (x \vee c) \leq (y \vee a) \wedge (m_\infty \vee b_0) \\
& \text{iff} \quad x \vee c \leq x \vee [(y \vee a) \wedge (m_\infty \vee b_0)] \\
& \text{iff} \quad c \leq x \vee [(y \vee a) \wedge (m_\infty \vee b_0)] \\
& \text{iff} \quad c \leq T(a, m, b) \\
& \text{iff} \quad c = T(a, m, b) \quad \text{by modularity.}
\end{aligned}$$

Claim 5. For any $p \in P$ and $l \in L$ either $p \leq l$, or $p \vee l = 1$ and $p \wedge l = 0$.

Proof. We will prove this claim only for $p = (y \vee a) \wedge (x \vee c)$ and $l = m_\infty \vee b_0$ where $c \neq T(a, m, b)$.

$$\begin{aligned}
p \vee l &= p \vee [(y \vee a) \wedge (m_\infty \vee b_0)] \vee m_\infty \\
&= p \vee [(y \vee a) \wedge (x \vee T(a, m, b))] \vee m_\infty \\
&= [(y \vee a) \wedge (x \vee c \vee T(a, m, b))] \vee m_\infty \\
&= y \vee a \vee m_\infty \quad \text{since } c \neq T(a, m, b) \\
&= 1 \\
p \wedge l &= (x \vee c) \wedge (y \vee a) \wedge (m_\infty \vee b_0) \\
&= (y \vee a) \wedge (x \vee c) \wedge (x \vee T(a, m, b)) \\
&= (y \vee a) \wedge x, \quad \text{since } c \neq T(a, m, b) \\
&= 0
\end{aligned}$$

Claim 6. The join (in M) of distinct points is a line.

Proof. We will consider the two non-trivial cases and leave the rest to the reader. If $p = (y \vee a) \wedge (x \vee b)$ and $q = (y \vee c) \wedge (x \vee d)$ are distinct then $(a, b) \neq (c, d)$. If $a = c$, $p \vee a = y \vee a \in L$ or if $b = d$, $p \vee q = x \vee b = z_\infty \vee b_0 \in L$. Therefore we may assume $a \neq c$ and $b \neq d$. With these assumptions one can easily show that

(i) $(y \vee z) \wedge (p \vee q) \in D_0$, as a complement of y
(ii) $(y \vee x) \wedge (p \vee q) \in D_\infty$ and
(iii) $p \vee q = [(y \vee z) \wedge (p \vee q)] \vee [(y \vee x) \wedge (p \vee q)] \in L$.

If $p = (y \vee a) \wedge (x \vee b)$ and $q = m_\infty$ then easily $(y \vee z) \wedge (p \vee q) \in D_0$ and $p \vee q = [(y \vee z) \wedge (p \vee q)] \vee q \in L$.

Claim 7. The meet (in M) of distinct lines is a point.

https://doi.org/10.4153/CMB-1983-046-1 Published online by Cambridge University Press

Proof. We have used already that $(y\vee a)\wedge(m_\infty\vee b_0)=(y\vee a)\wedge(x\vee T(a,m,b))$ and therefore are left with only one other non-trivial case: $m_\infty\vee b_0$ and $n_\infty\vee c_0$ with $(m,b)\neq(n,c)$. If however $m=n$, then the meet of the lines is m_∞. Therefore assume $m\neq n$. We complete the proof by showing that $(m_\infty\vee b_0)\wedge(n_\infty\vee c_0)\in A\subseteq P$.

$$\begin{aligned} h\wedge(m_\infty\vee b_0)\wedge(n_\infty\vee c_0) &= [m_\infty\vee(b_0\wedge h)]\wedge[n_\infty\vee(c_0\wedge h)]\\ &= m_\infty\wedge n_\infty\\ &= 0 \quad \text{as} \quad m\neq n\\ h\vee[(m_\infty\vee b_0)\wedge(n_\infty\vee c_0)] &= (m_\infty\vee n_\infty)\vee[(m_\infty\vee b_0)\\ &\qquad \wedge(n_\infty\vee c_0)] \quad \text{as} \quad m\neq n\\ &= [m_\infty\vee n_\infty\vee b_0]\wedge[n_\infty\vee m_\infty\vee c_0]\\ &= 1 \end{aligned}$$

We have therefore that $\{0,1\}\cup P\cup L$ is a sublattice of M containing $D\cup\{x,y\}$. Since M is assumed to be generated by $D\cup\{x,y\}$, $M=\{0,1\}\cup P\cup L$ and M is a projective plane.

§3. **The case** $n\geq 4$. The proof this case is by induction on the statement: If $\mathbf{y}=(y_2,\ldots,y_n,z,t)$ is an n-diamond in a modular lattice M with

(1) $h_{\mathbf{y}}=\bigvee(y_i:2\leq i\leq n)$

(2) $w_{\mathbf{y}}=h_{\mathbf{y}}\wedge(z\vee t)$

(3) $A_{\mathbf{y}}=\{p\in M: p\vee h_{\mathbf{y}}=z\vee h_{\mathbf{y}},\ p\wedge h_{\mathbf{y}}=0\}$

(4) $D_{\mathbf{y}}=\{p\in A_{\mathbf{y}}: p\leq z\vee t\}=\mathrm{Inv}(D_{\mathbf{y}})\cup\{z\}$

then the sublattice of M generated by $D\cup\{y_2,\ldots,y_n\}$ is a projective geometry of dimension $(n-1)$ whose point (qua geometry) set includes $A_{\mathbf{y}}\cup\{h_{\mathbf{y}}\wedge(p\vee q): p,q\in A_{\mathbf{y}}\}$.

This statement is true for $n=3$ so let $\mathbf{x}=(x_1,x_2,\ldots,x_n,z,t)$ be an $(n+1)$-diamond in a modular lattice M, and let $y_2=(x_1\vee x_2)\wedge(\bar{x}_{12}\vee z\vee t)$ and $y_i=x_i$ for $3\leq i\leq n$.

The reader may easily verify that $\mathbf{y}=(y_2,\ldots,y_n,z,t)$ is an n-diamond in M with

(1) $h_{\mathbf{y}}=h_{\mathbf{x}}\wedge(\bar{x}_{12}\vee z\vee t)=h\wedge(\bar{x}_{12}\vee z\vee t)$

(2) $w_{\mathbf{y}}=w_{\mathbf{x}}(=w)$

(3) $A_{\mathbf{y}}=\{(\bar{x}_{12}\vee a_2)\wedge\bigwedge(\bar{x}_i\vee a_i:3\leq i\leq n): a_2, a_i\in D\}$

(4) $D_{\mathbf{y}}=D_{\mathbf{x}}=D=\mathrm{Inv}(D)\cup\{z\}$

By induction hypothesis we have that $M_{12}=\langle D\cup\{y_2,\ldots,y_n\}\rangle$ is a projective geometry of dimension n. Also $M_{12}\leq[0,\bar{x}_{12}\vee z\vee t]$.

https://doi.org/10.4153/CMB-1983-046-1 Published online by Cambridge University Press

Now consider the projective isomorphism $\phi:[0, \bar{x}_{12} \vee z \vee t] \overset{x_1}{\underset{\wedge}{=}} [0, \bar{x}_1 \vee t] \underset{\wedge}{=} [0, h] \cdot \phi[M_{12}] = H$ is therefore a projective geometry of dimension $(n-1)$ generated by $\{h \wedge [z[(\bar{x}_1 \vee t) \wedge (x_1 \vee a)]]: a \in D\} \cup \{x_2, \ldots, x_n\}$. Note that this set includes $x_1 = \phi(z)$ and $w = \phi(t)$. Moreover the set $A_{\mathbf{y}}$ is mapped precisely onto the complements of $\bar{x}_1$ in $[0, h]$ since $\phi(h_{\mathbf{y}}) = \bar{x}_1$.

CLAIM 1. For $p, q \in A(=A_{\mathbf{x}})$, $h \wedge (p \vee q)$ is a point of H.

Proof. If $p, q \in A$ then there exists $\mathbf{a}, \mathbf{b} \in D^n$ with $p = \bigwedge (\bar{x}_i \vee a_i)$ and $q = \bigwedge (\bar{x}_i \vee b_i)$. Moreover if $a_1 \neq b_1$ $\bar{x}_1 \vee [h \wedge (p \vee q)] = h$ and $\bar{x}_1 \wedge (h \wedge (p \vee q)) = 0$. Therefore $h \wedge (p \vee q)$ is a point in H.

If $a_1 = b_1 = c$, then $p \vee q = [\bigwedge^{2,n} (\bar{x}_i \vee a_i) \vee \bigwedge^{2,n} (\bar{x}_i \vee b_i)] \wedge (\bar{x}_1 \vee c)$ and $h \wedge (p \vee q) = \bar{x}_1 \wedge [\bigwedge^{2,n} (\bar{x}_i \vee a_i) \vee \bigwedge^{2,n} (\bar{x}_i \vee b_i)] = (h_{\mathbf{y}} \wedge (p_{\mathbf{y}} \vee q_{\mathbf{y}}))$ where $p_{\mathbf{y}} = (\bar{x}_{12} \vee a_2) \wedge \bigwedge^{3,n} (\bar{x}_i \vee a_i)$ and $q_{\mathbf{y}}$ is similarly defined. This proves the claim.

Now let $U = \{p \vee s: p \in A \cup \{0\}$ and $s \in H\} \subseteq M$. We want that U is a sublattice of M and in fact a projective geometry of dimension n.

CLAIM 2. $p_1 \vee s_1 \leq p_2 \vee s_2$ if and only if $s_1 \leq s_2$ and $h \wedge (p_1 \vee p_2) \leq s_2$ for $p_1, p_2 \in A$ and $s_1, s_2 \in H$.

Proof. If $p_1 \vee s_1 \leq p_2 \vee s_2$ then meeting with h produces $s_1 \leq s_2$ and meeting $p_1 \vee p_2 \leq p_2 \vee s_2$ with h produces $h \wedge (p_1 \vee p_2) \leq s_2$. Conversely $p_2 \vee s_2 = p_2 \vee [h \wedge (p_1 \vee p_2)] \vee s_2 = p_1 \vee p_2 \vee s_2 \geq p_1 \vee s_1$.

COROLLARY. *For any* $q \in \{x_1, \ldots, x_n, z, t\}$, $[0, q] \cap U = \{0, q\}$.

Now since $(p_1 \vee s_1) \vee (p_2 \vee s_2) = p_1 \vee (s_1 \vee s_2 \vee [h \wedge (p_1 \vee p_2)])$ when $p_1 \neq 0$, U is closed under joins (as H is).

CLAIM 3. For distinct $p, q \in A$, $p \wedge q = 0$.

Proof. We have $\mathbf{a}, \mathbf{b} \in D^n$ with $p = \bigwedge (\bar{x}_i \vee a_i)$ and $q = \bigwedge (\bar{x}_i \vee b_i)$, and

$$\begin{aligned} p \wedge q &= \bigwedge (\bar{x}_i \vee a_i) \wedge (\bar{x}_i \vee b_i) \\ &= \bigwedge (\bar{x}_i \vee (a_i \wedge b_i)). \end{aligned}$$

Since $p \neq q, \mathbf{a} \neq \mathbf{b}$ and therefore $a_i \neq b_i$ for some i. For this i we obtain

$$\begin{aligned} p \wedge q &= \bar{x}_i \wedge \bigwedge_{j \neq i} (\bar{x}_j \vee (a_j \wedge b_j)) \\ &= 0. \end{aligned}$$

CLAIM 4. U is closed under meets.

Proof. Since $p_1 \wedge (p_2 \vee s_2) = p_1 \wedge (p_2 \vee (s_2 \wedge h \wedge (p_1 \vee p_2))) = p_1 \wedge p_2$ and $(p_1 \vee s_1) \wedge s_2 = s_1 \wedge s_2$ we may assume without loss of generality that we have $p_i \vee s_i \in U$, $i = 1, 2$ with $p_j \nleq p_i \vee s_i$ for $i \neq j$. Since H is a projective geometry this is equivalent to $s_1 \wedge (p_1 \vee p_2) = s_2 \wedge (p_1 \vee p_2) = 0$.

https://doi.org/10.4153/CMB-1983-046-1 Published online by Cambridge University Press

Now suppose there are points of H, $a_i \leq s_i$ such that $a_1 \vee a_2 = a_1 \vee [h \wedge (p_1 \vee p_2)] = a_2 \vee [h \wedge (p_1 \vee p_2)]$. Clearly $p = (a_1 \vee p_1) \wedge (a_2 \vee p_2) \in A$ and $p \vee (s_1 \wedge s_2) \leq (p_1 \vee s_1) \wedge (p_2 \vee s_2)$. However both of these expressions are complements of s_1 in $[s_1 \wedge s_2, s_1 \vee p]$. Therefore we have equality and $(p_1 \vee s_1) \wedge (p_2 \vee s_2) \in U$.

Now if no such $a_i \leq s_i$ exist, we can conclude, since H is a projective geometry, that $s_i \wedge (s_j \vee [h \wedge (p_1 \vee p_2)]) = 0$, $i \neq j$. These simplify to $s_1 \wedge (s_2 \vee p_1 \vee p_2) = s_2 \wedge (s_1 \vee p_1 \vee p_2) = 0$, which give

$$\begin{aligned}(p_1 \vee s_1) \wedge (p_2 \vee s_2) &= (p_1 \vee s_1) \wedge (p_2 \vee s_2) \wedge (s_1 \vee p_1 \vee p_2) \\ &\quad \wedge (s_2 \vee p_1 \vee p_2) \\ &= (p_1 \vee [s_1 \wedge (s_2 \vee p_1 \vee p_2)]) \\ &\quad \wedge (p_2 \vee [s_2 \wedge (s_1 \vee p_1 \vee p_2)]) \\ &= p_1 \wedge p_2.\end{aligned}$$

This completes the proof.

§5. **Applications.** Since the concept of an n-diamond is a projective configuration (Huhn [9], see also [3]) one can form "equations" of the form "If $\mathbf{d} = (d_1, \ldots, d_{n-1})$ is an n-diamond then $p(d_1, \ldots, d_{n+1}) = q(d_1, \ldots, d_{n+1})$" where p and q are lattice terms in $(n+1)$ variables. If $(x_1, \ldots, x_{n-1}, z, t)$ is an n-diamond in a modular lattice one can define the natural number terms:

$$\mathbf{0} = z$$

$$\mathbf{k+1} = \mathbf{k} \oplus t$$

This allows one to define (among other things) the characteristic of an n-diamond by $(x_1, \ldots, x_{n-1}, z, t)$ is of characteristic k if $\mathbf{k} = \mathbf{0}$. Versions of these characteristic equations have been given in Herrmann and Huhn [8] and Freese [6]. Freese also showed in [7] that:

THEOREM. *For any $n, k \in \mathbb{N}$ $FM(nD[k])$, the free modular lattice generated by an n-diamond of characteristic k, is a projective modular lattice.*

Let M be a modular lattice with n-diamond $\mathbf{x} = (x_1, \ldots, x_{n-1}, z, t)$. If $n \geq 4$ we have from von Neumann that $(D; \oplus, z, \otimes, t)$ is a ring (cf. Artmann [1]). If $n \geq 3$ and M is Arguesian we also have from Day and Pickering [5] that $(D; \oplus, z, \otimes, t)$ is a ring. If $\mathbf{x}$ is an n-diamond of characteristic p, for prime p, then $\mathbb{Z}_p \leq D$, $\mathrm{Inv}(\mathbb{Z}_p) = \mathbb{Z}_p \backslash \{z\}$ and $T[\mathbb{Z}_p^3] \subseteq \mathbb{Z}_p$. We can now apply the lemma to obtain:

THEOREM. *For p prime $FM(nD[p])$ is a projective geometry for $n \geq 4$ and $FA(nD[p])$ is a projective geometry for $n \geq 3$.*

COROLLARY ([7]). $FM(nD[p]) \cong \mathcal{L}(\mathbb{Z}_p^n)$, $n \geq 4$.

COROLLARY. $FA(nD[p]) \cong \mathcal{L}(\mathbb{Z}_p^n)$, $n \geq 3$.

https://doi.org/10.4153/CMB-1983-046-1 Published online by Cambridge University Press

REFERENCES

1. B. Artmann, *On coordinates in modular lattices*, Illinois J. Math. **12** (1968), 626–648.

2. P. Crawley and R. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, N. J. 1973.

3. A. Day, *Equational theories of projective geometries*, Colloq. Janos Bolyai (Szeged), 1980.

4. ——, *Modular lattices and projective geometry*, Lecture notes, Lakehead Univ., 1980.

5. —— and D. Pickering, *The coordinatization of Arguesian lattices*. Trans. Amer. Math. Soc. (to appear).

6. R. Freese, *The variety of modular lattices is not generated by its finite members*, Trans. Amer. Math. Soc., **255** (1979), 277–300.

7. ——, *Projective geometries as projective modular lattices*, Trans. Amer. Math. Soc., **251** (1979), 329–342.

8. C. Herrmann and A. Huhn, *Zum Begriff der Charakteristik modularer Verbande*, Math. Z. **144** (1975), 185–194.

9. A. Huhn, *Weakly distributive lattices*, Doctoral dissertation Szeged 1972.

LAKEHEAD UNIVERSITY
THUNDER BAY, ONTARIO.

https://doi.org/10.4153/CMB-1983-046-1 Published online by Cambridge University Press