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FINITE-DIMENSIONAL SIMPLE MODULES OVER QUANTISED WEYL ALGEBRAS

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We classify finite-dimensional simple modules over quantised n-th Weyl algebras $A_n^{\overline{q},\Lambda}$ over an algebraically closed field under a certain condition on the parameters.

0. INTRODUCTION

Several authors have proposed various algebras as q-analogues to the Weyl algebras. See, for example, [**3, 1, 5, 2**]. Since the n-th Weyl algebra is the algebra of differential operators on the n-dimensional affine space, these q-analogues to the n-th Weyl algebara have been regarded as the algebras of quantised differential operators on n-dimensional quantum affine spaces. In this paper we deal with the quantised Weyl algebras $A_n^{\overline{q},\Lambda}$ studied in [**1, 5**] et cetera.

Although the Weyl algebras (over a field of characteristic 0) have no non-zero finite-dimensional module, the quantised Weyl algebras have them. The purpose of this paper is to classify finite-dimensional simple modules over the quantised Weyl algebras $A_n^{\overline{q},\Lambda}$ under a certain condition on the parameters. For this end, the classification result for $n = 1$ due to Jordan [**4**] is crucial.

Throughout this paper, let k be an algebraically closed field of arbitrary characteristic.

1. QUANTISED WEYL ALGEBRAS $A_n^{\overline{q},\Lambda}$

DEFINITION 1.1: ([**1**].) Let $\Lambda = (\lambda_{ij})$ be an $n \times n$ matrix over the multiplicative group $k^\times$ of k such that $\lambda_{ii} = 1$ for each i and such that $\lambda_{ij} = \lambda_{ji}^{-1}$ for each i, j, and let $\overline{q} = (q_1, \cdots, q_n)$ be an n-tuple of elements of $k \setminus \{0, 1\}$. The n-th *quantised Weyl algebra* $A_n^{\overline{q},\Lambda}$ is by definition the k-algebra generated by $2n$ elements $y_1, \cdots, y_n, x_1, \cdots, x_n$ with relations

$$x_i x_j = q_i \lambda_{ij} x_j x_i,$$

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$$
(1.2)\qquad
\begin{aligned}
y_iy_j &= \lambda_{ij}y_jy_i,\\
x_iy_j &= \lambda_{ji}y_jx_i,\\
y_ix_j &= q_i^{-1}\lambda_{ji}x_jy_i,\\
x_jy_j - q_jy_jx_j &= 1 + \sum_{l=1}^{j-1}(q_l-1)y_lx_l,\\
&(x_1y_1 - q_1y_1x_1 = 1),
\end{aligned}
$$

where $1 \leqslant i < j \leqslant n$. When $n = 1$, $\Lambda = (1)$ and $\bar{q} = (q_1)$, $A_1^{\bar{q},\Lambda}$ is abbreviated to A_1^q, where $q = q_1$.

For $1 \leqslant i \leqslant n$, let $z_i = 1 + \sum_{j=1}^{i}(q_j - 1)y_jx_j$. These elements of $A_n^{\bar{q},\Lambda}$ are called the *Casimir elements*, and play an important role in investigating the quantised Weyl algebras. By a direct computation we get the following result (see [**5**, 2.8]).

Lemma 1.3. *The Casimir elements $z_1, \cdots, z_n$ of $A_n^{\bar{q},\Lambda}$ satisfy the following relations:*

$$
z_jy_i = \begin{cases} y_iz_j & \text{if } j < i,\\ q_iy_iz_j & \text{if } j \geqslant i,\end{cases}\qquad
z_jx_i = \begin{cases} x_iz_j & \text{if } j < i,\\ q_i^{-1}x_iz_j & \text{if } j \geqslant i,\end{cases}\qquad
z_iz_j = z_jz_i
$$

for $1 \leqslant i, j \leqslant n$.

For $1 \leqslant i \leqslant n$, let $\mathcal{Y}_i = \{y_i^j\}_{j\geqslant 1}$, $\mathcal{X}_i = \{x_i^j\}_{j\geqslant 1}$ and $\mathcal{Z}_i = \{z_i^j\}_{j\geqslant 1}$ in $A_n^{\bar{q},\Lambda}$. Note that $\mathcal{Y}_1, \cdots, \mathcal{Y}_n, \mathcal{X}_1, \cdots, \mathcal{X}_n, \mathcal{Z}_1, \cdots, \mathcal{Z}_n$ and the product $\mathcal{Z} = \mathcal{Z}_1 \cdots \mathcal{Z}_n$ are Ore sets in $A_n^{\bar{q},\Lambda}$. We denote by $B_n^{\bar{q},\Lambda}$ the localisation of $A_n^{\bar{q},\Lambda}$ at $\mathcal{Z}$. It is proved in [**5**, Theorem 3.2] that, if no q_i is a root of unity, then $B_n^{\bar{q},\Lambda}$ is simple, so that $B_n^{\bar{q},\Lambda}$ has no non-zero finite-dimensional module, since $B_n^{\bar{q},\Lambda}$ is infinite-dimensional over k.

2. Finite-dimensional simple modules over $A_n^{\bar{q},\Lambda}$

Lemma 2.1. *Fix $1 \leqslant i \leqslant n$. Suppose that q_i is not a root of unity. Let V be a finite-dimensional $A_n^{\bar{q},\Lambda}$-module. If V is $\mathcal{Z}_j$-torsion-free for some $j \geqslant i$, then the endomorphisms induced by x_i and y_i on V are nilpotent.*

Proof: If x_i does not act on V as a nilpotent endomorphism, there is a non-zero eigenvalue $\mu \in k$ for the action of x_i on V. Let $v \in V$ be a eigenvector with the eigenvalue μ. It follows from the assumption that $vz_j^m \neq 0$ for each $m \geqslant 0$. Hence by Lemma 1.3 one sees that x_i has infinitely many eigenvalues $\{q_i^{-m}\mu\}_{m\geqslant 0}$ on V, which contradicts the fact that V is of finite dimension. The same argument is valid for y_i. ∎

In [**4**] Jordan classified finite-dimensional simple modules over certain iterated skew polynomial rings, which include the first quantised Weyl algebra A_1^q. We shall describe the classification result for A_1^q when q is not a root of unity.

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DEFINITION 2.2: [4] Let $q \in k \setminus \{0,1\}$, $R = A_1^q$. For $\mu \in k^\times$, denote by $C(\mu)$ the right R-module

$$R/(zR + (y-\mu)R).$$

If we denote by v the image of 1 via the canonical surjection $R \to C(\mu)$, one sees that

$$C(\mu) = kv, \qquad vy = \mu v, \qquad vx = \frac{1}{\mu(1-q)}v.$$

(In [4] the R-module $C(\mu)$ is denoted by $C(0,\mu)$.)

PROPOSITION 2.3. ([4].) *Suppose that q is not a root of unity. Then every finite-dimensional simple module over A_1^q is isomorphic to $C(\mu)$ for some $\mu \in k^\times$.*

Next we consider finite-dimensional simple modules over n-th quantised Weyl algebras $A_n^{\bar{q},\Lambda}$ for $n \geqslant 2$.

LEMMA 2.4. *Suppose that q_1 is not a root of unity. Let V be a finite-dimensional simple module over $A_n^{\bar{q},\Lambda}$.*

(i) *Both x_1y_1 and y_1x_1 act on V as the scalar $(1-q_1)^{-1}$,*

(ii) *$Vx_iy_i = Vy_ix_i = 0$ for $2 \leqslant i \leqslant n$.*

PROOF: Since $A_1^{q_1}$ is a subalgebra of $A_n^{\bar{q},\Lambda}$, V contains $C(\mu)$ for some $\mu \in k^\times$ by Proposition 2.3. Thus there is a non-zero element $v \in V$ such that $vy_1 = \mu v$, $vx_1 = (\mu(1-q_1))^{-1}v$. In particular it follows that y_1 is not nilpotent on V, so that by Lemma 2.1, V is $\mathcal{Z}_j$-torsion, equivalently $Vz_j = 0$ for $j = 1, \cdots, n$. By using the relations (1.2), the lemma follows. ∎

COROLLARY 2.5. *If q_1 is not a root of unity, then there exists no non-zero finite-dimensional module over $B_n^{\bar{q},\Lambda}$.*

PROOF: Suppose that there is a finite-dimensional non-zero $B_n^{\bar{q},\Lambda}$-module V. Since z_1 is a unit in $B_n^{\bar{q},\Lambda}$, V is $\mathcal{Z}_1$-torsion-free, so that x_1 and y_1 act nilpotently on V by Lemma 2.1. On the other hand, it follows from Lemma 2.4(i) that V contains a non-$\mathcal{X}_1$-torsion element, which is a contradiction. ∎

From relation (1.2) and Lemma 2.4, we get the following lemma in the same way as the proof of [6, Lemma 4].

LEMMA 2.6. *Suppose that q_1 is not a root of unity. Let V be a finite-dimensional simple module over $A_n^{\bar{q},\Lambda}$. Then the endomorphisms on V induced by $x_1, \cdots, x_n, y_1, \cdots, y_n$ are diagonalisable.*

LEMMA 2.7. *Suppose that q_1 is not a root of unity. Let V be a finite-dimensional simple module over $A_n^{\bar{q},\Lambda}$. Fix $1 \leqslant i < j \leqslant n$.*

(i) *If $\lambda_{ij}^m \neq 1$ for any positive integer $m \leqslant \dim V$, then $Vy_i = Vx_i = 0$ or $Vy_j = 0$.*

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(ii) *If* $(q_i\lambda_{ij})^m \neq 1$ *for any positive integers* $m \leqslant \dim V$, *then* $Vy_i = Vx_i = 0$ *or* $Vx_j = 0$.

PROOF: Let W be a $A_n^{\bar{q},\Lambda}$-module. For $r \in A_n^{\bar{q},\Lambda}$, $\mu \in k$, write

$$W(r;\mu) = \{w \in W \mid wr = \mu w\},$$

the eigenspace of r corresponding to the eigenvalue μ. By a direct computation using relations (1.2) it follows that for $m \geqslant 0$

$$\begin{aligned} W(x_i;\mu)x_j^m &\subset W\Big(x_i; (q_i\lambda_{ij})^{-m}\mu\Big), & W(y_i;\mu)x_j^m \subset W\Big(y_i; (q_i\lambda_{ij})^m\mu\Big),\\ W(x_i;\mu)y_j^m &\subset W\Big(x_i; \lambda_{ij}^m\mu\Big), & W(y_i;\mu)y_j^m \subset W\Big(y_i; \lambda_{ji}^m\mu\Big),\end{aligned}$$

where $i < j$. By taking W to be V in the above, the lemma follows immediately. ∎

Put $R = A_n^{\bar{q},\Lambda}$. For an n-tuple $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)$ of elements of k with $\mu_1 \neq 0$, denote by $D(\boldsymbol{\mu})$ (respectively $D^\dagger(\boldsymbol{\mu})$) the right R-module

$$R\Big/\Big(\sum_{i=1}^n (y_i - \mu_i)R + \Big(x_1 - (\mu_1(1-q_1))^{-1}\Big)R + \sum_{i=2}^n x_iR\Big)$$
$$\Big(\text{respectively } R\Big/\Big((y_1 - \mu_1)R + \sum_{i=2}^n y_iR + \sum_{i=1}^n (x_i - \mu_i)R\Big)\Big).$$

These modules are of dimension $\leqslant 1$. Clearly $D(\mu_1, 0, \cdots, 0) = D^\dagger(\mu_1, 0, \cdots, 0)$ is 1-dimensional. From Lemma 2.4 and Lemma 2.7 we deduce easily the following.

COROLLARY 2.8. *Suppose that* q_1 *is not a root of unity. If neither* λ_{1j} *nor* $q_1\lambda_{1j}$ *is a root of unity for each* $j \geqslant 2$, *then every finite-dimensional simple module over* $A_n^{\bar{q},\Lambda}$ *is isomorphic to* $D(\mu, 0, \cdots, 0)$ *for some* $\mu \in k^\times$.

COROLLARY 2.9. *Suppose that* q_1 *is not a root of unity. If* $\lambda_{ij} = 1$ *for all* i, j, *then every finite-dimensional simple module over* $A_n^{\bar{q},\Lambda}$ *is isomorphic to* $D(\boldsymbol{\mu})$ *for some* $\boldsymbol{\mu} \in k^n$ *with* $\mu_1 \neq 0$.

PROOF: Since $y_1, \cdots, y_n, x_1$ commute with each other, the endomorphism induced by $y_1, \cdots, y_n, x_1$ on V are simultaneously diagonalisable by Lemma 2.6. Then the result follows easily. ∎

Finally we shall consider the case when $n = 2$.

We say that $\mu \in k^\times$ is a root of unity *of order* m if m is the least positive integer such that $\mu^m = 1$.

Put $R = A_2^{\bar{q},\Lambda}$, $\lambda = \lambda_{12}$. For $\mu, \alpha \in k^\times$ and a positive integer m, we denote by $E(\mu, m, \alpha)$ (respectively $E^\dagger(\mu, m, \alpha)$) the right R-module

$$R\Big/\Big((y_1 - \mu)R + \Big(x_1 - (\mu(1-q_1))^{-1}\Big)R + (y_2^m - \alpha)R + x_2R\Big)$$
$$\Big(\text{respectively } R\Big/\Big((y_1 - \mu)R + \Big(x_1 - (\mu(1-q_1))^{-1}\Big)R + y_2R + (x_2^m - \alpha)R\Big)\Big).$$

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Note that $E(\mu,1,\alpha) = D(\mu,\alpha)$, $E^{\dagger}(\mu,1,\alpha) = D^{\dagger}(\mu,\alpha)$. It is easy to see that $E(\mu,m,\alpha)$ (respectively $E^{\dagger}(\mu,m,\alpha)$) is simple if and only if $\lambda = 1$ (respectively $\lambda = q_1^{-1}$), $\alpha = 0$ or λ (respectively $q_1\lambda$) is a root of unity of order m. We remark that $E^{(\dagger)}(\mu,m,0) \cong E^{(\dagger)}(\mu,1,0)$ for $m \geqslant 1$. For $\mu,\mu',\alpha,\alpha' \in k^{\times}$, if λ (respectively $q_1\lambda$) is a root of unity of order $m \geqslant 2$, then the simple R-module $E^{(\dagger)}(\mu,m,\alpha)$ is isomorphic to $E^{(\dagger)}(\mu',m,\alpha')$ if and only if $\alpha = \alpha'$ and $\mu' = \lambda^d\mu$ (respectively $\mu' = (q_1\lambda)^d\mu$) for some non-negative integer $d \leqslant m-1$.

Theorem 2.10. *Suppose that q_1 is not a root of unity. Put $\lambda = \lambda_{12}$.*

(i) If neither λ nor $q_1\lambda$ is a root of unity, then every finite-dimensional simple module over $A_2^{\bar{q},\Lambda}$ is isomorphic to $E(\mu,1,0)\big(= E^{\dagger}(\mu,1,0)\big)$ for some $\mu \in k^{\times}$.

(ii) If λ is a root of unity of order m, then every finite-dimensional simple module over $A_2^{\bar{q},\Lambda}$ is isomorphic to either $E(\mu,1,0)$ for some $\mu \in k^{\times}$ or $E(\mu,m,\alpha)$ for some $\mu,\alpha \in k^{\times}$.

(iii) If $q_1\lambda$ is a root of unity of order m, then every finite-dimensional simple module over $A_2^{\bar{q},\Lambda}$ is isomorphic to either $E^{\dagger}(\mu,1,0)$ for some $\mu \in k^{\times}$ or $E^{\dagger}(\mu,m,\alpha)$ for some $\mu,\alpha \in k^{\times}$.

Proof:

(i) This is a special case of Corollary 2.8.

(ii) Put $R = A_2^{\bar{q},\Lambda}$. Let V be a finite-dimensional simple R-module. Since $q_1\lambda$ is not a root of unity, it follows from Lemma 2.7(ii) that $Vx_2 = 0$. Suppose that V is not of the form $E(\mu,1,0)$. In particular, $Vy_1 \neq 0$ by Lemma 2.5. Thus it suffices to show that y_2^m acts on V as a non-zero scalar α. Note that V is a simple module over $S = R/(x_iy_i - y_ix_i \mid 1 \leqslant i \leqslant n)$ by Lemma 2.5. From relations (1.2), the image of y_2^m in S is contained in the centre of S, which shows the above claim.

(iii) Similar to (ii). ∎

Remark 2.11. For arbitrary parameters $\bar{q}$ and Λ, no finite-dimensional module over the quantised Weyl algebra $A_n^{\bar{q},\Lambda}$ is semisimple. For right $A_n^{\bar{q},\Lambda}$-modules V and W, we denote by $\operatorname{Ext}(V,W)$ the group of all equivalence classes of extensions of W by V. This additive group $\operatorname{Ext}(V,W)$ is naturally a k-vector space. One can directly see that, for $\mu \in k^{\times}$

$$\dim_k \operatorname{Ext}(C(\mu),C(\mu)) = 1 \qquad \text{(when } n = 1\text{)},$$

$$\dim_k \operatorname{Ext}(D(\mu,0,\cdots,0),D(\mu,0,\cdots,0)) = r \qquad \text{(when } n \geqslant 2\text{)},$$

where r is the number of i such that $\lambda_{i1} = 1$ or q_1. In the case when $n = 1$, moreover, it is easy to see that, for $\mu,\mu' \in k^{\times}$ such that $\mu' \neq \mu$ and $\mu' \neq q\mu$,

$$\dim_k \operatorname{Ext}(C(\mu),C(q\mu)) = 1, \qquad \dim_k \operatorname{Ext}(C(\mu),C(\mu')) = 0.$$

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