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WEAKNESS OF THE TOPOLOGY OF A JB^*-ALGEBRA

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ABSTRACT. The main purpose of this paper is to prove, that the topology of any (non-complete) algebra norm on a JB^*-algebra is stronger than the topology of the usual norm. The proof of this theorem consists of an adaptation of the recent Rodriguez proof [8] that every homomorphism from a complex normed (associative) Q-algebra onto a B^*-algebra is continuous.

1. **Previous concepts and results.** Let us recall that a complex unital normed Jordan algebra A is a complex Jordan algebra with product $a \circ b$, having a unit 1, and a norm $\| \ \|$, such that A with the norm $\| \ \|$ is a normed space, $\|1\| = 1$, and for all a and b in A $\|a \circ b\| \leq \|a\|\|b\|$.

As we shall only be considering complex unital normed Jordan algebras, we shall use "normed Jordan algebra" in place of "complex unital normed Jordan algebra". A Banach Jordan algebra is a normed Jordan algebra $(A, \|.\|)$ such that the normed linear space A with norm $\|.\|$ is complete (*i.e.* every Cauchy sequence converges).

A JB^*-algebra is a Banach Jordan algebra A, with an involution $*$ such that, for all a in A

$$\|U_a(a^*)\| = \|a\|^3,$$

where $U_a(b) = 2a \circ (a \circ b) - a^2 \circ b$.

Let $(A, \|.\|)$ be a B^*-algebra. A JC^*-algebra J of A is a complex Banach subspace of A satisfying:

i) J is a self-adjoint set (*i.e.* $a \in J \Longrightarrow a^* \in J$),

ii) $1 \in J$,

iii) $a, b \in J \Longrightarrow a \circ b = \frac{1}{2}(ab + ba) \in J$, where ab is the associative product.

It is easy to prove that every JC^*-algebra is a JB^*-algebra. However, in [9] it is shown that JC^*-algebras are not the only examples of JB^*-algebras. Thus, the converse of the preceding result is not true.

One should also note that if A is an associative algebra over the complex field which is a Banach space in the norm $\|.\|$ and where, in terms of the Jordan multiplication $a \circ b = \frac{1}{2}(ab+ba)$, $\|a \circ b\| \leq \|a\|\|b\|$ for all a, b in A; then it is not necessary that the associative product be continuous. An example is given in [5] of such an A.

Let $(A, \|.\|)$ be a normed Jordan algebra (completeness is not assumed). The spectral radius of an element a in A, denoted by $r_{\|.\|}(a)$ (or simply r(a), when it is clear to which norm it refers), is defined by

$$r_{\|.\|}(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}}.$$

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An element a of A is invertible with inverse b if $a \circ b = 1$ and $a^2 \circ b = a$. The spectrum of a, denoted by $\mathrm{Sp}(A, a)$, is defined by

$\mathrm{Sp}(A, a) = \{\lambda \in C : \lambda - a$ is not invertible in $A\}$.

An element a of A has the quasi-inverse b if $(1-a)$ has the inverse $(1-b)$. An element that has a quasi inverse is said to be quasi-invertible.

The normed Jordan algebra $(A, \|.\|)$ is called a Jordan Q-algebra if the set of quasi-invertible elements of A is open.

In what follows we will use without comment, the fact that $(A, \|.\|)$ is a Jordan Q-algebra if and only if

$$r(a) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(A, a)\}.$$

(See [10], lemma 2.1).

The proofs of many results on Banach Jordan algebras depend only on the fact that Banach Jordan algebras are Jordan Q-algebras, and this is the case of the following result: Let A and B be Jordan Q-algebras and F be homomorphism from A into B. Then

$$r(F(a)) \leq r(a),$$

for all a in A.

The notion of Jacobson radical for associative algebras has been generalized by K. Mc Crimmon to Jordan algebras (see [4]). In a Jordan algebra we say that an ideal I is quasi-invertible if all its elements are quasi-invertibles. Mc Crimmon proved that in any Jordan algebra there exists a unique quasi-invertible ideal containing every quasi-invertible ideal.

By definition, this ideal is the Mc Crimmon radical of A and is denoted by $\mathrm{rad}\, A$. A is said to be semi-simple if $\mathrm{rad}\, A = \{0\}$.

In the case of Banach algebras $\mathrm{rad}\, A = \{q \in A : aq$ is quasi-invertible for all a in $A\}$.

In the case of Banach Jordan algebras a similar result is not true. It is the reason why the proof of proposition 25.10 in [1] cannot be adapted to the Jordan case. Nevertheless, we shall give an alternate proof of that result.

NOTATION. If A and B are normed Jordan algebras and F is a linear mapping from A into B we denote by $S(F)$ (the separating subspace for F) the set of those b in B for which there is a sequence $\{a_n\}$ in A such that $0 = \lim\{a_n\}$ and $b = \lim\{F(a_n)\}$.

PROPOSITION 1. *Let A be a Jordan Q-algebra and B be a semi-simple Banach Jordan algebra. Suppose that F is homomorphism from A onto B. Then*

i) $r(b) = 0$, for every b in $S(F)$,

ii) The kernel of F is closed.

PROOF. i) The proof of $r(b) = 0$ in [6] remains valid in the Jordan case.

ii) It is straightforward to check that $F(\overline{\ker F})$ is an ideal of B.

Given $b \in F(\overline{\ker F})$, we have $b = F(a)$ for some a in $\overline{\ker F}$, and so there exists $\{a_n\}$ in $\ker F$ such that $a = \lim\{a_n\}$. Since $F(a_n) = 0$, we obtain $0 = \lim\{a - a_n\}$ and $\lim\{F(a - a_n)\} = F(a)$. Therefore F(a) is in S(F) and therefore, by (i), $r(F(a)) = 0$. Thus b is quasi-invertible, $F(\overline{\ker F})$ is a quasi-invertible ideal of B, and so $F(\overline{\ker F}) \subset \mathrm{rad}\, B = \{0\}$ and $\overline{\ker F} \subset \ker F$. Therefore, $\ker F$ is closed. ∎

PROPOSITION 2. *The quotient of a Jordan Q-algebra by a closed ideal is also a Jordan Q-algebra.*

PROOF. Let J be a closed ideal of a Jordan Q-algebra A. Let π be the canonical projection of A onto the normed Jordan algebra A/J, π is open and $\pi(G(A)) \subset G(A/J)$, where $G(X)$ denote the set of invertible elements in X. Let $a \in G(A)$, then $\pi(a)$ is an interior point of $G(A/J)$. Choose b in $G(A/J)$. Then the linear operator U_b is a homeomorphism on A/J and it leaves invariant the set $G(A/J)$ (see [3], Theorem 1.3, p.52), so $U_b(\pi(a))$ is a interior point of $G(A/J)$. Since the mapping $x \longmapsto U_x(\pi(a))$, $x \in A/J$, is continuous, it follows that there is some number $r > 0$ such that

$$U_x\big(\pi(a)\big) \in G(A/J),$$

so $x \in G(A/J)$ whenever $\|x - b\| < r$. Hence $G(A/J)$ is open. Since the mapping $x \longmapsto 1 - x$ is continuous mapping of A/J into A/J, then the set of quasi-invertible elements is also open. ∎

2. **Minimum topologies.** We say that $(A, \|.\|)$ has the property of minimality of norm topology if, whenever $|||.|||$ is an algebra norm on A with $|||.||| \leq k\|.\|$ for some non negative number k, we have that $|||.|||$ and $\|.\|$ are equivalent norms.

The proof of our main result is strongly based on the following lemma proved by Rodriguez.

LEMMA 1. *Let A be a Jordan Q-algebra and B be a semi-simple Banach Jordan algebra with minimality of norm topology. Then every homomorphism from A onto B is continuous.*

PROOF. We repeat the proof of the main result of [8] for Jordan algebras and use propositions 1 and 2. ∎

PROPOSITION 3. *Let $(A, \|.\|)$ be a B^*-algebra, $(J, \|.\|)$ a JC^*-algebra of A, and $|||.|||$ is any algebra norm on J. Then*

$$\|a\|^2 \leq \sqrt{6}\, |||a^*|||\, |||a|||,$$

for all a in J.

PROOF. We first prove that $r_{\|.\|}(h) = r_{|||.|||}(h)$ for every self-adjoint (i.e $h^* = h$) element h in J. Let $h \in J$ such that $h^* = h$ and let $Q(h, 1)$ denote the closed (relative to $\|.\|$) subalgebra of J generated by h and 1. As every Jordan algebra is power associative (see [3]), theorem 8 p.36) and multiplication $(a \circ b)$ is continuous, $Q(h, 1)$ is commutative Banach algebra. Moreover, as the involution on A is an isometry and h is self-adjoint $Q(h, 1)$ is a self-adjoint subset. Hence, $Q(h, 1)$ is a B^*-algebra. So, by the Corollary 4.8.4 of [7] we obtain

$$r_{\|.\|}(h) \leq r_{|||.|||}(h).$$

Since the reverse inequality holds for any algebra norm ($|||.|||$), we thus have proved that

$$r_{||.||}(h) = r_{|||.|||}(h),$$

for every $h \in J$ satisfying $h^* = h$.

Let, now, $a \in J$. Then,

$$\frac{1}{2}||a||^4 = \frac{1}{2}||a^*a||^2 = \frac{1}{2}||(a^*a)^2||.$$

It is known (see Theorem 7 and Lemma 6 of [11]) that if x and y are self-adjoint elements of a JB^*-algebra, then

$$||x^2|| \leq ||x^2 + y^2||.$$

Now we apply the above mentioned result to obtain

$$||(a^*a)^2|| \leq ||(a^*a)^2 + (aa^*)^2||.$$

Since $(a^*a)^2 + (aa^*)^2$ is self-adjoint, then

$$||(a^*a)^2 + (aa^*)^2|| = r_{||.||}((a^*a)^2 + (aa^*)^2).$$

Combining these estimates with the first part of this proof we deduce that

$$\begin{aligned}\frac{1}{2}||a||^4 &\leq r_{|||.|||}(\frac{1}{2}\{(a^*a)^2 + (aa^*)^2\})\\ &= r_{|||.|||}(\frac{1}{2}\{a^*(aa^*a) + (aa^*a)a^*\})\\ &= r_{|||.|||}(a^* \circ (aa^*a))\\ &= r_{|||.|||}(a^* \circ U_a(a^*))\\ &\leq |||a^* \circ U_a(a^*)|||\\ &\leq 3\,|||a^*|||^2\,|||a|||^2.\end{aligned}$$

It follows that $||a||^2 \leq \sqrt{6}\,|||a^*|||\,|||a|||$. ∎

PROPOSITION 4. *Let $(A, ||.||)$ be a JB^*-algebra and let $|||.|||$ be any algebra norm on A. Then*

$$||a||^2 \leq \sqrt{6}\,|||a^*|||\,|||a|||, \forall a \in A.$$

PROOF. Let $a \in A$ and B be the closure (relative to $||.||$) of the Jordan algebra generated by 1, $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$. Then, by corollary 3 of [12], we know that there exists a B^*-algebra $(X, |.|)$, a JC^*-algebra $(J, |.|)$ of X, and a isometric linear bijection F of B onto J satisfying

i) $F(x \circ y) = F(x) \circ F(y)$,

ii) $F(x^*) = \big(F(x)\big)^*$, for every x and y in B.

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We define a mapping P of J into R by $P(j) = |\!|\!|F^{-1}(j)|\!|\!|$. It is straightforward to check that P is an algebra norm on J. Therefore by proposition 3,

$$|j|^2 \leq \sqrt{6}\, P(j^*)P(j), \forall j \in J.$$

Hence,

$$||a||^2 = |F(a)|^2 \leq \sqrt{6}\, P((F(a))^*)P(F(a)) = \sqrt{6}\, |\!|\!|a^*|\!|\!|\, |\!|\!|a|\!|\!|. \blacksquare$$

THEOREM 1. *Every JB^*-algebra has the property of minimality of norm topology.*

PROOF. For any algebra norm, $|\!|\!|.|\!|\!|$, on a JB^*-algebra $(A, ||.||)$ we have

$$||a||^2 \leq \sqrt{6}\, |\!|\!|a^*|\!|\!|\, |\!|\!|a|\!|\!|$$

for all a in A by proposition 4.

Therefore, if $|\!|\!|.|\!|\!| \leq k||.||$ for some non-negative number k, we have

$$||a||^2 \leq k\sqrt{6}\, ||a^*||\, |\!|\!|a|\!|\!| = k\sqrt{6}\, ||a||\, |\!|\!|a|\!|\!|,$$

(the last equality follows from [11], lemma 4), so that, $||.|| \leq k\sqrt{6}\, |\!|\!|.|\!|\!|$, and so $|\!|\!|.|\!|\!|$ and $||.||$ are equivalent norms. ∎

LEMMA 2. *If $|\!|\!|.|\!|\!|$ is any algebra norm on a JB^*-algebra A, then $(A, |\!|\!|.|\!|\!|)$ is a Jordan Q-algebra.*

PROOF. By proposition 4 we have, $||a||^2 \leq \sqrt{6}\, |\!|\!|a^*|\!|\!|\, |\!|\!|a|\!|\!|$ for all a in A. We deduce that for all $n \geq 1$ and all a in A

$$||a^n||^2 \leq \sqrt{6}\, |\!|\!|(a^*)^n|\!|\!|\, |\!|\!|a^n|\!|\!|.$$

Taking *nth* roots and letting $n \longrightarrow \infty$, it follows that

$$[r_{||.||}(a)]^2 \leq r_{|\!|\!|.|\!|\!|}(a^*)r_{|\!|\!|.|\!|\!|}(a).$$

Since $r_{|\!|\!|.|\!|\!|}(x) \leq r_{||.||}(x)$ and $r_{||.||}(x^*) = r_{||.||}(x)$ for all x in A, we have

$$r_{||.||}(a) = r_{|\!|\!|.|\!|\!|}(a)$$

for all a in A. But $(A, ||.||)$ is a Banach Jordan algebra, so $r_{||.||}(a) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(A, a)\}$. Therefore $r_{|\!|\!|.|\!|\!|}(a) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(A, a)\}$ and $(A, |\!|\!|.|\!|\!|)$ is a Jordan Q-algebra, as required. ∎

We now come to the main result.

THEOREM 2. *The topology of any algebra norm on a JB^*-algebra is stronger than the topology of the usual norm.*

PROOF. Let $(A, ||.||)$ be a JB^*-algebra and let $|\!|\!|.|\!|\!|$ be any algebra norm on A. Then, by lemma 2, $(A, |\!|\!|.|\!|\!|)$ is a Jordan Q-algebra and, by theorem 1, $(A, ||.||)$ is a semi-simple Banach Jordan algebra with minimality of norm topology. Therefore, by lemma 1 the mapping $x \longmapsto x$ from $(A, |\!|\!|.|\!|\!|)$ onto $(A, ||.||)$ is continuous. ∎

REMARK. We recall that a normed Jordan algebra $(A, ||.||)$ has the property of minimality of the norm if, whenever $|\!|\!|.|\!|\!|$ is an algebra norm on A with $|\!|\!|.|\!|\!| \leq ||.||$, we have $|\!|\!|.|\!|\!| = ||.||$. Lemma 1 of [8] and theorem 1 suggest the following question. Does every JB^*-algebra have the property of minimality of the norm?

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