BULL. AUSTRAL. MATH. SOC. 90C25
VOL. 14 (1976), 129-136.

On the complex nonlinear complementary problem

J. Parida and B. Sahoo

The complex nonlinear complementarity problem considered here is the following: find z such that

$$g(z) \in S^* , \quad z \in S ,$$
$$\mathrm{Re}\left(g(z), z\right) = 0 ,$$

where S is a polyhedral cone in C^n , S^* the polar cone, and g is a mapping from C^n into itself. We study the extent to which the existence of a $z \in S$ with $g(z) \in S^*$ (feasible point) implies the existence of a solution to the nonlinear complementarity problem, and extend, to nonlinear mappings, known results in the linear complementarity problem on positive semi-definite matrices.

1. Introduction

Given $g : C^n \to C^n$, the nonlinear complementarity problem consists of finding a z such that

$$\begin{aligned} & g(z) \in S^* , \quad z \in S , \\ & \mathrm{Re}\left(g(z), z\right) = 0 , \end{aligned} \tag{1.1}$$

where S is a polyhedral cone in C^n and S^* the polar cone of S .

Problems of the form (1.1), where $g(z)$ is the affine transformation $Mz + q$, have already appeared in the literature. McCallum [4] showed that

Received 12 November 1975.

when $M \in C^{n\times n}$ is positive semi-definite, S and S^* are sectors in complex space, and the constraints are feasible; then a solution exists to the corresponding linear complementarity problem. Mond [5] extended this result to the more general complex linear complementarity problem, where the constraints are restricted to polyhedral cones.

In this paper, we have studied the existence of a solution to (1.1) under feasibility assumptions. Theorems analogous to those proved by McCallum [4] and Mond [5] in the complex linear case, and Moré [6] and Cottle [2] in the real case are obtained by considering monotone functions. These mappings are nonlinear versions of positive semi-definite matrices.

2. Notations and preliminaries

Denote by C^n $[R^n]$ n-dimensional complex [real] space; denote by $C^{m\times n}$ $[R^{m\times n}]$ the vector space of all $m \times n$ complex [real] matrices; denote by $R^n_+ = \left\{x \in R^n : x_i \geq 0, 1 \leq i \leq n\right\}$ the non-negative orthant of R^n; and for any $x, y \in R^n$, $x \geq y$ denotes $x - y \in R^n_+$. If A is a complex matrix or vector, then A^T, $\bar{A}$, and A^H denote its transpose, complex conjugate, and conjugate transpose. For $x, y \in C^n$, $(x, y) \equiv y^H x$ denotes the inner product of x and y.

A nonempty set $S \subset C^n$ is a polyhedral cone if for some positive integer k and $A \in C^{n\times k}$,

$$S = \{Ax : x \in R^k_+\} .$$

The polar of S is the cone S^* defined by

$$S^* = \{y \in C^n : x \in S \Rightarrow \operatorname{Re}(x, y) \geq 0\} ,$$

or equivalently by

$$S^* = \{y \in C^n : \operatorname{Re}\left(A^H y\right) \geq 0\} .$$

The interior of S^*, int S^*, is given by

$$\operatorname{int} S^* = \{y \in S^* : \operatorname{Re}\left(A^H y\right) > 0\} .$$

A mapping $g : C^n \to C^n$ is said to be monotone on S if $\operatorname{Re}\left(g(z^1)-g(z^2), z^1-z^2\right) \geq 0$ for each $z^1, z^2 \in S$, and strictly monotone if the strict inequality holds whenever $z^1 \neq z^2$.

We shall make use of the following definition of convexity [1] of a complex-valued function with respect to a cone.

A mapping $g : C^n \to C^n$ is concave with respect to the polyhedral cone S if, for all $z^1, z^2 \in C^n$ and for all $\lambda \in [0, 1]$,

$$g\left(\lambda z^1+(1-\lambda)z^2\right) - \lambda g(z^1) - (1-\lambda)g(z^2) \in S .$$

Given a mapping $g : C^n \to C^n$, $\operatorname{Re} z^H g(z)$ is convex with respect to R_+ if, for all $z^1, z^2 \in C^n$ and $\lambda \in [0, 1]$,

$$\lambda \operatorname{Re}\left(g(z^1), z^1\right) + (1-\lambda)\operatorname{Re}\left(g(z^2), z^2\right) - \operatorname{Re}\left(g\left(\lambda z^1+(1-\lambda)z^2\right), \lambda z^1+(1-\lambda)z^2\right) \geq 0 .$$

3. Solutions of variational inequalities

Hartman and Stampacchia [3] have proved the following result on variational inequalities: if $F : R^n \to R^n$ is a continuous mapping on the nonempty, compact, convex set $K \subset R^n$, then there is an x^0 in K such that

$$\left(F(x^0), x-x^0\right) \geq 0 \tag{3.1}$$

for all $x \in K$. Since C^n can be identified with R^{2n}, a natural extension of this result to complex space can be obtained as follows.

THEOREM 3.1. *If $g : C^n \to C^n$ is a continuous mapping on the nonempty, compact, convex set $S \subset C^n$, then there is a z^0 in S with*

$$\operatorname{Re}\left(g(z^0), z-z^0\right) \geq 0 \tag{3.2}$$

for all $z \in S$.

A polyhedral cone is a closed, convex set, but not bounded. We shall show that Theorem 3.1 holds for polyhedral cones under a very weak

https://doi.org/10.1017/S0004972700024898 Published online by Cambridge University Press

restriction on the growth of the mapping g .

Let S be a polyhedral cone in C^n . Then there is a positive integer k and a matrix $A \in C^{n\times k}$ such that $S = \{Ax : x \in R_+^k\}$. For a constant $p > 0$, we denote $z(p) = \{Ax : x_i = p, 1 \le i \le k\}$ and for any $z = Ax \in S$, we write $z \le z(p)$ if $\|x\|_\infty \le p$, where $\|x\|_\infty = \max\{|x_i| : 1 \le i \le k\}$.

LEMMA 3.2. *Let* $y^0 \in C^n$ *be given, and assume* S *is a polyhedral cone in* C^n . *Then an element* $z^0 \in S$ *satisfies*

$$\operatorname{Re}(y^0, z-z^0) \ge 0 \tag{3.3}$$

for all $z \in S$ *provided there is a vector* $z(p) > z^0$ *in* S *such that* (3.3) *holds for all*

$$z \in S_p = \{z \in S : z \le z(p)\} .$$

Proof. Let $z \in S$, and write $u = \lambda z + (1-\lambda)z^0$ for $0 < \lambda < 1$. Since S is a polyhedral cone, $u \in S$, and also it follows that there exist $A \in C^{n\times k}$ and $x, x^0 \in R_+^k$ such that $z = Ax$ and $z^0 = Ax^0$. Then $u \in S_p$ if $\|\lambda x+(1-\lambda)x^0\|_\infty \le p$. Since $\|x^0\|_\infty < p$, we can choose λ sufficiently small so that u lies in S_p . Then

$$0 \le \operatorname{Re}(y^0, u-z^0) = \lambda \operatorname{Re}(y^0, z-z^0) ,$$

and consequently, z^0 satisfies (3.3) for all $z \in S$.

THEOREM 3.3. *Let* $g : C^n \to C^n$ *be a continuous mapping on the polyhedral cone* S . *If there are vectors* $z(p)$, $u \in S$, *with* $z(p) > u$ *such that* $\operatorname{Re}(g(z), z-u) \ge 0$ *for all* $z = z(p)$ *in* S , *then there is a* $z^0 \le z(p)$ *in* S *with*

$$\operatorname{Re}(g(z^0), z-z^0) \ge 0 \tag{3.4}$$

for all $z \in S$.

Proof. Consider the set $S_p = \{z \in S : z \le z(p)\}$. Since S is a

https://doi.org/10.1017/S0004972700024898 Published online by Cambridge University Press

polyhedral cone, we can write $S_p = \{Ax : x \in R_+^k, \|x\|_\infty \le p\}$ which is obviously a compact, convex set in C^n. Therefore by Theorem 3.1, there is a $z^0 \in S_p$ satisfying (3.4) for all $z \in S_p$. If $z^0 < z(p)$, then taking $y^0 = g(z^0)$ in Lemma 3.2, we get the desired result. If $z^0 = z(p)$, then by the hypothesis, $\operatorname{Re}(g(z^0), z^0-u) \ge 0$. Since $\operatorname{Re}(g(z^0), z-z^0) \ge 0$ for all $z \in S_p$, it follows that $\operatorname{Re}(g(z^0), z-u) \ge 0$ for all $z \in S_p$. But $u < z(p)$, and thus by Lemma 3.2, $\operatorname{Re}(g(z^0), z-u) \ge 0$ for all $z \in S$. Also $u \in S_p$, and so $\operatorname{Re}(g(z^0), u-z^0) \ge 0$. Now adding the last two inequalities, we obtain $\operatorname{Re}(g(z^0), z-z^0) \ge 0$ for all $z \in S$.

4. Solvability of the complementarity problem

We now prove a lemma which gives the connection between variational inequalities discussed in Section 3 and the nonlinear complementarity problem (1.1).

LEMMA 4.1. *Let S be a polyhedral cone in C^n, and let $g : C^n \to C^n$ be continuous on S. If there is a $z^0 \in S$ such that*

$$\operatorname{Re}(g(z^0), z-z^0) \ge 0 \tag{4.1}$$

for all $z \in S$, then

$$g(z^0) \in S^* \text{ and } \operatorname{Re}(g(z^0), z^0) = 0 .$$

Thus z^0 is a solution to (1.1).

Proof. If $\operatorname{Re}(g(z^0), z-z^0) \ge 0$ for all $z \in S$, then $\operatorname{Re}(g(z^0), z) \ge \operatorname{Re}(g(z^0), z^0)$ for all $z \in S$. Since S is a polyhedral cone, $z + z^0 \in S$ for all $z \in S$. Then $\operatorname{Re}(g(z^0), z+z^0) \ge \operatorname{Re}(g(z^0), z^0)$ for all $z \in S$ and consequently, $\operatorname{Re}(g(z^0), z) \ge 0$ for all $z \in S$ and, in particular, $\operatorname{Re}(g(z^0), z^0) \ge 0$. So $g(z^0) \in S^*$. Since $0 \in S$, from

(4.1) we get $\operatorname{Re}(g(z^0), z^0) \le 0$, and hence $\operatorname{Re}(g(z^0), z^0) = 0$.

THEOREM 4.2. *Let* $g : C^n \to C^n$ *be a continuous monotone function on* S *, a polynedral cone in* C^n *. If there is a* $u \in S$ *with* $g(u) \in \operatorname{int} S^*$ *, then* (1.1) *has a solution* $z^0 \in S$ *.*

Proof. Since $g(z)$ is monotone on S ,

$$\operatorname{Re}(g(z), z-u) \ge \operatorname{Re}(g(u), z-u) .$$

If $z = Ax$, $u = Ay$, $x, y \in R^k_+$, then

$$\operatorname{Re}(g(u), z-u) = (x-y)^T \operatorname{Re}(A^H g(u)) .$$

Since $g(u) \in \operatorname{int} S^*$, $\operatorname{Re}(A^H g(u)) > 0$. It is then clear that there is a vector $z(p) > u$ in S such that $(x-y)^T \operatorname{Re}(A^H g(u)) \ge 0$ for all $z = z(p)$ in S . Theorem 3.3 with Lemma 4.1 now gives the result.

REMARKS 4.3. $M \in C^{n\times n}$ is said to be positive semi-definite if $\operatorname{Re} z^H Mz \ge 0$ for all $z \in C^n$. If $g(z)$ is defined by $g(z) = Mz + q$ for some matrix M and q in C^n , then g is monotone on S if M is positive semi-definite. Thus Theorem 4.2 is a generalization to nonlinear mappings of the results of McCallum [4, Theorem 4.5.1] and Mond [5, Theorem 5] in the complex linear complementarity problem on positive semi-definite matrices.

Recently, Moré [6] has extended the result of Cottle [2] on linear complementarity problem in real space to nonlinear mappings. If $S = R^n_+$ and $g : R^n \to R^n$ is a continuous mapping on R^n_+ , then Theorem 4.2 reduces to the result of Moré [6, Theorem 3.2].

If g is strictly monotone on S , then there is at most one $z^0 \in S$ which satisfies (1.1). For if z^0 and w^0 are two solutions, then $\operatorname{Re}(g(z^0)-g(w^0), z^0-w^0) = -\operatorname{Re}(g(z^0), w^0) - \operatorname{Re}(g(w^0), z^0) \le 0$, and consequently, $z^0 = w^0$.

LEMMA 4.4. *Let* S *be a polyhedral cone in* C^n *. If* $g : C^n \to C^n$

https://doi.org/10.1017/S0004972700024898 Published online by Cambridge University Press

is a continuous function concave with respect to S^* *and* $\mathrm{Re}\, z^H g(z)$ *is convex with respect to* R_+ *, then* $g(z)$ *is monotone on* S .

Proof. Concavity of $g(z)$ with respect to S^* and $z^1, z^2 \in S$ imply that for $\lambda \in (0, 1)$,

$$(4.2)\quad \mathrm{Re}\left(g\left(\lambda z^1+(1-\lambda)z^2\right),\ \lambda z^1+(1-\lambda)z^2\right) - \mathrm{Re}\left(\lambda g\left(z^1\right)+(1-\lambda)g\left(z^2\right),\ \lambda z^1+(1-\lambda)z^2\right) \geq 0 .$$

Convexity of $\mathrm{Re}\, z^H g(z)$ gives

$$(4.3)\quad \lambda\ \mathrm{Re}\left(g\left(z^1\right),\ z^1\right)+(1-\lambda)\mathrm{Re}\left(g\left(z^2\right),\ z^2\right) - \mathrm{Re}\left(g\left(\lambda z^1+(1-\lambda)z^2\right),\ \lambda z^1+(1-\lambda)z^2\right) \geq 0 .$$

From (4.2) and (4.3),

$$\lambda(1-\lambda)\mathrm{Re}\left(g\left(z^1\right)-g\left(z^2\right),\ z^1-z^2\right) \geq 0 ,$$

and consequently, $g(z)$ is monotone on S .

Now we are able to give a different version of Theorem 4.2.

THEOREM 4.5. *Let* $g : C^n \to C^n$ *be continuous on* S *and concave with respect to* S^* *on* C^n . *Let* $\mathrm{Re}\, z^H g(z)$ *be convex with respect to* R_+ *on* C^n . *If there is a* $u \in S$ *with* $g(u) \in \mathrm{int}\, S^*$ *, then* (1.1) *has a solution* z^0 *in* S .

REMARKS 4.6. It is proved by the first author [7] that if g , in addition to satisfying the hypotheses of Theorem 4.5, is analytic, then the nonlinear program

$$\text{(P):}\quad \text{minimize}\quad \mathrm{Re}\, z^H g(z) \qquad \text{subject to}\quad g(z) \in S^* ,\quad z \in S ,$$

is a self-dual problem with zero optimal value. Thus an optimal point of (P) under the said restrictions on the growth of g is a solution to (1.1).

Moreover, any feasible solution to (P) which makes the objective function vanish is necessarily a solution to (1.1). So a critical study of (P) may shed more light on this problem of existence of a solution to (1.1) under feasibility assumptions.

References

[1] Robert A. Abrams, "Nonlinear programming in complex space: sufficient conditions and duality", *J. Math. Anal. Appl.* 38 (1972), 619-632.

[2] Richard W. Cottle, "Note on a fundamental theorem in quadratic programming", *J. Soc. Indust. Appl. Math.* 12 (1964), 663-665.

[3] P. Hartman and G. Stampacchia, "On some nonlinear elliptic different differential functional equations", *Acta Math.* 115 (1966), 271-310.

[4] Charles J. McCallum, Jr, "Existence theory for the complex linear complementarity problem", *J. Math. Anal. Appl.* 40 (1972), 738-762.

[5] Bertram Mond, "On the complex complementarity problem", *Bull. Austral. Math. Soc.* 9 (1973), 249-257.

[6] Jorge J. Moré, "Classes of functions and feasibility conditions in nonlinear complementarity problems", *Math. Programming* 6 (1974), 327-338.

[7] J. Parida, "Self-duality in complex mathematical programming", *Cahiers Centre Études Recherche Opér.* (to appear).

Department of Mathematics,
Regional Engineering College,
Rourkela,
Orissa,
India.

https://doi.org/10.1017/S0004972700024898 Published online by Cambridge University Press