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Differential-free Characterisation of Smooth Mappings with Controlled Growth

Marijan Marković

Abstract. In this paper we give some generalizations and improvements of the Pavlović result on the Holland–Walsh type characterization of the Bloch space of continuously differentiable (smooth) functions in the unit ball in $\mathbf{R}^m$.

1 Introduction and the Main Result

We consider the space $\mathbf{R}^m$ equipped with the standard norm $|\zeta|$ and the scalar product $\langle\zeta,\eta\rangle$ for $\zeta\in\mathbf{R}^m$ and $\eta\in\mathbf{R}^m$. We denote by $\mathbf{B}^m$ the unit ball in $\mathbf{R}^m$. Let $\Omega\subseteq\mathbf{R}^m$ be a domain. For a differentiable mapping $f\colon\Omega\to\mathbf{R}^n$, denote by $Df(\zeta)$ its differential at $\zeta\in\Omega$, and by

$$\|Df(\zeta)\| = \sup_{\ell\in\partial\mathbf{B}^m}|Df(\zeta)\ell|$$

the norm of the linear operator $Df(\zeta)\colon\mathbf{R}^m\to\mathbf{R}^n$.

This paper is mainly motivated by the following surprising result of Pavlović [4].

Proposition 1.1 (*cf.* [4]) *A continuously differentiable complex-valued function $f(\zeta)$ in the unit ball $\mathbf{B}^m$ is a Bloch function,* i.e.,

$$\sup_{\zeta\in\mathbf{B}^m}(1-|\zeta|^2)\|Df(\zeta)\|$$

is finite if and only if the following quantity is finite:

$$\sup_{\zeta,\eta\in\mathbf{B}^m,\zeta\neq\eta}\sqrt{1-|\zeta|^2}\sqrt{1-|\eta|^2}\,\frac{|f(\zeta)-f(\eta)|}{|\zeta-\eta|}.$$

Moreover, these numbers are equal.

As Pavlović observed in [4], the above result is actually two-dimensional. Namely, if one proves it for continuously differentiable functions $\mathbf{B}^2\to\mathbf{C}$, then the general case (the case of continuously differentiable functions $\mathbf{B}^m\to\mathbf{C}$) follows from it. We give a proof of Proposition 1.1 in the next section following our main result.

Since for an analytic function $f(z)$ in the unit disc $\mathbf{B}^2$ we have $\|Df(z)\| = |f'(z)|$ for every $z\in\mathbf{B}^2$, the first part of Proposition 1.1 (without the equality statement) is the

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Holland–Walsh characterization of analytic functions in the Bloch space in the unit disc. See [3, Theorem 3], which says that $f(z)$ is a Bloch function if and only if

$$\sqrt{1-|z|^2}\sqrt{1-|w|^2}\frac{|f(z)-f(w)|}{|z-w|}$$

is bounded as a function of two variables $z \in \mathbf{B}^2$ and $w \in \mathbf{B}^2$ for $z \neq w$. This characterisation of analytic Bloch functions in the unit ball is given by Ren and Tu [5].

Our aim here is to obtain a characterisation result similar to Proposition 1.1 of continuously differentiable mappings that satisfy a certain growth condition. Before we formulate our main theorem we need to introduce some notation.

Let $\mathbf{w}(\zeta)$ be an everywhere positive continuous function in a domain $\Omega \subseteq \mathbf{R}^m$ (a weight function in Ω). We will consider continuously differentiable mappings in Ω that map this domain into $\mathbf{R}^n$ and satisfy the growth condition

$$\|f\|_{\mathbf{w}}^{\mathbf{b}} := \sup_{\zeta\in\Omega} \mathbf{w}(\zeta)\|Df(\zeta)\| < \infty.$$

We say that $\|f\|_{\mathbf{w}}^{\mathbf{b}}$ is the $\mathbf{w}$-Bloch semi-norm of the mapping f (it is easy to check that it has indeed all semi-norm properties). We denote by $\mathcal{B}_{\mathbf{w}}$ the space of all continuously differentiable mappings $f\colon \Omega \to \mathbf{R}^n$ with the finite $\mathbf{w}$-Bloch semi-norm. The space $\mathcal{B}_{\mathbf{w}}$ we call $\mathbf{w}$-Bloch space. If $\Omega = \mathbf{B}^m$ and $\mathbf{w}(\zeta) = 1-|\zeta|^2$ for $\zeta \in \mathbf{B}^m$, we just say the Bloch space, and denote it by $\mathcal{B}$.

In the sequel we will consider the $\mathbf{w}$-distance between $\zeta \in \Omega$ and $\eta \in \Omega$, which is obtained in the following way:

$$d_{\mathbf{w}}(\zeta,\eta) = \inf_{\gamma} \int_{\gamma} \frac{|d\omega|}{\mathbf{w}(\omega)},$$

where the infimum is taken over all piecewise smooth curves $\gamma \subseteq \Omega$ connecting ζ and η. It is well known that $d_{\mathbf{w}}(\zeta,\eta)$ is a distance function in the domain Ω.

One of our aims in this paper is to give a differential-free description of the $\mathbf{w}$-Bloch space and a differential-free expression for $\mathbf{w}$-Bloch semi-norm. In order to do that, for a given $\mathbf{w}(\zeta)$ in a domain Ω, we now introduce a new everywhere positive function $\mathbf{W}(\zeta,\eta)$ on the product domain $\Omega\times\Omega$ that satisfies the following four conditions. For every $\zeta \in \Omega$ and $\eta \in \Omega$,

$$
\begin{aligned}
&(W_1) \quad \mathbf{W}(\zeta,\eta) = \mathbf{W}(\eta,\zeta);\\
&(W_2) \quad \mathbf{W}(\zeta,\zeta) = \mathbf{w}(\zeta);\\
&(W_3) \quad \liminf_{\eta\to\zeta} \mathbf{W}(\zeta,\eta) \geq \mathbf{W}(\zeta,\zeta) = \mathbf{w}(\zeta);\\
&(W_4) \quad d_{\mathbf{w}}(\zeta,\eta)\mathbf{W}(\zeta,\eta) \leq |\zeta-\eta|.
\end{aligned}
$$

We say that $\mathbf{W}(\zeta,\eta)$ is admissible for $\mathbf{w}(\zeta)$.

Of course, one can pose the existence question concerning $\mathbf{W}(\zeta,\eta)$ if $\mathbf{w}(\zeta)$ is given. In the sequel we will prove that the following functions $\mathbf{W}(\zeta,\eta)$ are admissible for the given functions $\mathbf{w}(\zeta)$.

(a) The function

$$\mathbf{W}(\zeta,\eta)=\begin{cases}\mathbf{w}(\zeta), & \text{if } \zeta=\eta,\\ |\zeta-\eta|/d_{\mathbf{w}}(\zeta,\eta), & \text{if } \zeta\neq\eta,\end{cases}$$

in $\Omega\times\Omega$ is admissible for any given $\mathbf{w}(\zeta)$ in Ω;

(b) If $\mathbf{w}(\zeta)=1-|\zeta|^2$ for $\zeta\in\mathbf{B}^m$, then $d_{\mathbf{w}}(\zeta,\eta)$ is the hyperbolic distance in the unit ball $\mathbf{B}^m$. One of the admissible functions is

$$\mathbf{W}(\zeta,\eta)=\sqrt{1-|\zeta|^2}\sqrt{1-|\eta|^2}.$$

This is shown in the next section. From this fact we deduce the Pavlović result stated in the above proposition.

(c) If Ω is a convex domain and if $\mathbf{w}(\zeta)$ is a decreasing function in $|\zeta|$, then

$$\mathbf{W}(\zeta,\eta)=\min\{\mathbf{w}(\zeta),\mathbf{w}(\eta)\}$$

is admissible for $\mathbf{w}(\zeta)$. It would be of interest to find such simple admissible functions for more general domains Ω and/or more general functions $\mathbf{w}$.

For a mapping $f\colon\Omega\to\mathbf{R}^n$ introduce now the quantity

$$\|f\|^{\mathbf{l}}_{\mathbf{W}}:=\sup_{\zeta,\eta\in\Omega,\zeta\neq\eta}\mathbf{W}(\zeta,\eta)\frac{|f(\zeta)-f(\eta)|}{|\zeta-\eta|}.$$

We call it the $\mathbf{W}$-Lipschitz semi-norm (it is also an easy task to check that it is indeed a semi-norm). The space of all continuously differentiable mappings $f\colon\Omega\to\mathbf{R}^n$ for which its $\mathbf{W}$-Lipschitz semi-norm $\|f\|^{\mathbf{l}}_{\mathbf{W}}$ is finite is denoted by $\mathcal{L}_{\mathbf{W}}$. Note that if $\mathbf{W}(\zeta,\eta)$ is not symmetric, we can replace it by $\widetilde{\mathbf{W}}(\zeta,\eta)=\max\{\mathbf{W}(\zeta,\eta),\mathbf{W}(\eta,\zeta)\}$ which produces the same Lipschitz type semi-norm.

Our main result in this paper shows that for any continuously differentiable mapping $f\colon\Omega\to\mathbf{R}^n$, we have $\|f\|^{\mathbf{b}}_{\mathbf{w}}=\|f\|^{\mathbf{l}}_{\mathbf{W}}$; *i.e.*, the $\mathbf{w}$-Bloch semi-norm is equal to the $\mathbf{W}$-Lipschitz semi-norm of the mapping f. As a consequence we have the coincidence of the two spaces $\mathcal{B}_{\mathbf{w}}=\mathcal{L}_{\mathbf{W}}$. Thus, the space $\mathcal{B}_{\mathbf{w}}$ may be described as

$$\mathcal{B}_{\mathbf{w}}=\Big\{f\colon\Omega\to\mathbf{R}^n:\sup_{\zeta,\eta\in\Omega,\zeta\neq\eta}\mathbf{W}(\zeta,\eta)|f(\zeta)-f(\eta)|/|\zeta-\eta|<\infty\Big\},$$

where $\mathbf{W}(\zeta,\eta)$ is any admissible function for $\mathbf{w}(\zeta)$. This is the content of the following theorem.

Theorem 1.2 *Let $\Omega\subseteq\mathbf{R}^m$ be a domain and let $f\colon\Omega\to\mathbf{R}^n$ be a continuously differentiable mapping. Let $\mathbf{w}(\zeta)$ be positive and continuous in Ω, and let $\mathbf{W}(\zeta,\eta)$ be an admissible function for $\mathbf{w}(\zeta)$. If one of the numbers $\|f\|^{\mathbf{b}}_{\mathbf{w}}$ and $\|f\|^{\mathbf{l}}_{\mathbf{W}}$ is finite, then both numbers are finite and equal.*

Proof For one direction, assume that $\mathbf{W}$-Lipschitz semi-norm of the mapping f is finite, *i.e.*, that the quantity

$$\sup_{\zeta,\eta\in\Omega,\zeta\neq\eta}\mathbf{W}(\zeta,\eta)\frac{|f(\zeta)-f(\eta)|}{|\zeta-\eta|}$$

is finite. We will show that $\|f\|^{\mathbf{b}}_{\mathbf{w}}\le\|f\|^{\mathbf{l}}_{\mathbf{W}}$, which implies that $\|f\|^{\mathbf{b}}_{\mathbf{w}}$ is also finite.

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If we have in mind that

$$\limsup_{\omega\to\zeta} \frac{|f(\zeta)-f(\omega)|}{|\zeta-\omega|} = \|Df(\zeta)\|$$

for every $\zeta \in \Omega$, we obtain

$$\begin{aligned}\|f\|^{\mathbf{l}}_{\mathbf{W}} &= \sup_{\eta,\omega\in\Omega,\eta\neq\omega} \mathbf{W}(\eta,\omega)\frac{|f(\eta)-f(\omega)|}{|\eta-\omega|} \geq \limsup_{\omega\to\zeta} \mathbf{W}(\zeta,\omega)\frac{|f(\zeta)-f(\omega)|}{|\zeta-\omega|}\\ &\geq \liminf_{\omega\to\zeta} \mathbf{W}(\zeta,\omega) \limsup_{\omega\to\zeta} \frac{|f(\zeta)-f(\omega)|}{|\zeta-\omega|} = \mathbf{W}(\zeta,\zeta)\|Df(\zeta)\|\\ &= \mathbf{w}(\zeta)\|Df(\zeta)\|.\end{aligned}$$

It follows that

$$\|f\|^{\mathbf{l}}_{\mathbf{W}} \geq \sup_{\zeta\in\Omega} \mathbf{w}(\zeta)\|Df(\zeta)\| = \|f\|^{\mathbf{b}}_{\mathbf{w}},$$

which we aimed to prove.

Assume now that $\|f\|^{\mathbf{b}}_{\mathbf{w}}$ is finite. We will prove the reverse inequality $\|f\|^{\mathbf{l}}_{\mathbf{W}} \leq \|f\|^{\mathbf{b}}_{\mathbf{w}}$. Let $\zeta \in \Omega$ and $\eta \in \Omega$ be arbitrary and different and let $\gamma \subseteq \Omega$ be any piecewise smooth curve parameterized by $t \in [0,1]$ that connects ζ and η, *i.e.,* for which $\gamma(0) = \zeta$ and $\gamma(1) = \eta$. Since $\|f\|^{\mathbf{b}}_{\mathbf{w}}$ is finite, we obtain

$$\begin{aligned}|f(\zeta)-f(\eta)| = |(f\circ\gamma)(1)-(f\circ\gamma)(0)| &= \Big|\int_0^1 ((f\circ\gamma)(t))'dt\Big|\\ &= \Big|\int_0^1 Df(\gamma(t))\gamma'(t)dt\Big| \leq \int_0^1 \big|Df(\gamma(t))\gamma'(t)\big|dt\\ &\leq \int_0^1 \big\|Df(\gamma(t))\big\||\gamma'(t)|dt \leq \|f\|^{\mathbf{b}}_{\mathbf{w}} \int_0^1 \frac{|\gamma'(t)|dt}{\mathbf{w}(\gamma(t))}\\ &= \|f\|^{\mathbf{b}}_{\mathbf{w}} \int_\gamma \frac{|d\omega|}{\mathbf{w}(\omega)}.\end{aligned}$$

If we take infimum over all such curves γ, we obtain

$$|f(\zeta)-f(\eta)| \leq \|f\|^{\mathbf{b}}_{\mathbf{w}} d_{\mathbf{w}}(\zeta,\eta).$$

Because of our conditions posed on the function $\mathbf{W}(\zeta,\eta)$, we have

$$\mathbf{W}(\zeta,\eta)\frac{|f(\zeta)-f(\eta)|}{|\zeta-\eta|} \leq \mathbf{W}(\zeta,\eta)\frac{d_{\mathbf{w}}(\zeta,\eta)}{|\zeta-\eta|}\|f\|^{\mathbf{b}}_{\mathbf{w}} \leq \|f\|^{\mathbf{b}}_{\mathbf{w}}.$$

Therefore,

$$\|f\|^{\mathbf{l}}_{\mathbf{W}} = \sup_{\zeta,\eta\in\Omega,\zeta\neq\eta} \mathbf{W}(\zeta,\eta)\frac{|f(\zeta)-f(\eta)|}{|\zeta-\eta|} \leq \|f\|^{\mathbf{b}}_{\mathbf{w}},$$

which we needed to prove. ■

Remark 1.3 Let $\mathbf{w}(\zeta)$ be a weight in a domain $\Omega \subseteq \mathbf{R}^m$. Observe that we have

$$\sup_{\zeta\in\Omega} \mathbf{w}(\zeta) = \sup_{\substack{\zeta,\eta\in\Omega\\ \zeta\neq\eta}} \mathbf{W}(\zeta,\eta),$$

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where $\mathbf{W}(\zeta,\eta)$ is admissible for $\mathbf{w}(\zeta)$. This remark is a direct consequence of the fact that we can set the identity $f(\zeta) = \mathrm{Id}(\zeta)$ in our main theorem.

2 On the Pavlović Result

As we have previously stated, if we take $\mathbf{w}(\zeta) = 1 - |\zeta|^2$ for $\zeta \in \mathbf{B}^m$, then $\mathbf{w}$-distance is the hyperbolic distance. For the hyperbolic distance between $\zeta \in \mathbf{B}^m$ and $\eta \in \mathbf{B}^m$ we will use the usual notation $\rho(\zeta,\eta)$.

It is well known that the hyperbolic distance is invariant under Möbius transforms of the unit ball; *i.e.,* if $T\colon \mathbf{B}^m \to \mathbf{B}^m$ is a Möbius transform, then we have

$$\rho\big(T(\zeta), T(\eta)\big) = \rho(\zeta,\eta)$$

for every $\zeta \in \mathbf{B}^m$ and $\eta \in \mathbf{B}^m$.

Up to an orthogonal transform, a Möbius transform of the unit ball $\mathbf{B}^m$ onto itself can be represented as

$$T_\zeta(\eta) = \frac{-(1-|\zeta|^2)(\zeta-\eta) - |\zeta-\eta|^2\zeta}{[\zeta,\eta]^2}, \quad \eta \in \mathbf{B}^m$$

for $\zeta \in \mathbf{B}^m$, where

$$[\zeta,\eta]^2 = 1 - 2\langle\zeta,\eta\rangle + |\zeta|^2|\eta|^2.$$

It is known that

$$|T_\zeta\eta| = \frac{|\zeta-\eta|}{[\zeta,\eta]} \quad \text{and} \quad 1 - |T_\zeta\eta|^2 = \frac{(1-|\zeta|^2)(1-|\eta|^2)}{[\zeta,\eta]^2}$$

for every $\zeta \in \mathbf{B}^m$ and $\eta \in \mathbf{B}^m$.

Particularly, one easily calculates

$$\rho(0,\omega) = \frac{1}{2}\log\frac{1+|\omega|}{1-|\omega|}$$

for $\omega \in \mathbf{B}^m$. Because of the invariance with respect to the group of Möbius transforms of the unit ball, the hyperbolic distance between $\zeta \in \mathbf{B}^m$ and $\eta \in \mathbf{B}^m$ can be expressed as

$$\rho(\zeta,\eta) = \frac{1}{2}\log\frac{1+|T_\zeta\eta|}{1-|T_\zeta\eta|} = \operatorname{atanh}|T_\zeta(\eta)|.$$

For all mentioned facts and identities above, we refer the reader to Ahlfors [1] or Vuorinen [6].

Proposition 1.1 can be seen as a consequence of our main result and the following elementary lemma, which proves that $\mathbf{W}(\zeta,\eta) = \sqrt{1-|\zeta|^2}\sqrt{1-|\eta|^2}$ has W_4-property, and therefore it is admissible for $\mathbf{w}(\zeta) = 1 - |\zeta|^2$.

Lemma 2.1 *The function $\mathbf{W}(\zeta,\eta) = \sqrt{1-|\zeta|^2}\sqrt{1-|\eta|^2}$ satisfies the inequality $\rho(\zeta,\eta)\mathbf{W}(\zeta,\eta) \le |\zeta-\eta|$ for every $\zeta \in \mathbf{B}^m$ and $\eta \in \mathbf{B}^m$.*

Proof We will first establish the following special case of the inequality we need:

$$\rho(0,\omega)\sqrt{1-|\omega|^2} \le |\omega|$$

for $\omega \in \mathbf{B}^m$.

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Since

$$\rho(0,\omega)=\frac{1}{2}\log\frac{1+|\omega|}{1-|\omega|},$$

if we take $t=|\omega|$, the above inequality is equivalent to the following one:

$$\frac{1}{2}\log\frac{1+t}{1-t}\le\frac{t}{\sqrt{1-t^2}},$$

where $0\le t<1$. Denote the difference of the left-hand side minus the right-hand side by $F(t)$. Then we have

$$F'(t)=-\frac{1}{(1-t^2)^{3/2}}+\frac{1}{1-t^2},\quad 0<t<1.$$

Since $F'(t)<0$ for $0<t<1$, it follows that $F(t)$ is a decreasing function in $[0,1)$. Therefore, $F(t)\le F(0)=0$, which implies the inequality we aimed to prove.

In the inequality we have just proved, let us take $\omega=T_\zeta\eta$, where $\zeta\in\mathbf{B}^m$ and $\eta\in\mathbf{B}^m$ are arbitrary. Then we have

$$\rho(0,\omega)=\rho(T_\zeta\zeta,T_\zeta\eta)=\rho(\zeta,\eta),$$

$$\sqrt{1-|\omega|^2}=\sqrt{1-|T_\zeta\eta|^2}=\frac{\sqrt{1-|\zeta|^2}\sqrt{1-|\eta|^2}}{[\zeta,\eta]},$$

as well as

$$|\omega|=|T_\zeta\eta|=\frac{|\zeta-\eta|}{[\zeta,\eta]}.$$

If we substitute all above expressions, we obtain the inequality in the statement of our lemma. ■

Remark 2.2 One more expression for the hyperbolic distance in the unit ball is given by

$$\sinh^2\rho(\zeta,\eta)=\frac{|\zeta-\eta|^2}{(1-|\zeta|^2)(1-|\eta|^2)}$$

(see [6]). Using the elementary inequality $t\le\sinh t$, as suggested by the referee, one deduces the inequality in the above lemma.

3 Some Other Consequences of the Main Theorem

In this section we will derive some new consequences of our main result.

Corollary 3.1 *Let $\mathbf{w}(\zeta)$ be an everywhere positive, continuous, and decreasing function of $|\zeta|$ in a convex domain $\Omega\subseteq\mathbf{R}^m$. Then we have*

$$\sup_{\zeta\in\Omega}\mathbf{w}(\zeta)\|Df(\zeta)\|=\sup_{\zeta,\eta\in\Omega,\zeta\neq\eta}\min\{\mathbf{w}(\zeta),\mathbf{w}(\eta)\}\frac{|f(\zeta)-f(\eta)|}{|\zeta-\eta|}$$

for every continuously differentiable mapping $f\colon\Omega\to\mathbf{R}^n$.

Proof Let

$$\mathbf{W}(\zeta,\eta)=\min\{\mathbf{w}(\zeta),\mathbf{w}(\eta)\},$$

for $(\zeta,\eta)\in\Omega\times\Omega$. We have only to check if $\mathbf{W}(\zeta,\eta)$ satisfies conditions (W_1)–(W_4) and to apply our main theorem.

It is clear that $\mathbf{W}(\zeta,\eta)$ is symmetric and that $\mathbf{W}(\zeta,\zeta)=\mathbf{w}(\zeta)$. Since $\mathbf{W}(\zeta,\eta)$ is continuous in $\Omega\times\Omega$, the (W_3)-condition for $\mathbf{W}(\zeta,\eta)$ obviously holds. Therefore, it remains to check if the following inequality is true:

$$d_{\mathbf{w}}(\zeta,\eta)\min\{\mathbf{w}(\zeta),\mathbf{w}(\eta)\}\le|\zeta-\eta|$$

for every $(\zeta,\eta)\in\Omega\times\Omega$.

Let $\zeta\in\Omega$ and $\eta\in\Omega$ be arbitrary and fixed and let $\gamma\subseteq\Omega$ be among piecewise smooth curves that join ζ and η. We have

$$\begin{aligned} d_{\mathbf{w}}(\zeta,\eta)&=\inf_{\gamma}\int_{\gamma}\frac{|d\omega|}{\mathbf{w}(\omega)}\le\int_{[\zeta,\eta]}\frac{|d\omega|}{\mathbf{w}(\omega)}\le\int_{[\zeta,\eta]}\max_{\omega\in[\zeta,\eta]}\Big\{\frac{1}{\mathbf{w}(\omega)}\Big\}|d\omega|\\ &\le\max\Big\{\frac{1}{\mathbf{w}(\zeta)},\frac{1}{\mathbf{w}(\eta)}\Big\}\int_{[\zeta,\eta]}|d\omega|=\max\Big\{\frac{1}{\mathbf{w}(\zeta)},\frac{1}{\mathbf{w}(\eta)}\Big\}|\zeta-\eta|\\ &=\min\{\mathbf{w}(\zeta),\mathbf{w}(\eta)\}^{-1}|\zeta-\eta|,\end{aligned}$$

where we have used in the fourth step our assumption that $\mathbf{w}(\omega)$ is decreasing in $|\omega|$ and that the maximum modulus of points on a line segment is attained at an endpoint. The inequality we need follows. ∎

Remark 3.2 Since the function $\mathbf{w}(\zeta)=1-|\zeta|^2$ is decreasing in $|\zeta|$ in the unit ball $\mathbf{B}^m$, the above corollary produces a new Holland–Walsh type characterisation of continuously differentiable Bloch mappings. Notice that $\min\{A,B\}\le\sqrt{A}\sqrt{B}$ for all non-negative numbers A and B. Because of this inequality, it seems that Corollary 3.1 improves the Pavlović result stated at the beginning of the paper as Proposition 1.1.

Corollary 3.3 *Let $\mathbf{w}(\zeta)$ be an everywhere positive and continuous function in a domain Ω and let $d_{\mathbf{w}}(\zeta,\eta)$ be the $\mathbf{w}$-distance in Ω. Then we have*

$$\sup_{\zeta\in\Omega}\mathbf{w}(\zeta)\|Df(\zeta)\|=\sup_{\zeta,\eta\in\Omega,\zeta\ne\eta}\frac{|f(\zeta)-f(\eta)|}{d_{\mathbf{w}}(\zeta,\eta)}$$

for any continuously differentiable mappings $f\colon\Omega\to\mathbf{R}^n$.

Proof For $\zeta\in\Omega$ and $\eta\in\Omega$, let

$$\mathbf{W}(\zeta,\eta)=\begin{cases}\mathbf{w}(\zeta), & \text{if } \zeta=\eta,\\ |\zeta-\eta|/d_{\mathbf{w}}(\zeta,\eta), & \text{if } \zeta\ne\eta.\end{cases}$$

It is enough to show that $\mathbf{W}(\zeta,\eta)$ is admissible for $\mathbf{w}(\zeta)$. It is clear that $\mathbf{W}(\zeta,\eta)$ is symmetric. The (W_4)-condition for $\mathbf{W}(\zeta,\eta)$ is obviously satisfied, and here it is optimal in some sense. Therefore, we have only to check if $\mathbf{W}(\zeta,\eta)$ satisfies the (W_3)-condition:

$$\liminf_{\eta\to\zeta}\mathbf{W}(\zeta,\eta)\ge\mathbf{W}(\zeta,\zeta).$$

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This means that we need to show that

$$\liminf_{\eta\to\zeta} \frac{|\zeta-\eta|}{d_{\mathbf{w}}(\zeta,\eta)} \geq \mathbf{w}(\zeta).$$

If we invert both sides, we obtain that we have to prove

$$\limsup_{\eta\to\zeta} \frac{d_{\mathbf{w}}(\zeta,\eta)}{|\zeta-\eta|} \leq \frac{1}{\mathbf{w}(\zeta)}.$$

for every $\zeta \in \Omega$.

Since this is a local question, we may assume that η is in a convex neighborhood of ζ. Let γ be among piecewise smooth curves in Ω connecting ζ and η. We have

$$\begin{aligned}\limsup_{\eta\to\zeta} \frac{1}{|\zeta-\eta|} \inf_{\gamma} \int_{\gamma} \frac{|d\omega|}{\mathbf{w}(\omega)} &\leq \limsup_{\eta\to\zeta} \frac{1}{|\zeta-\eta|} \int_{[\zeta,\eta]} \frac{|d\omega|}{\mathbf{w}(\omega)} \\ &= \lim_{\eta\to\zeta} \frac{1}{|\zeta-\eta|} \int_{[\zeta,\eta]} \frac{|d\omega|}{\mathbf{w}(\omega)} = \frac{1}{\mathbf{w}(\zeta)},\end{aligned}$$

which we wanted to prove. The equalities above follow, because of continuity of the function $\mathbf{w}(\zeta)$. ∎

Remark 3.4 In the case $\mathbf{w}(\zeta) = (1-|\zeta|^2)^\alpha$ for $\zeta \in \mathbf{B}^2$, where $\alpha > 0$ is a constant, Corollary 3.3 is proved by Zhu in [8] for analytic functions (see [8, Theorem 19]). A variant of this corollary is obtain in [7] (see also [7, Theorem 1] for analytic functions).

As a special case of the above corollary, we have the following one (certainly very well known for analytic Bloch functions in the unit disc).

Corollary 3.5 *A continuously differentiable mapping $f\colon \mathbf{B}^m \to \mathbf{R}^n$ is a Bloch mapping (*i.e., *$f \in \mathcal{B}$) if and only if it is a Lipschitz mapping with respect to the Euclidean and hyperbolic distance in $\mathbf{R}^n$ and $\mathbf{B}^m$. In other words, for the mapping f, there holds*

$$|f(\zeta)-f(\eta)| \leq C\rho(\zeta,\eta)$$

for a constant C, if and only if $f \in \mathcal{B}$. Moreover, the optimal constant C is

$$C = \sup\{\,(1-|\zeta|^2)\|Df(\zeta)\| : \zeta \in \mathbf{B}^m\}$$

(for a given $f \in \mathcal{B}$).

Remark 3.6 The result of the last corollary is proved for harmonic mappings of the unit disc into itself by Colonna in [2], where it is also found that the constant C is always less or equal to $4/\pi$ for such type of mappings.

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Faculty of Natural Sciences and Mathematics, University of Montenegro, Cetinjski put b.b., 81000 Podgorica, Montenegro
e-mail: marijanmmarkovic@gmail.com