$\lambda(n)$-PARAMETER FAMILIES

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I is an interval of R, the set of real numbers, n is a positive integer and $F \subset C^j(I)$ for $j \geq 0$ large enough so that the following definitions are possible:

(i) Let $\lambda(n) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $k, \lambda_1, \lambda_2, \ldots, \lambda_k$ are positive integers and $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$. Then $\lambda(n)$ is an _ordered partition of n_. The set of all such partitions of n is denoted by P(n).

(ii) Let $\lambda(n) \in P(n)$ be given. The family F of real valued functions on I is said to be a _$\lambda(n)$-parameter family_ on I in case for any choice of points $x_1 < x_2 < \ldots < x_k$ in I and any set of n real numbers $y_i^{(j)}$ there is a unique $f \in F$ satisfying

$$(1) \qquad f^{(j)}(x_i) = y_i^{(j)}, \quad j = 0, 1, 2, \ldots, \lambda_i - 1, \quad i = 1, 2, \ldots, k.$$

If F is a $\lambda(n)$-parameter family for $\lambda(n) = (1, 1, \ldots, 1)$, then F is called an _n-parameter family_. (See [5].) If F is a $\lambda(n)$-parameter family on I for $\lambda(n) = n$, i.e., all conditions are specified at one point, then we will say that _initial value problems are uniquely solvable_ in F on I. If F is a $\lambda(n)$-parameter family on I for all $\lambda(n) \in P(n)$, then F is called an _unrestricted n-parameter family_ on I. (See [1].)

P. Hartman [1] proved the following:

THEOREM. _A family $F \subset C^{n-1}(I)$, where I is an open interval of R, is an unrestricted n-parameter family on I if and only if F is an n-parameter family on I and initial value problems are uniquely solvable in F on I._

Z. Opial [4] gave a very nice short proof of Hartman's result in the case that F is the solution set for an n^{th} order homogeneous linear differential equation with summable coefficients. Opial's proof uses the linearity of F (F is an n-dimensional real vector space) and the continuity with respect to initial values.

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What happens if I is not open? In [1] Hartman poses that question for I closed, but does not resolve it. We give here an example to show that neither Hartman's Theorem nor Opial's Theorem is valid if I is closed. Before giving the example we have the following definition and lemma:

(iii) Let $L_n[y] = y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$ be a homogenous linear differential equation with continuous coefficients on I. For each $s \in I$ let $K(x, s)$ be the solution to the initial value problem

$$L_n[y] = 0, \quad y(s) = y'(s) = \ldots = y^{(n-2)}(s) = 0 = y^{(n-1)}(s) - 1 .$$

K is called the Cauchy function for L_n.

(iv) L_n (or $L_n[y] = 0$) is said to be disconjugate on I in case no non-trivial solution to $L_n[y] = 0$ has more than $n-1$ zeros (counting multiplicity) on I.

Clearly L_n being disconjugate on I implies that

$$\text{(2)} \qquad \operatorname{sgn} K(x, s) = \operatorname{sgn} (x-s)^{n-1} \quad \text{for all } x \text{ and } s \text{ in } I .$$

$$(\operatorname{sgn} x = 0 \text{ if } x = 0,\ 1 \text{ if } x > 0,\ -1 \text{ if } x < 0.)$$

The lemma in [2] states that the converse is also true for $n = 3$. We state that lemma here and supply it with a quite different proof.

LEMMA 1. *Let* a_0, a_1 *and* a_2 *be continuous on* I. *If* $K(x, s) > 0$ *for all* x *and* s *in* I *with* $x \neq s$, *then* $L_3[y] = 0$ *is disconjugate on* I.

Proof. Let s_1, s_2 and s_3 be three distinct points in I. Then $K(x, s_1)$, $K(x, s_2)$ and $K(x, s_3)$ are linearly independent on I. Suppose that $c_1 K(x, s_1) + c_2 K(x, s_2) + c_3 K(x, s_3) = 0$ for all x in I. Put $x = s_1$, s_2 and s_3. The resulting homogeneous system of equations in c_1, c_2 and c_3 has only the solution $c_1 = c_2 = c_3 = 0$.

Now suppose that $y = y_0(x)$ is a non-trivial solution to $L_3[y] = 0$ on I, and let $y_0(x)$ have three zeros in I. Clearly $y_0(x)$ cannot have a double zero, so let the zeros be s_1, s_2 and s_3. Then there are constants

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c_1, c_2 and c_3 so that $y_0(x) = c_1 K(x, s_1) + c_2 K(x, s_2) + c_3 K(x, s_3)$ for all x in I. But then $c_1 = c_2 = c_3 = 0$ as above. Hence L_3 is disconjugate.

Example: (This example is from the author's doctoral dissertation written under the direction of Professor L.K. Jackson at the University of Nebraska; see [3].) Let F be the set of solutions to the differential equation

$$(3) \qquad x^3 y''' + 4x^2 y'' + 3xy' + y = 0$$

on the interval $[1, x_0]$ where x_0 is the first zero of $\frac{1}{x} + \sin \log x - \cos \log x$ to the right of 1. We will show that F is a 3-parameter family on $[1, x_0]$ but F is not an unrestricted 3-parameter family on $[1, x_0]$. The Cauchy function for (3) is given by $K(x, s) = \frac{s^2}{2}\left(\frac{s}{x} + \sin \log \frac{x}{s} - \cos \log \frac{x}{s}\right)$, so $K(x_0, 1) = 0$ and $K(x, 1) > 0$ for $1 < x < x_0$. $K(x, s) = s^2 K\left(\frac{x}{s}, 1\right)$, so $K(x, s) > 0$ for $s < x < x_0 s$. Also, one can show, using derivatives, that $K(x, s) > 0$ if $1 < s \le x_0$ and $x < s$. Hence (3) is disconjugate on $[1, x_0)$ and $(1, x_0]$ by Lemma 1. Let $y = y_0(x)$ be a solution to (3) satisfying $0 = y_0(1) = y_0(x_0) = y_0(c)$, $1 < c < x_0$. Then $y_0(x) = c_1 u_1(x) + c_2 K(x, 1)$ where c_1 and c_2 are constants and $u_1(x) = \frac{1}{2x} + \frac{3}{2} \sin \log x - \frac{1}{2} \cos \log x$. Now $\exp(5\pi/4) < x_0 < \exp(4\pi/3)$, so that $u_1(x_0) < 0$ and therefore $y_0(c) = 0$ implies that $c_2 = 0$; so by the linearity of F we conclude that F is a 3-parameter family on $[1, x_0]$, but clearly F is not an unrestricted 3-parameter family on $[1, x_0]$ since $y \equiv 0$ and $y = K(x, 1)$ both satisfy $0 = y(1) = y'(1) = y(x_0)$.

Is Hartman's Theorem valid on a half open interval? If F is linear and $n = 3$, 4 or 5, then the answer is yes. The author conjectures that if F is linear and I is half open, then Hartman's Theorem is correct. A proof of this for the general case (n arbitrary) has not yet been given.

A natural question which arises is whether Lemma 1 generalizes, and, if so, how. It is easy to give an example to show that (2) does not imply disconjugacy for L_n if $n > 3$; for instance, for $L_n[y] \equiv y^{(n)} + y^{(n-2)}$, (2) is satisfied on $(-\infty, \infty)$, but L_n is clearly not disconjugate on $(-\infty, \infty)$.

One generalization of Lemma 1 is the following theorem:

THEOREM 1. Let F be a linear $\lambda(n)$ - parameter family on I for $\lambda(n) = (2, 1, 1, \ldots, 1)$, $\lambda(n) = (1, 1, \ldots, 1, 2)$ and $\lambda(n) = (2, 1, 1, \ldots, 1, 2)$. Then F is an n-parameter family on I.

Proof. Let f be a non-trivial member of F satisfying $f(x_1) = f(x_2) = \ldots = f(x_n) = 0$ where $x_1 < x_2 < \ldots < x_n$ are points in I. Define $g \in F$ by $g(x_1) = g'(x_1) - f'(x_1) = g(z_3) = g(z_4) = \ldots = g(z_{n-2}) = g(x_n) = g'(x_n) = 0$, where $z_i = (x_i + x_{i+1})/2$ for $i = 3, 4, \ldots, n-2$. By hypothesis we must have $f'(x) \neq 0$ for $x = x_1, x_2, x_{n-1}$ and x_n. But $g'(x_1) = f'(x_1) \neq 0$, so $g(x)$ does not vanish identically on I. g has a double zero at x_n and zeros at $x_1, z_3, z_4, \ldots, z_{n-2}$, so these points are the only zeros of g. If f changes sign at all the points x_i, then $f - g$ changes sign in each of the intervals (x_3, z_3), (z_3, z_4), $(z_4, z_5), \ldots, (z_{n-3}, z_{n-2})$. $f - g$ also changes sign in (z_{n-2}, x_n) since $g'(x_n) = 0 \neq f'(x_n)$, $f(x_{n-1}) = f(x_n) = 0$, f changes sign at x_{n-1} and $g(z_{n-2}) = g(x_n) = 0$. So $f - g$ has a double zero at x_1 and $n - 2$ other zeros. This contradicts the uniqueness of solutions to (1) in F for $\lambda(n) = (2, 1, 1, \ldots, 1)$. If f does not change sign at all the points x_i, let s be the number of points x_i, $i = 3, 4, \ldots, n-1$ at which f changes sign and let d be the number of points x_i, $3 \leq i \leq n-2$, at which f does not change sign. Then $s + d = n - 3$. To each zero x_i at which f changes sign there corresponds a point p_i, $z_{i-1} < p_i < z_i$, $i = 4, 5, \ldots, n-2$ and $z_{n-2} < p_{n-2} < x_n$, such that $f - g$ changes sign. Hence we have at least s changes of sign of $f - g$. Let d_1 be the number of double zeros x_i of f (i.e. f does not change sign at x_i) such that f and g have the same sign in $(x_i - \delta, x_i + \delta)$ for $\delta > 0$ sufficiently small. There are two zeros of $f - g$ in (z_{i-1}, z_i) for each of these d_1 double zeros of f. $f - g$ has a double zero at x_1 and as we have shown above, at least $s + 2d_1 + 1$ other zeros in I. But since f and g are not identical, we must have $s + 2d_1 + 1 < n - 2$, i.e., $2d_1 < n - s - 3 = d$, and then $2(d - d_1) > d$. At the remaining $d - d_1$ points x_i at which f does not change sign we must have a $\delta > 0$ so that $f(x)$ and $g(x)$ have opposite signs for $0 < |x - x_i| < \delta$. Then for $\varepsilon > 0$ small enough the graphs of f and $-\varepsilon g$ will intersect at two points (in (z_{i-1}, z_i)) separated by x_i. Hence for $\varepsilon > 0$ small enough there will be two points in (z_{i-1}, z_i) at which $f + \varepsilon g$ changes sign. This will give $2(d - d_1)$ changes of sign for $f + \varepsilon g$. Also to each p_i there corresponds a q_i, $z_{i-1} < q_i < z_i$

and $z_{n-2} < q_{n-2} < x_n$, at which $f + \varepsilon g$ changes sign. This follows since $f(x_i) = 0$, $f'(x_i) \neq 0$, $f(x) \neq 0$ for x in (z_{i-1}, z_i) with $x \neq x_i$, and $-\varepsilon g(z_{i-1}) = -\varepsilon g(z_i) = 0$. $f + \varepsilon g$ must vanish in (z_{n-2}, x_n), since $f(x_{n-1}) = f(x_n) = g(x_n) = g(z_{n-2}) = 0$ and $f'(x_n) \neq 0 = g'(x_n)$. So we will have $2(d - d_1) + s \geq d + s + 1 = n - 2$ changes of sign in (x_1, x_n). This is impossible (as we showed in the first part of this proof) since $f + \varepsilon g$ will also have zeros (simple) at x_1 and x_n, and we know that $f + \varepsilon g$ is in F because F is a linear family on I. Hence no such non-trivial $f \in F$ can exist. This shows that uniqueness of solutions of (1) in F for $\lambda(n) = (1, 1, \ldots, 1)$. The existence of solutions of (1) in F for the same $\lambda(n)$ follows immediately from uniqueness since F is linear. To show this let $\{f_1, f_2, \ldots, f_n\}$ be a basis for F. Then there exist n constants $c_1, c_2, \ldots, c_n$ so that $f = \sum_{i=1}^{n} c_i f_i$. But the $n \times n$ system of linear equations generated by (1) from this representation of f must (by uniqueness) have a non-zero coefficient determinant, and hence that system has a solution. This proves the theorem. We here note that in general for a linear family F uniqueness of solutions of (1) in F for a given $\lambda(n)$ implies the existence of solutions of (1) in F for that $\lambda(n)$.

COROLLARY. _If F is a linear $\lambda(n)$-parameter family on the open interval I for $\lambda(n) = n$ and the values of $\lambda(n)$ as given in Theorem 1, then F is an unrestricted n-parameter family on I._

The corollary follows directly from Theorem 1 and Hartman's Theorem.

An affirmative answer to the question Q below would yield another generalization of Lemma 1.

Q. If F is a linear $\lambda(n)$-parameter family on I for $|\lambda(n)| \leq 2$, is F an unrestricted n-parameter family on I? ($|\lambda(n)|$ denotes the length of the partition $\lambda(n)$.) That the answer to Q is yes for $n = 4$ follows from Lemma 1 and from Lemma 2 below. Q is as yet unsolved for $n \geq 5$.

LEMMA 2. _If F is a linear $\lambda(n)$-parameter family for $\lambda(n) = (n-1, 1)$ and $(n-2, 2)$ (or for $\lambda(n) = (1, n-1)$ and $(2, n-2)$), then F is an $(n-2, 1, 1)$-parameter (or $(1, 1, n-2)$-parameter) family on_ I.

Proof. Let f be a non-trivial member of F with $f^{(i)}(x_1) = 0$ for $i = 0, 1, 2, \ldots, n-3$, $f(x_2) = f(x_3) = 0$ where $x_1 < x_2 < x_3$ are three points in I. Let $p = (x_2 + x_3)/2$ and pick $g \in F$ satisfying $g^{(i)}(x_1) = 0$ for $i = 0, 1, 2, \ldots, n-2$ and $g(p) = f(p)/2 \neq 0$. (We assume without loss of generality that $f(x) < 0$ in (x_1, x_2) and $f(x) > 0$ in (x_2, x_3).) $g(x) > 0$ for $x > x_1$, so there are points in (x_2, x_3) at which $f(x) - g(x) > 0$. Let M be the set of real numbers γ such that $f(x) - \gamma g(x) > 0$ for some points in (x_2, x_3). Let $\gamma_0 = \sup M$. Then there is a point x_0 in (x_2, x_3) such that $h = f - \gamma_0 g$ satisfies $h(x_0) = h'(x_0) = 0$. Also $h^{(i)}(x_1) = 0$ for $i = 0, 1, 2, \ldots, n-3$. F is a $(n-2, 2)$-parameter family, so $h(x) = 0$ for all x in I. This of course is impossible since $h(x_2) = -\gamma_0 g(x_2) < 0$. The other half of the lemma follows in a similar fashion.

In terms of boundary value problems for ordinary differential equations, question Q can be phrased as follows:

> If every two point boundary value problem is solvable, is every boundary value problem solvable?

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Added in proof. The author has recently become aware of two papers ([6] and [7]) which answer the last question in the affirmative for a linear differential equation with continuous coefficients.

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