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INFINITE QUASI-NORMAL MATRICES

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1. Introduction. If A is a finite matrix with complex elements, and if $A = A^T$ (where A^T denotes the transpose of A), it is known (see [**8**]) that there exists a unitary matrix U such that $UAU^T = D$ is a real diagonal matrix with non-negative elements which is a canonical form for A relative to the given U, U^T transformation. If A is a quasi-normal matrix, i.e. a complex matrix such that $AA^{CT} = A^TA^C$ (where A^C denotes the complex conjugate of A and A^{CT} denotes the complex conjugate transpose), it is known by [**6**; **10**] that a necessary and sufficient condition for this to occur is that there exist a unitary matrix U such that UAU^T is a direct sum of non-negative real numbers and of matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where a and b are non-negative real numbers. If $A = -A^T$, the a's are 0 and a special case of this form results (see [**9**], also). Here the analogous normal forms are obtained for the case of infinite matrices which represent completely continuous operators in Hilbert space l^2. The point of view involves operator theory to some extent, but ultimately the matrix point of view since the results are concerned essentially with normal forms of matrices.

2. The transpose operator. First, the following facts are recalled relating to a completely continuous operator $\mathscr{A}$ in complex Hilbert space l^2. Let $A = (a_{ij})$ be an infinite matrix which represents $\mathscr{A}$ relative to a given orthonormal basis. It is known that $\mathscr{A}$ has a polar decomposition $\mathscr{A} = \mathscr{U}\mathscr{P}$ where $\mathscr{P}$ is a positive Hermitian operator, $\mathscr{U}$ is a partially isometric operator whose initial set is the closure of the range of $\mathscr{P}$ and whose final set is the closure of the range of $\mathscr{A}$, and where $\mathscr{U}$ is unique. (See [**7**; **3**], or [**4**].) It is also true that $\mathscr{P} = \mathscr{U}^*\mathscr{A}$ (where $\mathscr{U}^*$ denotes the adjoint of $\mathscr{U}$) is completely continuous, that $\mathscr{U}^*\mathscr{U}\mathscr{P} = \mathscr{P}$ (see [**4**, p. 264, solution 105] or [**7**, p. 5]), and that $\mathscr{A} = \mathscr{U}\mathscr{P} = \mathscr{Q}\mathscr{U}$ where $\mathscr{U}\mathscr{P}\mathscr{U}^* = \mathscr{Q}$ is positive Hermitian. Relative to the given basis let $\mathscr{A} = \mathscr{U}\mathscr{P} = \mathscr{Q}\mathscr{U}$ have the matrix representation $A = UP = QU$ where $A = (a_{ij})$, $U = (u_{ij})$, $P = (p_{ij})$, and $Q = (q_{ij})$. This is true since to the product of operators there corresponds the product of corresponding matrices relative to the same basis. $U = (u_{ij})$ is not necessarily unitary.

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The transpose of $\mathscr{A}$, denoted by $\mathscr{A}^T$, is a linear transformation on the conjugate space with a matrix representation which is the transpose of A, relative to the appropriate basis. For purposes here, a transpose of $\mathscr{A}$ is defined which is an operator on the same space l^2 but also has a matrix representation which is the transpose of A. This is done as follows and the author is indebted to William L. Armacost who supplied the details in the following four paragraphs when the question of defining such a transpose arose in this work.

Let $\{e_i\}$ be the usual orthonormal basis of l^2 and let $x = \sum_{i=1}^{\infty} x_i e_i \in l^2$. Define a transformation $\mathscr{C}$ as follows:

$$\mathscr{C}x = \sum_{i=1}^{\infty} \bar{x}_i e_i,$$

which is in l^2. This is a conjugate-linear (or "semilinear") transformation from l^2 onto l^2 which is one-to-one and has the following properties (see [**5**, p. 357]): $\mathscr{C}^2 = \mathscr{I}$ so $\mathscr{C}^{-1} = \mathscr{C}$; $\mathscr{C}(x + y) = \mathscr{C}x + \mathscr{C}y$; $\mathscr{C}(ax) = \bar{a}\mathscr{C}(x)$ for any complex scalar a where $\bar{a}$ denotes the conjugate of a. Also if (x, y) denotes scalar or inner product, then $(\mathscr{C}x, y) = \overline{(x, \mathscr{C}y)}$ and $(\mathscr{C}x, \mathscr{C}y) = \overline{(x, y)} = (y, x)$; the latter implies $(\mathscr{C}x, \mathscr{C}x) = ||\mathscr{C}x||^2 = ||x||^2$ so $||Cx|| = ||x||$ (where $||x||$ denotes the norm of x) for any x in l^2. This means $\mathscr{C}$ is an isometric mapping.

Let $\mathscr{A}$ be a bounded linear operator mapping l^2 into l^2 with $(\mathscr{A}e_j, e_i) = a_{ij}$ relative to a basis $\{e_i\}$. Let $\bar{\mathscr{A}}$ be the operator defined by $\bar{\mathscr{A}} = \mathscr{C}\mathscr{A}\mathscr{C}$. Then $\bar{\mathscr{A}}$ is a linear operator mapping l^2 into l^2 (since $\bar{\mathscr{A}}(x + y) = \mathscr{C}\mathscr{A}\mathscr{C}(x + y) = (\mathscr{C}\mathscr{A}\mathscr{C}x) + (\mathscr{C}\mathscr{A}\mathscr{C}y)$ and since

$$\bar{\mathscr{A}}(ax) = (\mathscr{C}\mathscr{A}\mathscr{C})(ax) = (\mathscr{C}\mathscr{A})(\bar{a}\mathscr{C}x) = \mathscr{C}(\bar{a}\mathscr{A}\mathscr{C}x) = \bar{\bar{a}}\mathscr{C}\mathscr{A}\mathscr{C}x = a\bar{\mathscr{A}}x).$$

$\bar{\mathscr{A}}$ is bounded. (For

$$||\bar{\mathscr{A}}|| = \sup_{||x||=1}||\bar{\mathscr{A}}x|| = \sup_{||x||=1}||\mathscr{C}\mathscr{A}\mathscr{C}x|| = \sup_{||x||=1}||\mathscr{A}\mathscr{C}x||$$

since $\mathscr{C}$ is an isometry. The latter, since $\mathscr{C}$ is a one-to-one onto isometry, is equal to

$$\sup_{||\mathscr{C}x||=1}||\mathscr{A}\mathscr{C}x|| = \sup_{||x||=1}||\mathscr{A}x|| = ||A|| < \infty$$

so $\bar{\mathscr{A}}$ is bounded.) And, finally, $(\bar{\mathscr{A}}e_j, e_i) = \bar{a}_{ij}$ since

$$(\bar{\mathscr{A}}e_j, e_i) = (\mathscr{C}\mathscr{A}\mathscr{C}e_j, e_i) = \overline{(\mathscr{A}\mathscr{C}e_j, \mathscr{C}e_i)} = \overline{(\mathscr{A}e_j, e_i)} = \bar{a}_{ij}.$$

Next, define the **T**-transpose of $\mathscr{A}$, denoted by $\mathscr{A}^{\mathbf{T}}$, to be the linear transformation $\mathscr{A}^{\mathbf{T}} = \bar{\mathscr{A}}^* = \mathscr{C}\mathscr{A}^*\mathscr{C}$ where $\mathscr{A}^*$ is the adjoint of $\mathscr{A}$. $\mathscr{A}^{\mathbf{T}}$ is a linear transformation on l^2 into l^2 for which the usual transpose properties hold:

$$(\mathscr{A}\mathscr{B})^{\mathbf{T}} = \mathscr{B}^{\mathbf{T}}\mathscr{A}^{\mathbf{T}}, \ (\mathscr{A} + \mathscr{B})^{\mathbf{T}} = \mathscr{A}^{\mathbf{T}} + \mathscr{B}^{\mathbf{T}}$$

and $(\mathscr{A}^{\mathbf{T}})^{\mathbf{T}} = \mathscr{A}$. Also,

$$(\mathscr{A}^{\mathbf{T}}e_j, e_i) = (\bar{\mathscr{A}}^*e_j, e_i) = \overline{(e_j, \mathscr{A}e_i)} = (\mathscr{A}e_i, e_j) = a_{ji}$$

so the matrix of $\mathscr{A}^{\mathbf{T}}$ is the transpose of the matrix of $\mathscr{A}$.

Other properties of $\mathscr{A}^{\mathbf{T}}$ are as follows: $(\mathscr{A}^*)^{\mathbf{T}} = \mathscr{C}\mathscr{A}\mathscr{C} = \bar{\mathscr{A}}$. $\mathscr{A}^{\mathbf{T}} = \bar{\mathscr{A}}^* = (\bar{\mathscr{A}})^*$ since $(\mathscr{C}\mathscr{A}\mathscr{C})^* = \mathscr{C}\mathscr{A}^*\mathscr{C}$ because

$$((\mathscr{C}\mathscr{A}\mathscr{C})x, y) = \overline{(\mathscr{A}\mathscr{C}x, \mathscr{C}y)} = \overline{(\mathscr{C}x, \mathscr{A}^*\mathscr{C}y)} = \overline{\overline{(x, \mathscr{C}\mathscr{A}^*\mathscr{C}y)}} = (x, \mathscr{C}\mathscr{A}^*\mathscr{C}y)$$

for all x and y in l^2. If $\mathscr{A}$ is Hermitian, so is $\mathscr{A}^{\mathbf{T}}$ since $(\mathscr{A}^{\mathbf{T}})^* = (\mathscr{C}\mathscr{A}^*\mathscr{C})^* = (\mathscr{C}\mathscr{A}\mathscr{C})^* = \mathscr{C}\mathscr{A}^*\mathscr{C} = \mathscr{A}^{\mathbf{T}}$. If $\mathscr{A}$ is unitary, so is $\mathscr{A}^{\mathbf{T}}$, since if $\mathscr{A}$ is unitary, $\mathscr{A}\mathscr{A}^* = \mathscr{A}^*\mathscr{A} = \mathscr{I}$ and so

$$(\mathscr{A}^{\mathbf{T}})^*\mathscr{A}^{\mathbf{T}} = (\mathscr{C}\mathscr{A}^*\mathscr{C})^*(\mathscr{C}\mathscr{A}^*\mathscr{C}) = (\mathscr{C}\mathscr{A}\mathscr{C})(\mathscr{C}\mathscr{A}^*\mathscr{C}) = \mathscr{C}\mathscr{A}\mathscr{A}^*\mathscr{C} = \mathscr{C}\mathscr{C} = \mathscr{I}$$

and similarly $\mathscr{A}^{\mathbf{T}}(\mathscr{A}^{\mathbf{T}})^* = \mathscr{I}$. Similarly, if $\mathscr{A}$ is a partial isometry, then so is $\mathscr{A}^{\mathbf{T}}$. (It is also evident that if $\mathscr{A}$ is unitary so is $\bar{\mathscr{A}}$, and if $\mathscr{A}$ is Hermitian, so is $\bar{\mathscr{A}}$.) Also, $(\mathscr{A}^{\mathbf{T}})^* = (\mathscr{A}^*)^{\mathbf{T}}$.

Furthermore, if $\mathscr{A}$ is completely continuous, so is $\bar{\mathscr{A}}$, and therefore so is $\mathscr{A}^{\mathbf{T}} = (\bar{\mathscr{A}})^*$. It is to be shown that if $x_n \rightharpoonup x$, i.e., if x_n converges to x weakly, then $\bar{\mathscr{A}}x_n \to \bar{\mathscr{A}}x$, i.e., $\bar{\mathscr{A}}x_n$ converges to $\bar{\mathscr{A}}x$ strongly. This will follow if it is shown that $x_n \rightharpoonup x$ implies $\mathscr{C}x_n \rightharpoonup \mathscr{C}x$ which in turn implies $\bar{\mathscr{A}}x_n \to \bar{\mathscr{A}}x$. If $x_n \rightharpoonup x$, $|(\mathscr{C}x_n - \mathscr{C}x, y)| = |(\mathscr{C}x_n - \mathscr{C}x, \mathscr{C}(\mathscr{C}y))| = |(\mathscr{C}(x_n - x), \mathscr{C}(\mathscr{C}y)| = |\overline{(x_n - x, \mathscr{C}y)}| = |(x_n - x, \mathscr{C}y)| \to 0$ for each y in l^2 since $z_n \rightharpoonup z$ if and only if $(z_n, y) \to (z, y)$ for each y in l^2. So $\mathscr{C}x_n \rightharpoonup \mathscr{C}x$. Since $\mathscr{A}$ is completely continuous, $\mathscr{A}\mathscr{C}x_n \to \mathscr{A}\mathscr{C}x$. Therefore,

$$||\bar{\mathscr{A}}x_n - \bar{\mathscr{A}}x|| = ||\mathscr{C}\mathscr{A}\mathscr{C}x_n - \mathscr{C}\mathscr{A}\mathscr{C}x|| = ||\mathscr{C}(\mathscr{A}\mathscr{C}x_n - \mathscr{A}\mathscr{C}x)|| = ||\mathscr{A}\mathscr{C}x_n - \mathscr{A}\mathscr{C}x|| \to 0$$

so $\bar{\mathscr{A}}$ is completely continuous.

Let $\mathscr{A}^{\mathbf{T}}$ denote the transpose of $\mathscr{A}$, as above, and let A^T denote the transpose of matrix A relative to a given basis. Since $\mathscr{A} = \mathscr{U}\mathscr{P} = \mathscr{Q}\mathscr{U}$, then $\mathscr{A}^{\mathbf{T}} = \mathscr{P}^{\mathbf{T}}\mathscr{U}^{\mathbf{T}} = \mathscr{U}^{\mathbf{T}}\mathscr{Q}^{\mathbf{T}}$, and so $A^T = P^TU^T = U^TQ^T$, where $\mathscr{U}^{\mathbf{T}}$ and U^T are partial isometries, $\mathscr{P}^{\mathbf{T}}$, $\mathscr{Q}^{\mathbf{T}}$, P^T and Q^T Hermitian.

3. The T-symmetric case. Let $\mathscr{A}$ be a completely continuous operator in l^2 such that relative to some orthonormal basis $\{e_i\}$, $(\mathscr{A}e_j, e_i) = (\mathscr{A}^{\mathbf{T}}e_j, e_i)$, i.e., $a_{ij} = a_{ji}$ for A determined by $\mathscr{A}$ relative to this basis. (That such operators and matrices do exist and non-trivially is evident if one takes a completely continuous operator $\mathscr{B}$, and forms $\mathscr{B} + \mathscr{B}^{\mathbf{T}}$ which is in l^2 and completely continuous, since $\mathscr{B}^{\mathbf{T}}$ is such.) Such an operator $\mathscr{A}$ for which the above is true will be called **T**-symmetric and denoted by $\mathscr{A} = \mathscr{A}^{\mathbf{T}}$.

The following analog for the finite case will be shown:

THEOREM 1. *If A is the matrix, relative to an orthonormal basis, of a completely continuous operator $\mathscr{A} = \mathscr{A}^{\mathbf{T}}$, i.e. such that $A = A^T$, there is a unitary matrix U such that $UAU^T = D$ is a real diagonal matrix.*

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If $\mathscr{A} = \mathscr{A}^{\mathrm{T}}$, then from the above it follows that $A = UP = A^T = U^TQ^T$ where $P = P^{CT}$ (where the latter denotes the complex conjugate-transpose of P) and $Q^T = (Q^T)^{CT} = Q^C$. This means that $P = Q^T$; for $\mathscr{P}^2 = \mathscr{A}^*\mathscr{A} = (\mathscr{A}^*)^{\mathrm{T}}\mathscr{A}^{\mathrm{T}} = (\mathscr{A}\mathscr{A}^*)^{\mathrm{T}} = (\mathscr{Q}^2)^{\mathrm{T}} = (\mathscr{Q}^{\mathrm{T}})^2$ and since $\mathscr{P}$ and Q^{T} are positive operators, it follows that $\mathscr{P} = \mathscr{Q}^{\mathrm{T}}$ and $P = Q^T$.

Therefore $A = UP = QU = P^TU$. Since $\mathscr{A}$ is completely continuous, $\mathscr{P}$ and $\mathscr{Q}$ are completely continuous Hermitian operators so there exist (infinite) unitary matrices W and V such that $WPW^{CT} = D$ and $VQV^{CT} = D_1$ are real diagonal matrices. In particular assume D is diagonal with real diagonal elements λ_i where $\lim \lambda_i = 0$ as i becomes infinite. Then it follows that $W^CAW^{CT} = W^CUW^{CT}WPW^{CT} = W^CP^TW^TW^CUW^{CT}$ or, if $X = W^CUW^{CT}$, $XD = DX$ and X is such that U is the matrix of a partial isometry and W^C and W^{CT} are unitary. Since $\mathscr{U}$ is a partial isometry, $\mathscr{U}^*\mathscr{U} = K$ is such that $\mathscr{K}^2 = \mathscr{K}$ is Hermitian (see [**2**, p. 153], for example) which means that $U^{CT}U = (U^{CT}U)^2$ so that

$$(WU^{CT}W^T)(W^CUW^{CT}) = (WU^{CT}W^T)(W^CUW^{CT})(WU^{CT}W^T)(W^CUW^{CT}) = (WU^{CT}W^TW^CUW^{CT})^2$$

or $X^{CT}X = (X^{CT}X)^2$. There are two possible cases. (a) If no $\lambda_i = 0$ in D, then since $\lim_{i\to\infty}\lambda_i = 0$, there can be only a finite number l of the x_i such that $\lambda_1 = \lambda_2 = \ldots \lambda_l$ is the λ_i of largest value. Since X commutes with D, $X = X_1 \dotplus X'$ where X_1 is of finite dimension l. Then $X^{CT}X = (X^{CT}X)^2$ which is Hermitian. (It follows that the roots of $X_1{}^{CT}X_1$ are either 1, 0 or both.) From $U^{CT}\,UP = P$ follows $WU^{CT}W^TW^CUW^{CT}WPW^{CT} = W^{CT}PW$ or $X^{CT}XD = D$. This means that $X_1{}^{CT}X_1\lambda_1 = \lambda_1I_1$ where $\lambda_1 \neq 0$ and so $X_1{}^{CT}X_1 = I_1$, where I_1 is a suitably finite-dimensional identity matrix, so that X_1 is unitary. X' can now be treated in the same way and by such successive steps involving λ_i of like value, $X = X_1 \dotplus X_2 \dotplus X_3 \dotplus \ldots$ where the latter is a direct sum of the X_i and conformable to the blocks of like diagonal elements in D and where each such X_i has the properties of X_1 above. (b) If some $\lambda_i = 0$ (either finite or infinite in number) by following the stepwise process described above D can become a direct sum of such λ_iI_i blocks, $\lambda_i \neq 0$, interspersed with direct sums of 0's. If $X = (x_{st})$ and if 0 appears in the i-i and j-j positions of D for $i \neq j$ then x_{ii}, x_{ij}, x_{ji}, and x_{jj} are not necessarily 0. (The matrix composed of all such x_{ij} is the matrix of a partial isometry.) But X is such that $W^CAW^{CT} = DX$ is a direct sum of λ_iX_i, $\lambda_i \neq 0$, interspersed with 0's along the diagonal. If for some k all $\lambda_i = 0$ for $i > k$, then D may be taken to be in the form $\lambda_1I_1 \dotplus \lambda_2I_2 \dotplus \ldots \dotplus \lambda_{t-1}I_{t-1} \dotplus 0$ (where 0 is an infinite matrix of 0's) with the $\lambda_i \neq 0$. In this case $W^CAW^{CT} = W^CUPW^{CT} = W^CUW^{CT}WPW^{CT} = XD = \lambda_1X_1 \dotplus \lambda_2X_2 \dotplus \ldots \dotplus \lambda_{t-1}X_{t-1} \dotplus 0$ where the latter is an infinite matrix of zeros.

From $A = UP = QU = A^T = U^TQ^T = U^TP$ or $A = UP = U^TP$ follows $W^CAW^{CT} = W^CUW^{CT}WPW^{CT} = W^CU^TW^{CT}WPW^{CT} = XD = X^TD$. This means that each X_i corresponding to $\lambda_i \neq 0$ is such that $\lambda_iX_i = \lambda_iX^T$ or that

$X_i = X_i^T$ where X_i is unitary. So for each X_i, for which $\lambda_i \neq 0$, there exists a real orthogonal matrix W_i such that $W_i X_i W_i^T$ is complex diagonal and unitary. (For if $M = M_1 + iM_2$, M_1 and M_2 real, is unitary and symmetric, M_1 and M_2 are commutative real symmetric matrices which can be diagonalized by the same real orthogonal matrix.) So there is a complex unitary matrix T (a direct sum of W_i and 1's corresponding to 0's in D) such that TAT^T is a diagonal matrix with diagonal elements of the form $\lambda\mu$, λ real (including 0) and approaching 0 as one moves along the diagonal and $|\mu| = 1$. If S is the diagonal matrix with diagonal elements $\mu^{-1/2}$ in corresponding position, when $\lambda \neq 0$ and 1 when $\lambda = 0$, S is unitary and $STAT^TS^T$ is a real diagonal matrix with diagonal elements λ_i where $\lambda_i \to 0$ as $i \to \infty$.

4. The T-skew symmetric case. Let $\mathscr{A}$ be a completely continuous operator in l^2 such that relative to some orthonormal basis $\{e_i\}$, $(\mathscr{A}e_j, e_i) = -(\mathscr{A}^{\mathbf{T}}e_j, e_i)$, i.e., $a_{ij} = -a_{ji}$ for $A = (a_{ij})$ determined by $\mathscr{A}$ relative to this basis. (That such operators and matrices do exist is evident if one takes a completely continuous operator $\mathscr{B}$ as before and forms $\mathscr{B} - \mathscr{B}^{\mathbf{T}}$ which is in l^2 and completely continuous.) Such an operator will be called **T**-skew-symmetric and denoted by $\mathscr{A}^{\mathbf{T}} = -\mathscr{A}$.

The following analog for the finite case is to be shown:

THEOREM 2. *If A is the matrix, relative to an orthonormal basis, of a completely continuous operator $\mathscr{A} = -\mathscr{A}^{\mathbf{T}}$, i.e. such that $A = -A^T$, there is a unitary matrix U such that UAU^T is a direct sum of 0's and of matrices of the form*

$$\begin{bmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{bmatrix}$$

where λ_i is real and $\lim \lambda_i = 0$ *as $i \to \infty$.*

The proof parallels that as found in the reference [**9**] given above. If $\mathscr{A} = -\mathscr{A}^{\mathbf{T}}$, then $\mathscr{A} = \mathscr{U}\mathscr{P} = \mathscr{Q}\mathscr{U} = -\mathscr{A}^{\mathbf{T}} = -\mathscr{P}^{\mathbf{T}}\mathscr{U}^{\mathbf{T}} = -\mathscr{U}^{\mathbf{T}}\mathscr{Q}^{\mathbf{T}}$ which means $A = UP = QU = -P^TU^T = -U^TQ^T$. As before, $P^2 = (Q^T)^2$ from which $P = Q^T$. ($P = -Q^T$ would not be possible since Q^T is positive and P must also be positive.) The proof proceeds, in the reference given, by considering the non-singular and singular case. Here the proof is along lines similar to the **T**-symmetric case as follows. Let $WPW^{CT} = D$ be real diagonal with diagonal elements λ_i where W is unitary. From $A = UP = QU = P^TU$ it follows, as above, that $X = W^CUW^{CT}$ is such that $XD = DX$, and if no $\lambda_i = 0$, X is a direct sum of $X_1, X_2, \ldots$, each finite dimensional; and if some $\lambda_i = 0$ then DX is a direct sum of finite dimensional $\lambda_i X_i$ interspersed with direct sums of 0's. As before, the finite-dimensional X_i are unitary and from $A = UP = -U^TP$, it follows that $XD = -X^TD$ and so each finite dimensional $X_i = -X_i^T$. By [**9**, Lemma 1, p. 438], if X is a (finite-dimensional) complex, unitary, skew-symmetric matrix, there exists a complex unitary V such that

$VXV^T = E$ is a direct sum of 2×2 matrices of the form

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

An inspection of the proof of that lemma reveals that the V described is actually a product of matrices ST where T is real orthogonal such that TXT^T is a direct sum of 2×2 blocks of the form

$$\text{(i)} \qquad \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$$

where α is non-zero complex and $\alpha\bar{\alpha} = 1$. In the present case if T_i is the real orthogonal matrix which performs this on the X_i, then if T is the corresponding direct sum of 1's (corresponding to 0's in the diagonal of D) and of these T_i, $TW^CAW^{CT}T^T$ is a direct sum of 0's and of $\lambda_i A_i$, $\lambda_i \neq 0$, where each A_i is a direct sum of 2×2 matrices of the form (i). If $\alpha = e^{i\theta}$ and S is an appropriate direct sum of 1's and of 2×2 matrices of the form

$$\begin{bmatrix} 0 & e^{-i\theta/2} \\ -e^{-i\theta/2} & 0 \end{bmatrix},$$

then $STW^CAW^{CT}T^TS^T$ is a direct sum of matrices of the form

$$\begin{bmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{bmatrix}$$

where $\lambda_i \to 0$ as $i \to \infty$.

5. The quasi-normal case. Let $\mathscr{A}$ be a completely continuous operator. Then so are $\mathscr{A}\mathscr{A}^*$, $\mathscr{A}^*\mathscr{A}$ and $(\mathscr{A}^*\mathscr{A})^{\mathbf{T}} = \mathscr{A}^{\mathbf{T}}(\mathscr{A}^*)^{\mathbf{T}} = \mathscr{A}^{\mathbf{T}}(\mathscr{A}^{\mathbf{T}})^*$. By definition, an operator $\mathscr{A}$ will be said to be quasi-normal if $\mathscr{A}\mathscr{A}^* = (\mathscr{A}^*\mathscr{A})^{\mathbf{T}} = \mathscr{A}^{\mathbf{T}}(\mathscr{A}^*)^{\mathbf{T}} = \mathscr{A}^{\mathbf{T}}\bar{\mathscr{A}}$. (That such operators do exist may be seen as follows. Let $\mathscr{B}$ be an operator which, relative to some orthonormal basis, has an infinite matrix which is a direct sum of matrices of the form

$$\text{(ii)} \qquad \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}$$

where the a_i and b_i are non-negative real numbers and $\sum_{i=1}^{\infty}(a_i^2 + b_i^2) < \infty$. Then $\mathscr{B}$ is a completely continuous operator (see [**2**, Exercise 1, p. 177]). Let $\mathscr{U}$ be a unitary operator, and so $\mathscr{U}^{\mathbf{T}}$ is also unitary. Then $\mathscr{U}\mathscr{B}\mathscr{U}^{\mathbf{T}}$ is completely continuous and meets, non-trivially, the above definition, as can be directly verified.)

The following theorem results:

THEOREM 3. *If A is the matrix, relative to an orthonormal basis, of a completely continuous quasi-normal operator $\mathscr{A}$, i.e., $\mathscr{A}\mathscr{A}^* = \mathscr{A}^{\mathbf{T}}(\mathscr{A}^*)^{\mathbf{T}}$, there is a unitary matrix U such that UAU^T is a direct sum of matrices of the form* (ii) *above,*

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where a_i and b_i are real and $a_i \to 0$ and $b_i \to 0$ as i becomes infinite, and of 1×1 real c_i where $c_i \to 0$ as i becomes infinite.

Let $\mathscr{A}$ be quasi-normal and completely continuous and let $\mathscr{A} = \mathscr{S} + \mathscr{T}$ where $\mathscr{S} = 1/2(\mathscr{A} + \mathscr{A}^{\mathbf{T}})$ and $T = 1/2(\mathscr{A} - \mathscr{A}^{\mathbf{T}})$ so that $\mathscr{S} = \mathscr{S}^{\mathbf{T}}$ and $\mathscr{T} = -\mathscr{T}^{\mathbf{T}}$. The proof proceeds as in the finite case as follows. Since $\mathscr{A}\mathscr{A}^* = \mathscr{A}^{\mathbf{T}}(\mathscr{A}^*)^{\mathbf{T}}$, it follows that $(\mathscr{S} + \mathscr{T})(\mathscr{S}^* + \mathscr{T}^*) = (\mathscr{S}^{\mathbf{T}} + \mathscr{T}^{\mathbf{T}})(\mathscr{S}^* + \mathscr{T}^*)^{\mathbf{T}}$ or

$$
\begin{aligned}
(\mathscr{S} + \mathscr{T})(\mathscr{S}^* + \mathscr{T}^*) &= (\mathscr{S} - \mathscr{T})([\mathscr{S}^{\mathbf{T}}]^* + [\mathscr{T}^{\mathbf{T}}]^*) \\
&= (\mathscr{S} - \mathscr{T})(\mathscr{S}^* - \mathscr{T}^*)
\end{aligned}
$$

so $\mathscr{S}\mathscr{S}^* + \mathscr{S}\mathscr{T}^* + \mathscr{T}\mathscr{S}^* + \mathscr{T}\mathscr{T}^* = \mathscr{S}\mathscr{S}^* - \mathscr{S}\mathscr{T}^* - \mathscr{T}\mathscr{S}^* + \mathscr{T}\mathscr{T}^*$ so $\mathscr{S}\mathscr{T}^* = -\mathscr{T}\mathscr{S}^*$. Relative to the given basis, the corresponding matrix product becomes $ST^{CT} = -TS^{CT}$ or $-ST^C = -TS^C$ or $ST^C = TS^C$. By Theorem 1 there exists a unitary matrix U such that $USU^T = D$ is a real diagonal matrix of the form described there. If $UTU^T = M$, then $MD = DM^C$. If the diagonal elements of D are d_i, and if $M = (t_{ij})$, then $t_{ij}d_j = d_i \bar{t}_{ij}$ where $t_{ji} = -t_{ij}$. Three possibilities may occur: if $d_j = d_i \neq 0$, then t_{ij} is real; if $d_j = d_i = 0$, t_{ij} is arbitrary (though $M = -M^T$ still holds); and if $d_j \neq d_i$, then $t_{ij} = 0$, for if $t_{ij} = a + ib$, then $(a + ib)d_j = d_i(a - ib)$ and $a(d_j - d_i) = 0$ implies $a = 0$ and $b(d_i + d_j) = 0$ implies $d_i = -d_j$ (which is not possible since the d_i are real and non-negative and $d_j \neq d_i$) or $b = 0$ so $t_{ij} = 0$.

So if $USU^T = D$ then the following two cases arise: (a) If no $d_i = 0$, the d_i may be arranged so $d_i \geqq d_{i+1}$ for $i = 1, 2, 3, \ldots$ and relabelled $d_1, d_2, d_3, \ldots$ with $d_i \neq d_j$ when $i \neq j$. Then $UTU^T = T_1 \dotplus T_2 \dotplus \ldots$ is a direct sum conformable to D where the T_i are real, finite dimensional, and infinite in number and $T_i = -T_i^T$; for each such T_i there exists a real orthogonal V_i such that $V_iT_iV_i^T$ is a direct sum of 0's and of matrices of the form

$$
\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}
$$

where b is real (see [**1**, p. 65] for the real even dimension skew-symmetric matrix case). (b) If some $d_i = 0$ (either finite or infinite in number) and if $UTU^T = (t_{ij})$,when 0 is in the i-i and j-j position for $i \neq j$, then t_{ii}, t_{ij}, t_{ji}, and t_{jj} are not necessarily 0 but the matrix T' composed of all such t_{rs} taken in order (which could be finite or infinite and distributed throughout UTU^T) is complex skew-symmetric. For such a T', finite or infinite, there exists a complex unitary W such that $WT'W^T$ is the direct sum described in Theorem 1 of [**9**] if T' is finite, or such that $WT'W^T$ is the direct sum described in Theorem 2 above if T' is infinite.

To examine each of these cases consider the following representative sample as a guide in which $UAU^T = USU^T + UTU^T =$

$$\begin{bmatrix} d_1I_1 & & & & & & \\ & d_2I_2 & & & & & \\ & & 0I_3 & & & & \\ & & & d_4I_4 & & & \\ & & & & 0I_5 & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & \cdot \end{bmatrix} + \begin{bmatrix} T_1 & 0 & & & & & \\ 0 & T_2 & & & & & \\ & & T_{11} & 0 & T_{12} & \cdots \\ & & 0 & T_4 & 0 & \\ & & -T_{12}^T & 0 & T_{22} & \cdots \\ & & \cdot & \cdot & & \\ & & \cdot & \cdot & & \\ & & \cdot & \cdot & & \end{bmatrix}$$

where the d_i are non-zero and $d_i > d_j$ for $i < j$.

If the number of 0's in the diagonal of USU^T is finite, so is the number of T_{ij}, and a simple rearrangement of the T_{ij} into a single complex skew-symmetric block in the given order is possible under a real orthogonal transformation and this block will correspond to a diagonal of 0's in USU^T while retaining the diagonal form in the latter. If V is a direct sum of real V_i acting on the T_i as in (a) and of the appropriate unitary matrix acting on (T_{ij}) of Theorem 1 of [**9**], $VUAU^TV^T$ gives the form desired.

If the number of diagonal 0's in USU^T is infinite so that $T' = (T_{ij})$ is infinite, by Theorem 2 in this article there is a unitary matrix W such that $WT'W^T = F$ is a direct sum of the form of Theorem 2. Let $W = (W_{ij})$ be sectioned according to the sectioning of $T' = (T_{ij})$. Let V be an infinite matrix formed as follows: in UTU^T let each T_i be replaced by the real orthogonal V_i as described in (a) above, and let each T_{ij} be replaced by W_{ij}. Then $VUSU^TV^T = USU^T$, and $VUTU^TV^T$ has a form in which each T_i in UTU^T is replaced by the form F_i described in (a) and each T_{ij} in UTU^T is replaced by F_{ij} where $F = (F_{ij})$ is sectioned as (T_{ij}) is. In the sample case above $VUTU^TV^T$ has the form

$$\begin{bmatrix} F_1 & 0 & & & & \\ 0 & F_2 & & & & \\ & & F_{11} & 0 & F_{12} & \cdots \\ & & 0 & F_4 & 0 & \\ & & -F_{12}^T & 0 & F_{22} & \cdots \\ & & \cdot & & \cdot & \\ & & \cdot & & \cdot & \\ & & \cdot & & \cdot & \end{bmatrix}.$$

Because of the form of F either F_{1j}, all $j \neq 1$, in the same row as F_{11} are 0 or not. But if not, because of the direct-sum form of F, at most one of them, namely F_{12}, can be different from 0 and if so, the only nonzero element a in an F_{12} must occur in the lower left corner position (and in the upper right corner position of $-F_{12}^T$). A suitable interchange of the row and column containing $-a$ and a and of F_4 can bring the upper left block of $VUTU^TV^T$ into the desired form without altering the diagonal form of USU^T. Repeating this process with each subsequent F_{ii} as needed brings the matrix A into the desired form under the required transformation.

In the case of a finite $n \times n$ matrix with complex elements it is true that every matrix is similar to its transpose. Here, since $\mathscr{A}^{\mathbf{T}} = \mathscr{C}\mathscr{A}^*\mathscr{C}$, it is true if A has a polar form $\mathscr{A} = \mathscr{U}\mathscr{P}$, $\mathscr{A}^{\mathbf{T}} = \mathscr{C}\mathscr{P}\mathscr{U}^*\mathscr{C} = \mathscr{C}\mathscr{U}^*\mathscr{U}\mathscr{P}\mathscr{U}^*\mathscr{C} = \mathscr{C}\mathscr{U}^*\mathscr{A}\mathscr{U}^*\mathscr{C}$ but this provides no matrix connection between A^T and A. For the case in which $\mathscr{A}$ is quasi-normal, the following observation holds. Let $UAU^T = F$ be the direct sum as described in the preceding theorem. Then $UA^TU^T = F^T$. If W is an appropriate direct sum of 2×2 matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and 1's, $WFW^T = F^T$ from which $U^{CT}WUAU^TW^TU^C = A^T$ or $VAV^T = A^T$ where $V = U^{CT}WU$ is unitary. Therefore, a linear operator $\mathscr{V}$ on l^2 does exist so that $\mathscr{V}\mathscr{A}\mathscr{V}^{\mathbf{T}} = \mathscr{A}^{\mathbf{T}}$.

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