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On Projectively Flat (α, β)-metrics

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Abstract. The solutions to Hilbert's Fourth Problem in the regular case are projectively flat Finsler metrics. In this paper, we consider the so-called (α, β)-metrics defined by a Riemannian metric α and a 1-form β, and find a necessary and sufficient condition for such metrics to be projectively flat in dimension $n \geq 3$.

1 Introduction

Projectively flat Finsler metrics on a convex open set in R^n are the solutions to Hilbert's Fourth Problem. Beltrami's theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. For Finsler metrics, the flag curvature is a natural extension of the sectional curvature. However the situation is much more complicated. It is well known that every locally projectively flat Finsler metric is of scalar flag curvature, namely, the flag curvature is a scalar function on the tangent bundle, which might not necessarily be constant as in the Riemannian case. Thus locally projectively flat Finsler metrics form a rich class of Finsler metrics. Below are two important examples defined by a Riemannian metric and a 1-form on the unit ball $B^n \subset R^n$: Let

$$\bar{\alpha} = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2},$$

$$\bar{\beta} = \frac{\langle x, y\rangle}{1-|x|^2} + \frac{\langle a, y\rangle}{1+\langle a, x\rangle},$$

$$\lambda = \frac{(1+\langle a, x\rangle)^2}{1-|x|^2},$$

where $a \in R^n$ is a constant vector with $|a| < 1$. Then

(a) $\bar{F} := \bar{\alpha} + \bar{\beta}$ is projectively flat on the unit ball $B^n(1) \subset R^n$ with constant flag curvature $\mathbf{K} = -1/4$ (see [8]).

(b) $F := (\alpha+\beta)^2/\alpha$, where $\alpha = \lambda\bar{\alpha}$ and $\beta = \lambda\bar{\beta}$, is projectively flat on the unit ball $B^n(1) \subset R^n$ with zero flag curvature $\mathbf{K} = 0$ (see [6]).

These two examples inspire us to study projectively flat Finsler metrics $F = \alpha\phi(\beta/\alpha)$ defined by a Riemannian metric α and a 1-form β. Metrics in this form are called (α, β)-*metrics*. When $\phi = 1 + s$, we get Randers metrics $F = \alpha + \beta$. Randers metrics are the simplest (α, β)-metrics.

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It is well known that a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed (see [1, 3]). For a general (α, β)-metric $F = \alpha\phi(\beta/\alpha)$, if β is parallel with respect to α, then F is locally projectively flat if and only if α is locally projectively flat. This can be easily seen from (2.3) below.

The main purpose of this paper is to study and characterize locally projectively flat (α, β)-metrics which are not of Randers type.

Theorem 1.1 *Let* $F = \alpha\phi(s)$, $s = \beta/\alpha$, *be an* (α, β)*-metric on an open subset* $\mathcal{U}$ *in the n-dimensional Euclidean space* R^n $(n \geq 3)$, *where* $\phi(0) = 1$, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ *and* $\beta = b_i(x)y^i \neq 0$. *Let* $b := \|\beta_x\|_\alpha$. *Suppose that the following conditions hold:*

(a) β *is not parallel with respect to* α;
(b) F *is not in the form* $F = \sqrt{\alpha^2 + k\beta^2} + \epsilon\beta$ *for some constants* k *and* ϵ;
(c) $db \neq 0$ *everywhere or* $b = constant$ *on* $\mathcal{U}$.

Then F is projectively flat on $\mathcal{U}$ *if and only if*

$$\left\{ 1 + (k_1 + k_2 s^2)s^2 + k_3 s^2 \right\} \phi''(s) = (k_1 + k_2 s^2)\left\{ \phi(s) - s\phi'(s) \right\}, \tag{1.1}$$

$$b_{i|j} = 2\tau\left\{ (1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j \right\}, \tag{1.2}$$

$$G^i_\alpha = \xi y^i - \tau\left(k_1\alpha^2 + k_2\beta^2\right) b^i, \tag{1.3}$$

where $\tau = \tau(x)$ *is a scalar function on* $\mathcal{U}$ *and* k_1, k_2 *and* k_3 *are constants with* $(k_2, k_3) \neq (0, 0)$.

When $(k_2, k_3) = (0, 0)$, the solution ϕ of (1.1) with $\phi(0) = 1$ is given by

$$\phi(s) = \sqrt{1 + k_1 s^2} + \epsilon s,$$

where ϵ is a constant. The (α, β)-metric defined by ϕ is of Randers type

$$F = \sqrt{\alpha^2 + k_1\beta^2} + \epsilon\beta.$$

For the above metric with $\epsilon \neq 0$, it is projectively flat if and only if β is closed and $\tilde{\alpha} := \sqrt{\alpha^2 + k_1\beta^2}$ is projectively flat, in other words, β is closed and α can be expressed as $\alpha = \sqrt{\tilde{\alpha}^2 - k_1\beta^2}$ where $\tilde{\alpha}$ is projectively flat. We do not consider this case in Theorem 1.1.

Consider the following functions:

$$\phi = e^s + \epsilon s, \qquad \phi = \frac{1}{1 - s} + \epsilon s,$$

where ϵ is a constant. Clearly, they do not satisfy (1.1). Thus $F = \alpha\exp(\beta/\alpha) + \epsilon\beta$ (the exponential metric) and $F = \alpha^2/(\alpha - \beta) + \epsilon\beta$ (the Matsumoto metric) are projectively flat on $\mathcal{U}$ if and only if β is parallel with respect to α (Cf. [10], [5]). We conjecture that these metrics are of scalar flag curvature if and only if α is of constant sectional curvature and β is parallel with respect to α. On the other hand,

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there are many functions ϕ satisfying (1.1) for some constants k_i. Below are the most important ones.

$$\phi = 1 + s, \qquad \phi = 1 + \epsilon s + s^2,$$
$$\phi = 1 + \epsilon s + s\arctan(s), \qquad \phi = 1 + \epsilon s + 2s^2 - \frac{1}{3}s^4, \tag{1.4}$$

where ϵ is a constant. See [6] and [9] for the metrics defined by $\phi = 1 + \epsilon s + s^2$, [11] for the metrics defined by $\phi = 1 + \epsilon s + s\arctan(s)$, and [7] for the metrics defined by $\phi = 1 + \epsilon s + 2s^2 - \frac{1}{3}s^4$.

Corollary 1.2 *If ϕ satisfies*

$$\phi(s) - s\phi'(s) = (p + rs^2)\phi''(s), \tag{1.5}$$

where $p \neq 0, r$ are constants, then it satisfies (1.1) with $k_1 = 1/p, k_2 = 0$ and $k_3 = (r-1)/p$. In this case, $F = \alpha\phi(\beta/\alpha)$ is projectively flat if and only if there is a scalar function $\tau = \tau(x)$ such that

$$b_{i|j} = \frac{2\tau}{p}\left\{ (p + b^2)a_{ij} + (r-1)b_i b_j \right\}, \tag{1.6}$$

$$G^i_\alpha = \xi y^i - \frac{\tau}{p}\alpha^2 b^i. \tag{1.7}$$

This corollary slightly generalizes the theorem in [2], where the authors assume that $\phi = \phi(s)$ is analytic in s. The functions in (1.4) are particular solutions of (1.5). For these functions, one can find some special non-trivial solutions to (1.6) and (1.7). If $\phi = \phi(s)$ satisfies (1.5) with $r \neq 0$, then the (α, β)-metric

$$F := \left(1 - \frac{r}{p}|x|^2\right)^{-\frac{1}{2r}} |y|\phi\left(\frac{\langle x, y\rangle}{|y|\sqrt{1 - \frac{r}{p}|x|^2}}\right)$$

is projectively flat on a ball around the origin in R^n. However, so far, we do not have any explicit examples satisfying (1.1)–(1.3) with $k_2 \neq 0$.

2 Preliminaries

Consider a Finsler metric $F = F(x, y)$ on an open domain $\mathcal{U} \subset R^n$. The geodesics of F satisfy the following equations:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x, y)$ are called the *geodesic coefficients* of F, which are given by

$$G^i = \frac{1}{4}g^{il}\left\{ [F^2]_{x^m y^l} y^m - [F^2]_{x^l} \right\}.$$

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F is said to be *projectively flat* in $\mathcal{U}$ if all geodesics are straight lines. This is equivalent to saying that the geodesic coefficients G^i of F take the following form

$$G^i = P(x, y)y^i. \tag{2.1}$$

There is another system of equations that characterizes projectively flat metrics. According to G. Hamel [4], F is projectively flat if and only if it satisfies

$$F_{x^m y^l} y^m - F_{x^l} = 0. \tag{2.2}$$

In the study of projectively flat (α, β)-metrics, (2.2) is more useful than (2.1).

Let $\phi = \phi(s)$, $|s| < b_o$, be a positive C^∞ function satisfying the following

$$\phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \qquad (|s| \le \rho < b_o),$$

Let $\alpha = \sqrt{a_{ij}y^i y^j}$ be a Riemannian metric and $\beta = b_i y^i$ a 1-form on a manifold M. Assume that $\|\beta_x\|_\alpha < b_o$, then the scalar function $F := \alpha\phi(s)$, $s = \beta/\alpha$, is a Finsler metric which is called an (α, β)-metric. (α, β)-metrics form a special class of Finsler metrics. Most important, they are "computable" although the computation sometimes runs into very complicated situations.

Let $\nabla\beta = b_{i|j} dx^i \otimes dx^j$ denote covariant derivative of β with respect to α. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j := b^j s_{ij}.$$

We can express the geodesic coefficients G^i of F in terms of the geodesic coefficients G^i_α of α and the covariant derivatives of β.

$$G^i = G^i_\alpha + Py^i + Q^i, \tag{2.3}$$

where

$$P = \alpha^{-1}\Theta(-2\alpha Q s_0 + r_{00}),$$
$$Q^i = \alpha Q s^i{}_0 + \Psi(-2\alpha Q s_0 + r_{00}) b^i,$$

and

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\big((\phi - s\phi') + (b^2 - s^2)\phi''\big)},$$
$$Q = \frac{\phi'}{\phi - s\phi'},$$
$$\Psi = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

We have the following trivial lemmas.

Lemma 2.1 *If $\phi(0) = 1$ and $Q = k_1 s$, where k_1 is independent of s, then $\phi = \sqrt{1 + k_1 s^2}$.*

Lemma 2.2 *If $\phi(0) = 1$ and $2\Psi = k_1/(1 + k_1 b^2)$, where k_1 is a number independent of s, then $\phi = \sqrt{1 + k_1 s^2} + \epsilon s$, where ϵ is a number independent of s.*

By (2.2), one can easily get the following.

Lemma 2.3 (see [9]) *An (α, β)-metric $F = \alpha\phi(s)$, where $s = \beta/\alpha$, is projectively flat on an open subset $\mathcal{U} \subset \mathrm{R}^n$ if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + \Psi\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0, \tag{2.4}$$

where $y_l := a_{lm}y^m$.

To simplify equation (2.4), we shall prove the following

Theorem 2.4 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an open subset $U \subset R^n$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Suppose that*

(a) *β is not parallel everywhere;*
(b) *F is not of Randers type at any point $x \in \mathcal{U}$;*
(c) *either $db \neq 0$ everywhere or $b = constant \neq 0$ on U.*

Then F is projectively flat if and only if the function $\phi = \phi(s)$ satisfies

$$\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''} = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\delta s^2 + \eta(b^2 - s^2)}, \tag{2.5}$$

$$d\beta = 0, \tag{2.6}$$

$$r_{00} = 2\tau\Big\{\delta\beta^2 + \eta(b^2\alpha^2 - \beta^2)\Big\}, \tag{2.7}$$

$$G^i_\alpha = \xi y^i - \tau\Big(\lambda\beta^2 + \mu(b^2\alpha^2 - \beta^2)\Big)b^i, \tag{2.8}$$

where $\lambda, \mu, \delta, \eta$ and τ are scalar functions on U, with $\delta = 0$ if b is constant.

3 The 1-form β is closed

In this section, we are going to prove the following

Lemma 3.1 *Suppose that Q/s is not independent of s. If an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is projectively flat on an open subset in $\mathcal{U}$ in R^n ($n > 2$) and $\beta \neq 0$, then β is closed.*

Proof Let $F = \alpha\phi(\beta/\alpha)$ be a projectively flat (α, β)-metric on $\mathcal{U}$, namely, its geodesics are straight lines. Fix an arbitrary point $x_o \in \mathcal{U} \subset R^n$. There is an affine transformation $\varphi = \mathbf{A}u + x_o \colon (u^i) \in R^n \to (x^i) \in R^n$ such that $\varphi(0) = x_o$ and $\alpha_{x_o} = \sqrt{a_{ij}v^i v^j}$, and $\beta_{x_o} = b_i v^i$ at $u = 0$ are given by

$$a_{ij} = \delta_{ij}, \qquad b_i = b_o\delta_{1i},$$

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where $b_o := \|\beta_{x_o}\|_\alpha \neq 0$. The above identities hold only at $u = 0$. Since φ is affine, in the new coordinate system (u^i) the geodesics of $F = F(u, v)$ are still straight lines. Thus (2.4) holds for F with (u^i, v^i) in place of (x^i, y^i). At $u = 0$, we have

$$(\delta_{ml}\alpha^2 - v_m v_l)G^m_\alpha + \alpha^3 Q s_{l0} + \Psi\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s v_l) = 0, \tag{3.1}$$

where $v_l := \delta_{lm} v^m$.

With x_o fixed, we make another change of coordinates: $(s, v^a) \to (v^i)$ by

$$v^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \qquad v^a = v^a,$$

where

$$\bar{\alpha} := \sqrt{\sum_{a=2}^{n} (v^a)^2}.$$

Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \qquad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}.$$

Let

$$\bar{r}_{10} := \sum_{a=2}^{n} r_{1a} v^a, \qquad \bar{r}_{00} := \sum_{a,b=2}^{n} r_{ab} v^a v^b,$$

$$\bar{s}_{10} = \sum_{a=2}^{n} s_{1a} v^a, \qquad \bar{s}_0 := \sum_{a=2}^{n} s_a v^a.$$

Note that

$$\bar{s}_0 = b\bar{s}_{10}, \qquad s_1 = b s_{11} = 0.$$

Express

$$G^i_\alpha = \frac{1}{2} G^i_{jk} v^j v^k, \qquad G^i_{jk} = G^i_{kj}.$$

Let

$$\bar{G}^a_{10} = G^a_{1b} v^b, \qquad \bar{G}^0_{10} = \bar{G}^0_{01} = G^a_{1b} v_a v^b, \qquad \bar{G}^0_{00} = G^a_{bc} v_a v^b v^c,$$

where $v_a = \delta_{ab} v^b$.

Plugging the above identities into (3.1) we get a system of equations in the form

$$\Phi_l + \Psi_l \bar{\alpha} = 0,$$

where Φ_l and Ψ_l are polynomials in v^a. We must have

$$\Phi_l = 0, \qquad \Psi_l = 0.$$

For $l = 1$, by (3.1) we get

$$s\bar{G}^0_{00} = -s\bar{C}_0\bar{\alpha}^2 + \Big\{ bQB\bar{s}_{10} + 2s\bar{A}_{10} \Big\}\bar{\alpha}^2, \tag{3.2}$$

$$s^2 A_{11}\bar{\alpha}^2 - 2s^2\bar{G}^0_{10} + (b^2 - s^2)\bar{A}_{00} = 0. \tag{3.3}$$

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For $l = a,\ 2 \le a \le n$, we get from (3.1) that

$$s\bar{G}^a_{00} = -sC_a\bar{\alpha}^2 + \{2s\bar{A}_{10} + bQB\bar{s}_{10}\}v^a, \tag{3.4}$$

$$\{2sb^2\bar{G}^a_{10} - s^3A_{11}v^a + b^3Q\bar{s}_{a0}\}\bar{\alpha}^2 = s(b^2 - s^2)\{2\bar{G}^0_{10} + \bar{A}_{00}\}v^a. \tag{3.5}$$

Here

$$A_{ij} := G^1_{ij} + b\Psi r_{ij}, \quad \Gamma := 1 - 2\Psi b^2, \quad C_a = \frac{s}{b^2 - s^2}\{G^a_{11}s - bQs_{1a}\}.$$

$$\bar{C}_0 = C_a v^a, \quad \bar{A}_{10} = A_{1a}v^a, \quad \bar{A}_{00} = A_{ab}v^a v^b.$$

Note that contracting (3.4) with v_a yields (3.2) and contracting (3.5) with v_a yields (3.3). We can use (3.3) to eliminate A_{11} and A_{00} in (3.5).

$$(2s\bar{G}^a_{10} + bQ\bar{s}_{a0})\bar{\alpha}^2 = 2s\bar{G}^0_{10}v^a. \tag{3.6}$$

Dividing (3.6) by $2s$ yields

$$\Big(\bar{G}^a_{10} + \frac{bQ}{2s}\bar{s}_{a0}\Big)\bar{\alpha}^2 = \bar{G}^0_{10}v^a. \tag{3.7}$$

Note that except for $bQ/(2s)$, other terms in (3.7) are independent of s . By assumption, Q/s is not independent of s. We conclude that $\bar{s}_{a0} = 0$, *i.e.*,

$$s_{ab} = 0. \tag{3.8}$$

In this case, (3.7) is reduced to

$$\bar{G}^a_{10}\bar{\alpha}^2 = \bar{G}^0_{10}v^a. \tag{3.9}$$

Differentiating (3.4) with respect to v^b and v^c, we get

$$2sG^a_{bc} = -2sC_a\delta_{bc} + \Big\{(2sA_{1b} + bQ\Gamma s_{1b})\delta^a_c + (2sA_{1c} + bQ\Gamma s_{1c})\delta^a_b\Big\}. \tag{3.10}$$

Taking trace in (3.10) over $a = b = 2, \ldots, n$ yields

$$2sA_{1c} + bQ\Gamma s_{1c} = \frac{2s}{n}\{G^m_{mc} + C_c\}. \tag{3.11}$$

Plugging (3.11) into (3.10), we get

$$G^a_{bc} - \frac{1}{n}\{G^m_{mb}\delta^a_c + G^m_{mc}\delta^a_b\} = -C_a\delta_{bc} + \frac{1}{n}\{C_b\delta^a_c + C_c\delta^a_b\}. \tag{3.12}$$

By assumption, $n > 2$. For any $2 \le a \le n$, one can take $b = c \ne a$. In this case, (3.12) becomes $G^a_{bc} = -C_a$. Note that $C_a = 0$ at $s = 0$. We get $G^a_{bc} = 0$ $(b = c \ne a)$. Thus $C_a = 0$, $(|s| \le b)$ By the definition of C_a, we get $G^a_{11} - \frac{bQ}{s}s_{1a} = 0$. By assumption, Q/s is not independent of s, we conclude that

$$s_{1a} = 0. \tag{3.13}$$

In this case, we also have

$$G^a_{11} = 0. \tag{3.14}$$

Since $s_{11} = 0$, it follows from (3.8) and (3.13) that $s_{ij} = 0$. ■

4 Determining r_{ij} and G^i_α

In this section, we are going to derive two formulas for r_{ij} and G^i_α. We shall always assume that

(a) F is projectively flat on $\mathcal{U}$;
(b) F is not of Randers type at any point;
(c) $b \neq 0$ at any point;
(d) $db \neq 0$ at any point or $b = constant$;
(e) β is not parallel everywhere.

We continue to use the coordinate system (u^i, v^i) at $u = 0$. Express $\alpha = \sqrt{a_{ij}v^i v^j}$ and $\beta = b_i v^i$. We have at $u = 0$, $a_{ij} = \delta_{ij}$, $b_i = b\delta_{1i}$.

In the previous section, we have shown that $C_a = 0$ and $s_{1b} = 0$ under the assumption that $n \geq 3$. Now (3.10) is reduced to

$$G^a_{bc} = A_{1b}\delta^a_c + A_{1c}\delta^a_b. \tag{4.1}$$

We can rewrite (4.1) as

$$G^a_{bc} - (G^1_{1b}\delta^a_c + G^1_{1c}\delta^a_b) = b\Psi(r_{1b}\delta^a_c + r_{1c}\delta^a_b). \tag{4.2}$$

Note that the left side is independent of s. If $r_{1c} \neq 0$ for some $2 \leq c \leq n$, then $b\Psi$ is independent of s. We can express Ψ as $2\Psi = \frac{k_1}{1+k_1 b^2}$ where k_1 is a number independent of s. By Lemma 2.2, ϕ is given by $\phi = \sqrt{1 + k_1 s^2} + \epsilon s$, where ϵ is a number independent of s. This case is excluded in the theorem. Thus we conclude that

$$r_{1b} = 0. \tag{4.3}$$

Then (4.2) is further reduced to the following

$$G^a_{bc} - (G^1_{1b}\delta^a_c + G^1_{1c}\delta^a_b) = 0. \tag{4.4}$$

It follows from (3.3) that

$$s^2\{G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a})\} + (b^2 - s^2)G^1_{ab} = -b\Psi\{s^2 r_{11}\delta_{ab} + (b^2 - s^2)r_{ab}\}. \tag{4.5}$$

Case I $db \neq 0$ at $u = 0$. Observe that at $u = 0$,

$$[b^2]_{u^j} = 2b^i b_{i|j} = 2b^i r_{ij} + 2b^i s_{ij} = 2br_{1j} = 2br_{11}\delta_{1j}.$$

Thus $r_{11} \neq 0$. By (4.5), there are numbers $\lambda, \mu, \delta \neq 0$ and η independent of s such that

$$2\Psi = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\delta s^2 + \eta(b^2 - s^2)}. \tag{4.6}$$

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Actually, we may take

$$\delta = -br_{11}, \qquad \eta = -br_{22}, \lambda = \frac{1}{2}(G^1_{11} - 2G^2_{12}), \qquad \mu = \frac{1}{2}G^1_{22}.$$

Plugging (4.6) into (4.5) yields

$$\delta\Big\{ G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a})\Big\} = -\frac{b\lambda}{2}r_{11}\delta_{ab}$$

$$\delta G^1_{ab} + \eta\Big\{ G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a})\Big\} = -\frac{b\mu}{2}r_{11}\delta_{ab} - \frac{b\lambda}{2}r_{ab}$$

$$\eta G^1_{ab} = -\frac{b\mu}{2}r_{ab}.$$

Let τ be a number such that $r_{11} = 2b^2\delta\tau$. If $\mu\delta - \eta\lambda = 0$, then $2\Psi = \lambda/\delta$ is independent of s. We can express Ψ as $2\Psi = k_1/(1 + k_1b^2)$ where k_1 is a number independent of s. Then $\phi = \sqrt{1 + k_1s^2} + \epsilon s$, where ϵ is a number independent of s. This is the case excluded in the theorem. Therefore we conclude that $\mu\delta - \eta\lambda \neq 0$. By this fact, we get from the above linear system that

$$r_{ab} = 2b^2\eta\tau\delta_{ab}, \tag{4.7}$$

$$G^1_{ab} = -b^3\mu\tau\delta_{ab}, \tag{4.8}$$

$$G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a}) = -b^3\lambda\tau\delta_{ab}, \tag{4.9}$$

Contracting (4.9) with v^a and v^b yields $\bar{G}^0_{10} = \frac{1}{2}(G^1_{11} + b^3\lambda\tau)\bar{\alpha}^2$. Plugging it into (3.9) gives $\bar{G}^a_{10} = \frac{1}{2}(G^1_{11} + b^3\lambda\tau)v^a$. Differentiating the above identity with respect to v^b, we get

$$G^a_{1b} = \frac{1}{2}(G^1_{11} + b^3\lambda\tau)\delta^a_b. \tag{4.10}$$

Finally, let us summarize what we have proved so far:

$$s_{11} = 0, \qquad s_{ab} = 0, \qquad s_{1a} = 0. \tag{4.11}$$

$$r_{11} = 2b^2\delta\tau, \qquad r_{ab} = 2b^2\eta\tau\delta_{ab}, \qquad r_{1a} = 0. \tag{4.12}$$

It is easy to see that (4.11) is equivalent to $s_{ij} = 0$, and (4.12) is equivalent to $r_{ij} = 2\tau\{\delta b_ib_j + \eta(b^2\delta_{ij} - b_ib_j)\}$. The above identities hold in (u^i) at $u = 0$. Back to the local system (x^i) at x_o, we get $r_{ij} = 2\tau\{\delta b_ib_j + \eta(b^2a_{ij} - b_ib_j)\}$. By (3.14), (4.4), (4.8) and (4.10), we get

$$G^a_{11} = 0, \qquad G^1_{ab} = -b^3\mu\tau\delta_{ab},$$

$$G^1_{11} = k_1 - b^3\mu\tau, \qquad G^a_{1b} = \frac{1}{2}k_1\delta^a_b, \qquad G^1_{1a} = k_a, \qquad G^a_{bc} = k_b\delta^a_c + k_c\delta^a_b,$$

where k_i are numbers independent of s. It is easy to verify that the above identities are equivalent to $G^i_\alpha = \xi v^i - \tau\{\lambda\beta^2 + \mu(b^2\alpha^2 - \beta^2)\}b^i$, where $\xi = k_jv^j$. The above identities hold in (u^i, v^i) at $u = 0$. Clearly, G^i_α take the same form in (x^i, y^i) at x_o (hence at any point x since x_o is chosen arbitrarily).

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Case II $b \neq 0$ is constant. In this case, $r_{11} = 0$. We have proved that $s_{ij} = 0$ and $r_{1a} = 0$. Since we assume that β is not parallel, $(r_{ab}) \neq 0$. By (4.5), there are numbers λ, μ and $\eta \neq 0$ independent of s such that

$$2\Psi = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\eta(b^2 - s^2)}. \tag{4.13}$$

Plugging (4.13) into (4.5) yields

$$G^1_{ab} = -\frac{b\mu}{2\eta} r_{ab}, \tag{4.14}$$

$$G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a}) = -\frac{b\lambda}{2\eta} r_{ab}. \tag{4.15}$$

In this case, there is no restriction on r_{ab}.

Contracting (4.15) with v_a and v^b, we obtain that

$$\bar{G}^0_{10} = \frac{1}{2}\Big(G^1_{11}\bar{\alpha}^2 + \frac{b\lambda}{2\eta}\bar{r}_{00}\Big). \tag{4.16}$$

Plugging (4.16) into (3.9) yields

$$\Big(\bar{G}^a_{10} - \frac{1}{2} G^1_{11} v^a \Big)\bar{\alpha}^2 = \frac{b\lambda}{4\eta}\bar{r}_{00} v^a. \tag{4.17}$$

By (4.17), there is a number τ independent of s such that

$$r_{ab} = 2b^2\tau\eta\delta_{ab}, \tag{4.18}$$

and $G^a_{1b} = \frac{1}{2}(G^1_{11} + b^3\lambda\tau)\delta^a_b$.

It follows from the fact $r_{11} = 0$, (4.3) and (4.18) that $r_{ij} = 2\tau\eta(b^2\delta_{ij} - b_i b_j)$.

Plugging (4.18) into (4.14) and (4.15) yields

$$G^1_{ab} = -b^3\mu\tau\delta_{ab}, \tag{4.19}$$

$$G^1_{11}\delta_{ab} - (G^a_{1b} + G^b_{1a}) = -b^3\lambda\tau\delta_{ab}. \tag{4.20}$$

Contracting (4.20) with v^a and v^b yields $\bar{G}^0_{10} = \frac{1}{2}(G^1_{11} + b^3\lambda\tau)\bar{\alpha}^2$. Plugging it into (3.9) gives $\bar{G}^a_{10} = \frac{1}{2}(G^1_{11} + b^3\lambda\tau)v^a$. Differentiating the above identity with respect to v^b, we get

$$G^a_{1b} = \frac{1}{2}(G^1_{11} + b^3\lambda\tau)\delta^a_b. \tag{4.21}$$

It follows from (4.4) and (4.21) that there are numbers k_1 and k_a such that

$$G^1_{11} = k_1 - b^3\lambda\tau, \qquad G^a_{1b} = \frac{1}{2}k_1\delta^a_b, \qquad G^1_{1a} = k_a, \qquad G^a_{bc} = k_b\delta^a_c + k_c\delta^a_b.$$

Together with (3.14) and (4.19) we get $G^i_\alpha = \xi v^i - \tau(\lambda\beta^2 + \mu(b^2\alpha^2 - \beta^2))b^i$,where $\xi = k_i v^i$.

5 The Equation on ϕ

To prove Theorem 1.1, we consider

$$2\Psi = \frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\delta s^2 + \eta(b^2 - s^2)}, \tag{5.1}$$

where λ, μ, δ and η are scalar functions with $(\lambda, \mu) \neq (0, 0)$ and $(\delta, \eta) \neq (0, 0)$, possibly depending on $b = \|\beta_x\|_\alpha$.

Lemma 5.1 *Assume that $\phi = \phi(s)$ with $\phi(0) = 1$ and $b \neq 0$ satisfies* (5.1)*. Then $\phi^{(3)}(0) = \phi^{(5)}(0) = 0$ and one of the following holds:*

(i) $\phi^{(4)}(0) + 3(\phi''(0))^2 \neq 0$ *and*

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{k_1 + k_2 s^2}{1 + k_1 b^2 + k_2 b^2 s^2 + k_3 s^2}, \tag{5.2}$$

where $k_1 = \phi''(0)$, k_2 and k_3 are constants depending on $\phi''(0)$, $\phi^{(4)}(0)$ and $\phi^{(6)}(0)$.

(ii) $\phi^{(4)}(0) + 3(\phi''(0))^2 = 0$, *and then*

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{k_1}{1 + k_1 b^2}, \tag{5.3}$$

where $k_1 = \phi''(0)$.

The equations (5.2) *and* (5.3) *can be rewritten as one equation independent of b:*

$$\{1 + k_1 s^2 + k_2 s^4 + k_3 s^2\}\phi''(s) = (k_1 + k_2 s^2)\{\phi(s) - s\phi'(s)\}.$$

Proof Rewrite (5.1) as

$$[\delta s^2 + \eta(b^2 - s^2)]\phi'' = [\lambda s^2 + \mu(b^2 - s^2)][\phi - s\phi' + (b^2 - s^2)\phi'']. \tag{5.4}$$

Let $\phi = 1 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6 + a_7 s^7 + o(s^7)$. Plugging the above Taylor expansion into (5.4), we get some linear equations on λ, μ, δ and η. We can actually solve these equations for λ, μ, δ and η based on the values of the following quantities:

$$a_2, \qquad 1 + 2a_2 b^2, \qquad 2a_4 + a_2^2.$$

Case 1 $a_2 = 0$ or $a_2 = -1/2b^2$. Then by a comparison on the coefficients of the polynomials on both sides of (5.4), we conclude that $2a_4 + a_2^2 \neq 0$ and

$$\mu = k_1 \epsilon,$$
$$\eta = (1 + k_1 b^2)\epsilon,$$
$$\lambda = (k_1 + k_2 b^2)\epsilon,$$
$$\delta = (1 + k_1 b^2 + k_2 b^4 + k_3 b^2)\epsilon,$$

where ϵ is a number with $\epsilon \neq 0$ and k_i are given by

$$k_1 := 2a_2,$$
$$k_2 := 2\frac{a_4 a_2^2 - 5a_6 a_2 + 12a_4^2}{2a_4 + a_2^2},$$
$$k_3 := -\frac{11a_4 a_2 + 5a_6 + 3a_2^3}{2a_4 + a_2^2}.$$

In this case,

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{k_1 + k_2 s^2}{1 + k_1 b^2 + k_2 b^2 s^2 + k_3 s^2}.$$

Case 2 $a_2 \neq 0, -1/2b^2$ and $2a_4 + a_2^2 = 0$. By a comparison on the coefficients of the polynomials on both sides of (5.4), we get $2a_6 - a_2^3 = 0$ and

$$\mu = k_1 \epsilon,$$
$$\eta = (1 + k_1 b^2)\epsilon,$$
$$\lambda = \frac{k_1}{1 + k_1 b^2}\delta,$$

where ϵ is a number with $\epsilon \neq 0$ and $k_1 = 2a_2$. In this case,

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{k_1}{1 + k_1 b^2}.$$

Case 3 $a_2 \neq 0,\ -\frac{1}{2b^2}$, and $2a_4 + a_2^2 \neq 0$. By a comparison on the coefficients of the polynomials on both sides of (5.4), we still get

$$\mu = k_1 \epsilon,$$
$$\eta = (1 + k_1 b^2)\epsilon,$$
$$\lambda = (k_1 + k_2 b^2)\epsilon,$$
$$\delta = (1 + k_1 b^2 + k_2 b^4 + k_3 b^2)\epsilon,$$

where ϵ is a number with $\epsilon \neq 0$ and k_i are given in Case 1. In this case

$$\frac{\phi''(s)}{\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)} = \frac{k_1 + k_2 s^2}{1 + k_1 b^2 + k_2 b^2 s^2 + k_3 s^2}.$$ ■

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