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# THE WEYL-VON NEUMANN THEOREM FOR MULTIPLIERS OF SOME $AF$-ALGEBRAS

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**Introduction.** A well known theorem of Weyl-von Neumann asserts that if $X$ is a self-adjoint operator acting on a separable Hilbert space, then there is a decomposition $1 = \sum e_n$ of the identity into finite rank projections so that we may write

$$X = \sum \lambda_n e_n + y,$$

where the $\lambda_n$ are scalars and $y$ is a *compact* operator with small norm. In other words, $X$ can be *approximately* diagonalized. In this paper we consider the following question: given an $AF$-algebra $I$ and a self-adjoint element $X$ of $\mathcal{M}(I)$, the multiplier algebra of $I$, can we express $X$ in the above form, where now the $e_n$ are projections in $I$ (and $\sum e_n = 1$ in the sense of strict convergence) and $y \in I$? This reduces to the Weyl-von Neumann Theorem in the case $I = \mathcal{K}$

We shall answer this question affirmatively in the case that $I$ is simple and has a unique trace (up to scaling). Our approach is based upon the observation, which seems to have been made by a number of people (see especially [9]), that the problem is equivalent to showing that $\mathcal{M}(I)$ has one of a number of basic structural properties. See Section 1. These properties can then be analyzed in terms of the ideal structure of $\mathcal{M}(I)$, which in the case at hand is very straightforward.

Our techniques would carry over to a somewhat larger class of $AF$- and other algebras $I$ (for example, simple $AF$-algebras with only finitely many extremal traces), and indeed we have no doubt that the answer to the question is affirmative for general $AF$-algebras. However, in order to make the exposition as clear as possible we shall consider only simple $I$ with unique trace.

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After this paper was first typed, we discovered that there is a considerable overlap between this article and work of others in this area (we are very grateful to S. Zhang for sending us preprints of his articles [9], [10] and to G. Pedersen for discussions and a draft of [3]). In fact, our Theorem 4.4 is contained within [9]. However, our arguments are,

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for the most part, rather different, and since they are perhaps simpler and more direct, we hope that our article is still worthy of the reader's attention.

1. **Equivalent formulations.** We shall make extensive use of the following result. For a proof see [2,3,8].

THEOREM 1.1. *The following three conditions on a $C^*$-algebra A are equivalent.*

*(i) Each hereditary subalgebra of A has an approximate unit consisting of projections.*

*(ii) Every self-adjoint element in A is a norm limit of invertible self-adjoint elements.*

*(iii) Every self-adjoint element in A is a norm limit of self-adjoint elements in A with finite spectrum.* ∎

(In (ii), if $A$ does not have a unit then replace $A$ by the $C^*$-algebra obtained by adjoining a unit.)

We shall say that $A$ has *property FS* (= *finite spectrum*, a reference to (iii)) if it satisfies one of the above conditions. We shall move from one condition to another without comment.

Lemma 1.3 below shows that the Weyl-von Neumann Theorem for $\mathcal{M}(I)$ reduces to showing that $\mathcal{M}(I)$ has property *FS*.

LEMMA 1.2. *Let I be a separable $C^*$-algebra with property FS, and let P be a projection in $\mathcal{M}(I)$. There is a sequence $\{p_n\}_{n=1}^{\infty}$ of mutually orthogonal projections in I such that*

$$P = \sum_{n=1}^{\infty} p_n,$$

*where the sum converges in the strict topology.*

PROOF. The $C^*$-algebra $PIP$ is a hereditary subalgebra of $I$, and so there is a sequence $\{p_n\}_{n=1}^{\infty}$ of mutually orthogonal projections in $PIP$ such that $P = \sum_{n=1}^{\infty} p_n$, the convergence being in the strict topology of $\mathcal{M}(PIP)$. But this implies strict convergence $\mathcal{M}(I)$, for given $x \in I$ we have that

$$\begin{aligned}\|\sum_{n=M}^{N} p_n x\|^2 &= \|\sum_{m=M}^{N}\sum_{n=M}^{N} p_m x x^* p_n\| \\ &= \|\sum_{m=M}^{N}\sum_{n=M}^{N} p_m P x x^* P p_n\| \\ &= \|\sum_{n=M}^{N} p_n (P x x^* P)^{\frac{1}{2}}\|^2\end{aligned}$$

and $(Pxx^*P)^{\frac{1}{2}} \in PIP$. ∎

LEMMA 1.3 (SEE [9]). *Let I be a separable $C^*$-algebra with property FS. The following are equivalent:*

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*(i) for every self-adjoint $X \in \mathcal{M}(I)$, every projection $p \in I$, and every $\varepsilon > 0$ there is a projection $q \in I$ such that $q \geq p$ and $\|[q, X]\| < \varepsilon$;*

*(ii) for every self-adjoint $X \in \mathcal{M}(I)$ and every $\varepsilon > 0$ there is a family $\{e_n\}_{n=1}^{\infty}$ of mutually orthogonal projections in I such that $\sum e_n = 1$ (convergence in the strict topology), such that we may write $X = \sum \lambda_n e_n + y$, where $\lambda_n \in \mathbb{R}, y \in I$ and $\|y\| < \varepsilon$; and*

*(iii) $\mathcal{M}(I)$ has property FS.*

PROOF. (i)⇒(ii) A simple induction argument shows that we can write $X = \sum f_k X f_k + y_1$ for some sequence $\{f_k\}$ of projections in $I$ with $\sum f_k = 1$, and where $y_1 \in I$, $\|y_1\| < \varepsilon/2$. Using the fact that $I$, and hence $f_k I f_k$, has property $FS$, we can perturb each $f_k X f_k$ by an element of norm less that $\varepsilon 2^{-(k+1)}$ to a self-adjoint element $x_k \in f_k I f_k$ with finite spectrum. The spectral projections of all the $x_k$ together then form a suitable family $\{e_n\}$.

(ii)⇒(iii) This is clear.

(iii)⇒(i) For fixed $p \in I$, the set of self-adjoint $X \in \mathcal{M}(I)$ satisfying (i) for all $\varepsilon > 0$ is norm closed, and so it suffices to prove (i) for $X$ with finite spectrum. Write $X = \sum_{i=1}^{n} \lambda_i P_i$, where $P_i \in \mathcal{M}(I)$ are projections, $P_1 + \cdots + P_n = 1$, and $\lambda_i \in \mathbb{R}$. From Lemma 1.2, we get $X = \sum_{j=1}^{\infty} \mu_j e_j$ where the $e_i \in I$ are projections, $\sum_{j=1}^{\infty} e_j = 1$ and $\mu_j \in \mathbb{R}$ (in fact, $\mu_j = \lambda_i$ for some $i$). Put $f_n = \sum_{j=1}^{n} e_j$. Then $\lim_{n\to\infty} \|(1 - f_n)p\| = 0$. For $n$ large enough, $f_n$ is equivalent to a projection $q \in I$ with $q \geq p$ and $\|f_n - q\|$ small. Since $[f_n, X] = 0$, $\|[q, X]\|$ is small. ∎

We note that the Weyl-von Neumann Theorem for *normal* elements is much more complicated, if it is true at all, in the general situations we are considering.

2. **Comparison theory in $\mathcal{M}(I)$.** From here on, $I$ will denote a non-unital, simple $AF$-algebra which has a unique semi-finite trace $\tau$, up to scaling. We may extend $\tau$ to a trace function on $\mathcal{M}(I)^+$ by the formula

$$\tau(X) = \sup \tau(e_n X e_n),$$

where $\{e_n\}_{n=1}^{\infty}$ is any approximate unit for $I$.

PROPOSITION 2.1. *Let P and Q be projections in $\mathcal{M}(I)$.*

*(i) If $\tau(P) < \tau(Q)$, then $P \precsim Q$; and*

*(ii) if neither of P and Q is in I, and if $\tau(P) = \tau(Q)$, then $P \sim Q$.*

PROOF. Write $P = \sum p_n$ and $Q = \sum q_n$, as in Lemma 1.2. In case (i), by regrouping the $q_n$ (that is, replacing the $q_n$ with sums of the form $\sum_M^N q_n$) we may assume that $\tau(p_n) < \tau(q_n)$. In case (ii), by regrouping both the $p_n$ and the $q_n$, we may assume that $\sum_{n=1}^{N} \tau(p_n) < \sum_{n=1}^{N} \tau(q_n)$, for every $N$, and also $\sum_{n=1}^{N} \tau(q_n) < \sum_{n=1}^{N+1} \tau(p_n)$ (this construction requires that both $\sum \tau(p_n)$ and $\sum \tau(q_n)$ have infinitely many non-zero terms, which is where we use the fact that $P, Q \notin I$, as well as the fact that $I$ is simple, so that $\tau$ is faithful). Now, it is well known (see [4]) that for projections $p$ and $q$ in $I$, if $\tau(p) < \tau(q)$

then $p \lesssim q$. Thus in case (i) there exist partial isometries $v_n \in I$ such that $v_n^* v_n = p_n$ and $v_n v_n^* \leq q_n$, whilst in case (ii) there exist $v_n$ such that

$$\begin{aligned} v_1^* v_1 &= p_1, & v_1 v_1^* &\leq q_1 \\ v_2^* v_2 &\leq p_2, & v_2 v_2^* &= q_1 - v_1 v_1^* \\ v_3^* v_3 &= p_2 - v_2^* v_2, & v_3 v_3^* &\leq q_2 \\ v_4^* v_4 &\leq p_3, & v_4 v_4^* &= q_2 - v_3 v_3^* \end{aligned}$$

and so on. In either case, the series $V = \sum_{n=1}^{\infty} v_n$ is strictly convergent. In case (i) we have $V^* V = P$, and $VV^* \leq Q$, and in case (ii), $V^* V = P$ and $VV^* = Q$. ∎

3. **Ideals in $\mathcal{M}(I)$ and quotients.** For the rest of the paper we shall denote by $J$ the norm-closure of the set of elements $X \in \mathcal{M}(I)$ with $\tau(X^*X) < \infty$. G. Elliot [5] and H. Lin [6] prove that $J$ is an ideal in $\mathcal{M}(I)$, that $0 \subseteq I \subseteq J \subseteq \mathcal{M}(I)$, and that $\mathcal{M}(I)$ has no other ideals than these. Moreover, $I \neq J$ if and only if $I$ is non elementary (i.e., $I \not\cong \mathcal{K}$); and $J \neq \mathcal{M}(I)$ if and only if $I$ is not finite (in the sense that $\tau$ is unbounded on the positive unit-ball of $I$), and this again is equivalent to $I$ being stable. These results hold because $I$ is assumed to be simple (and $AF$) with a *unique* trace.

LEMMA 3.1. *Let $X \in \mathcal{M}(I)$. There is an approximate unit $\{u_n\}_{n=1}^{\infty}$ for $I$ such that $\lim_{n\to\infty} \|u_n X - X u_n\| = 0$ and $u_n u_{n-1} = u_{n-1}$ for all $n$. In fact, we may choose $\{u_n\}_{n=1}^{\infty}$ so that there is an approximate unit $\{e_n\}_{n=1}^{\infty}$ of projections in $I$ such that for each $n$, $u_n e_n = e_n$ and $u_n e_{n+1} = u_n$.*

PROOF. Let $\{f_n\}_{n=1}^{\infty}$ be any approximate unit of projections for $I$. The argument of [1] (see also [7, Theorem 3.12.14]) shows that there is an approximate unit $\{w_n\}_{n=1}^{\infty}$ contained in $\operatorname{conv}\{f_n\}$ such that $\|w_n X - X w_n\| \to 0$. Thus we can choose some $u_1 \in \operatorname{conv}\{f_n\}$ with $\|u_1 X - X u_1\| < 2^{-1}$. Setting $e_1 = f_1$ we have $u_1 e_1 = e_1$. For sufficiently large $N$, any element $u$ of $\operatorname{conv}\{f_N, f_{N+1}, \ldots\}$ satisfies $u u_1 = u_1$, and so by the argument of [1] again, we can choose some $u_2$ in $\operatorname{conv}\{f_N, f_{N+1}, \ldots\}$ such that $\|u_2 X - X u_2\| < 2^{-2}$. For $e_2 = f_N$ we have $e_2 u_1 = u_1$ and $u_2 e_2 = u_2$. Iterating this procedure, we obtain the desired approximate unit. ∎

PROPOSITION 3.2. *If $X \geq 0$ is an element of $\mathcal{M}(I)$, but not of $J$, then the hereditary subalgebra of $\mathcal{M}(I)$ generated by $X$ contains an infinite trace projection.*

PROOF. Choose $\{u_n\}_{n=1}^{\infty}$ and $\{e_n\}_{n=1}^{\infty}$ as in Lemma 3.1, for which $\|u_n X - X u_n\|$ is so small that $\|d_n X - X d_n\| < 2^{-n}$, where $d_n = (u_n - u_{n-1})^{\frac{1}{2}}$ (and $u_0 = 0$). Then $\sum d_n X d_n = X + y_1$, where $y_1 = \sum_{n=1}^{\infty} d_n [X, d_n] \in I$. Let $p_n = e_{n+1} - e_{n-1}$ (where $e_0 = 0$). Note that $p_n d_n = d_n$ and that the projections $p_{2n-1} (n = 1, 2, \ldots)$ are pairwise disjoint, as are the projections $p_{2n} (n = 1, 2, \ldots)$. By perturbing each $d_n X d_n$, within $p_n I p_n$, by a suitable operator $z_n$, with say $\|z_n\| < 2^{-n}$, we may write

$$X + y = \sum x_n,$$

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where $y = y_1 + \sum z_n \in I$, and $x_n = d_nXd_n + z_n$ is a positive element in $p_nIp_n$ with finite spectrum. Since $X \notin J$ it follows that $X + y \notin J$, and so (at least) one of $X_e = \sum x_{2n}$ or $X_o = \sum x_{2n+1}$ is not an element of $J$. Let us say $X_e \notin J$, and show first that the hereditary subalgebra generated by $X_e$ contains an infinite trace projection. From $X_e \notin J$ it follows easily that $X_e$ is not a norm limit of elements of $\mathcal{M}(I)^+$ of finite trace. From this it follows that for small enough $\varepsilon > 0$, the spectral projection $P_\varepsilon$ of $X_e$ corresponding to $[\varepsilon, \infty)$ (defined in $\mathcal{M}(I)$ since $X_e$ is an orthogonal strict sum of elements of finite spectrum) has infinite trace. But all the $P_\varepsilon$ are in the hereditary subalgebra generated by $X_e$. Now, the hereditary subalgebra generated by $X_e$ is contained in the hereditary subalgebra $A'$ generated by $X_e + X_0 = X + y$, so $A'$ contains an infinite trace projection $P'$. The images in $\mathcal{M}(I)/I$ of the hereditary subalgebra $A$ generated by $X$, and of $A'$ are equal; therefore $A/A \cap I$ contains the image of $P'$. Since $A \cap I$ is an $AF$-algebra, this image lifts to a projection $P$ in $A$ (see [4]). It is easily verified that $\tau(P) = \infty$.

PROPOSITION 3.3. *If $X \geq 0$ is an element of $J$, but not of $I$, then the hereditary subalgebra of $J$ generated by $X$ contains a projection in $J \setminus I$.*

PROOF. Repeat the above decomposition of $X$ into the sum $X = X_e + X_0 + y$, with say $X_e \notin I$. For suitable $\varepsilon > 0$ we have $\operatorname{dist}(X_e, I) > \varepsilon$, and so since $\|X_e - X_eP_\varepsilon\| \leq \varepsilon$ it follows that $P_\varepsilon \notin I$ for such $\varepsilon$. The rest of the above argument now produces a projection $P$ in the hereditary subalgebra generated by $X$ such that $P - P_\varepsilon \in I$. Since $P_\varepsilon \notin I$ it follows that $P \notin I$. ■

These two propositions give more information than we actually need, which is the following corollary.

COROLLARY 3.4. *The $C^*$-algebra $\mathcal{M}(I)/J$ is purely infinite, as is $PJP/PIP$ for every finite trace projection $P \in \mathcal{M}(I) \setminus I$.*

PROOF. Recall that a unital $C^*$-algebra different from $\mathbb{C}$ is said to be *purely infinite* if every non-zero hereditary subalgebra contains a projection equivalent to 1. For $\mathcal{M}(I)/J$ this follows immediately from Propositions 3.2 and 2.1. As for $PJP/PIP$, by Proposition 3.3 every hereditary subalgebra contains a non-zero projection, the image of a projection $Q \in PJP \setminus PIP$. Now, it follows from Lemma 1.2 that there is a projection $p \in PIP$ such that $\tau(P - p) < \tau(Q)$, and so by Proposition 2.1 there is a partial isometry $W$ such that $WW^* \leq Q$ and $W^*W = P - p$. If $V$ denotes the image of $W$ in $PJP/PIP$ then $V$ is an isometry and so $VV^*$ is a suitable projection in the hereditary subalgebra. ■

The remaining two propositions in this section generalize to $J$ two basic properties of $AF$-algebras. We need the following lemma.

LEMMA 3.5. *Let $p \in I$ be a projection and let $x \in (pIp)^+$. For any $\varepsilon > 0$ there is a projection $q \leq p$ with $\|(1 - q)x\| < \varepsilon$ and $\tau(q) < \frac{3}{\varepsilon}\tau(x)$.*

PROOF. Using the fact that $\tau$ is norm continuous on $pIp$ and the fact that $pIp$ is $AF$, we can reduce to the case where $x$ lies in some finite dimensional $C^*$-subalgebra. Take $q$ to be the spectral projection for $x$ corresponding to $[\varepsilon/2, \infty)$. ■

PROPOSITION 3.6. *Suppose that $X \in \mathcal{M}(I)^+$ and $\tau(X) < \infty$. For any $\varepsilon > 0$ there exists a projection $Q \in \mathcal{M}(I)$ with $\|(1-Q)X\| < \varepsilon$ and $\tau(Q) < \infty$.*

PROOF. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of mutually orthogonal projections in $I$ such that $\sum_{n=1}^{\infty} p_n = 1$. By regrouping the $p_n$ (as in Proposition 2.1) we may assume that for all $n$, $\|p_n X \sum_{|m-n|>2} p_m\| < \varepsilon 2^{-n}$ (compare [5]). For $Y = \sum_{|m-n|\leq 2} p_n X p_m$ we have $\|X - Y\| < \varepsilon$. By Lemma 3.5, for each $n$ there is a projection $q_n \leq p_n$ such that $\|(1 - q_n)p_n X p_n\| < \varepsilon$ and $\tau(q_n) < \frac{3}{\varepsilon}\tau(p_n X p_n)$. The sum $Q = \sum q_n$ converges in the strict topology and

$$\tau(Q) < \frac{3}{\varepsilon}\sum_{n=1}^{\infty}\tau(p_n X p_n) = \frac{3}{\varepsilon}\tau(X) < \infty.$$

Furthermore,

$$\begin{aligned}
&\|(1-Q)Y\| \\
&\leq \|(1-Q)\textstyle\sum p_n X p_{n-1}\| + \|(1-Q)\sum p_n X p_n\| + \|(1-Q)\sum p_n X p_{n+1}\| \\
&= \sup_n \|(1-q_n)p_n X p_{n-1}\| + \sup_n \|(1-q_n)p_n X p_n\| + \sup_n \|(1-q_n)p_n X p_{n+1}\|.
\end{aligned}$$

The middle term is no more than $\varepsilon$, by construction of the $q_n$. As for the other two terms, we have that

$$\begin{aligned}
\|(1-q_n)p_n X p_{n\pm 1}\|^2 &= \|(1-q_n)p_n X p_{n\pm 1} X p_n (1-q_n)\| \\
&\leq \|(1-q_n)p_n X^2 p_n (1-q_n)\| \\
&\leq \|X\| \cdot \|(1-q_n)p_n X p_n (1-q_n)\| \\
&\leq \|X\|\varepsilon.
\end{aligned}$$

Thus $\|(1-Q)Y\| \leq (1 + 2\|X\|)\varepsilon$, and so $\|(1-Q)X\| \leq (2 + 2\|X\|)\varepsilon$. ∎

We remark that the classification of the ideals of $\mathcal{M}(I)$ follows easily from this and Corollary 3.4.

PROPOSITION 3.7. *Every projection in $\mathcal{M}(I)/J$ lifts to a projection in $\mathcal{M}(I)$.*

PROOF. Let $\bar{P}$ be a non-trivial projection in $\mathcal{M}(I)/J$. Applying Proposition 3.2 to any positive lifting of $\bar{P}^{\perp}$, we see that there is an infinite trace projection $Q \in \mathcal{M}(I)$ whose image in $\mathcal{M}(I)/J$ is orthogonal to $\bar{P}$. We can write $Q$ as an orthogonal sum $Q = Q_2 + Q_3 + \cdots$ of infinite trace projections, the sum converging in the strict topology. Setting $Q_1 = Q^{\perp}$, which is also of infinite trace, and fixing a system of partial isometries between $Q_1$ and $Q_n$, we shall represent elements of $\mathcal{M}(I)$ as infinite matrices, with repsect to the decomposition $1 = \sum Q_i$, with entries in $Q_1\mathcal{M}(I)Q_1$. Let $X$ be any lifting of $\bar{P}$ with $1 \geq X \geq 0$ (not necessarily a projection). Since $Q_1 X Q_1$ is also such a lifting, we may assume $X \in Q_1\mathcal{M}(I)Q_1$. Let $g_1, g_2, \ldots$ be a sequence of continuous, non-negative functions on $[0, 1]$ such that (i) $\text{supp}(g_n) \subset [x_{n+2}, x_n]$, where $1 = x_1, x_2, \ldots$ is a sequence of points in $(\frac{1}{2}, 1]$ decreasing to $\frac{1}{2}$; and (ii) $\sum g_n = 1$ on $(\frac{1}{2}, 1]$ (note that for any $x$, at most two of the $g_n(x)$ are non-zero). Define functions $f_n$ in terms of the $g_n$ by

$$f_n(x) = \begin{cases} \left(g_n(x) - g_n(x)^2\right)^{\frac{1}{2}}, & x_{n+2} \leq x \leq x_{n+1}, \\ 0, & \text{otherwise.} \end{cases}$$

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Since at each $x_k$ every $g_n$ is either 0 or 1, the $f_n$ are continuous on [0, 1], The following relations are easily verified:

$$f_n f_m = 0 \qquad \text{if } n \neq m$$
$$g_{n+1} f_n + g_n f_n = f_n,$$
$$f_{n+1}^2 + g_n^2 + f_n^2 = g_n.$$

From these it follows that the element

$$p = \begin{pmatrix} g_1(X) & f_1(X) & & \\ f_1(X) & g_2(X) & f_2(X) & \\ & f_2(X) & g_3(X) & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

is a projection (it is easily seen that the matrix does indeed define an element of $\mathcal{M}(I)$). Since $g_1(1) = 1$ and $g_1(0) = 0$, the element $g_1(X)$ is a lifting of $\bar{P}$, so it suffices to show that the element of $\mathcal{M}(I)$, obtained by removing the $g_1(X)$ term from $P$, is an element of $J$. In fact, since $f_1(1) = f_1(0) = 0$, we have that $f_1(X) \in J$, and so it suffices to show that the positive element $R$ obtained from $P$ by deleting the terms $g_1(X)$ and $f_1(X)$ is in $J$. We will show that $R$ is a norm limit of positive elements with finite trace. All the functions $f_n, g_n (n \geq 2)$ are supported within $[\frac{1}{2}, x_2]$, so there is a continuous function $h$ on [0, 1] with $h \geq 0$, and $hf_n = f_n$, $hg_n = g_n$ for all $n \geq 2$. We have that $h(X) \in J$, and so there is, for every $\varepsilon > 0$, an $X_\varepsilon \in Q_1 \mathcal{M}(I) Q_1^+$ with $\tau(X_\varepsilon^2) < \infty$ and $\|h(X) - X_\varepsilon\| < \varepsilon$. The element $R_\varepsilon$ obtained from $R$ by multiplying each entry on the left and right by $X_\varepsilon$ satisifies $\|R_\varepsilon - R\| < 2\varepsilon$ and $\tau(R_\varepsilon) = \sum_{n=2}^{\infty} \tau\big(X_\varepsilon g_n(X) X_\varepsilon\big)$. Since

$$\sum_{n=2}^{N} X_\varepsilon g_n(X) X_\varepsilon = X_\varepsilon \left( \sum_{n=2}^{N} g_n(X) \right) X_\varepsilon \leq X_\varepsilon^2$$

we see that $\tau(R_\varepsilon) < \infty$. ∎

4. **Property** *FS* **for** $\mathcal{M}(I)$. The following two quite general lemmas reduce the main theorem to the properties of $\mathcal{M}(I)$ and $J$ that we have already established.

LEMMA 4.1. ([9], Part III, Proposition 2.33). *Let $D$ be a unital $C^*$-algebra and let $L$ be an ideal in $D$ such that every projection in $D/L$ lifts to a projection in $D$. If $L$ and $D/L$ have property FS then so does $D$.*

PROOF. The fact that projections lift from $D/L$ to $D$ implies that every self-adjoint, invertible $\bar{s} \in D/I$ lifts to a self-adjoint invertible in $D$. Indeed, by polar decomposition we can write $\bar{s} = \bar{t}(\bar{p} - \bar{p}^\perp)\bar{t}$, $(\bar{t} = |\bar{s}|^{\frac{1}{2}})$, and since $\bar{t}$ certainly lifts to some positive invertible $t$, $\bar{s}$ lifts to $t(p - p^\perp)t$, where $p$ lifts $\bar{p}$. Given that $D/L$ has property *FS*, we see that any self-adjoint element $x \in D$ may be approximated by elements of the form $s + y$, with $s$ invertible and $y \in L$. Thus it suffices to approximate every $s + y$ by self-adjoint invertibles. Writing $s = (p - p^\perp)|s|$, we have that

$$s + y = |s|^{\frac{1}{2}}(p - p^\perp + |s|^{-\frac{1}{2}} y |s|^{-\frac{1}{2}})|s|^{\frac{1}{2}}$$

and so putting $z = |s|^{-\frac{1}{2}}y|s|^{-\frac{1}{2}}$ we see that it suffices to approximate each element of the form $p - p^{\perp} + z$, $(z \in L)$, by self-adjoint invertibles. Both $pLp$ and $p^{\perp}Lp^{\perp}$ are hereditary subalgebras of $L$, and so for any $\varepsilon > 0$ there exist projections $q_1 \in pLp$ and $q_2 \in p^{\perp}Lp^{\perp}$ such that $\|(1-q_1)pz^2p\| < \varepsilon^2$ and $\|(1-q_2)p^{\perp}z^2p^{\perp}\| < \varepsilon^2$. Let $q = q_1 + q_2$. This projection commutes with $p$, and almost supports $z$:

$$\begin{aligned}\|(1-q)z\| &\leq \|(1-q)pz\| + \|(1-q)p^{\perp}z\| \\ &= \|(1-q_1)pz\| + \|(1-q_2)p^{\perp}z\| < 2\varepsilon.\end{aligned}$$

Thus $\|z - qzq\| < 4\varepsilon$. Since $qLq$ has property $FS$, there is a self-adjoint invertible $qwq \in qLq$ such that $\|qwq - q(p-p^{\perp}+z)q\| < \varepsilon$. The element $r = q^{\perp}(p-p^{\perp})q^{\perp} + qwq$ is a self-adjoint invertible with $\|r - (p - p^{\perp} + z)\| < 5\varepsilon$. ∎

LEMMA 4.2. (cf. [10]). *If $E$ is a purely infinite $C^*$-algebra then $E$ has property FS.*

PROOF. Let $x \in E$ be self-adjoint and let $\varepsilon > 0$. Let $g \colon \mathbb{R} \longrightarrow \mathbb{R}^+$ be a continuous function, supported within $(-\varepsilon/3, \varepsilon/3)$, and equal to 1 near 0. If $g(x) = 0$ then $x$ is invertible (and so is certainly approximable by invertibles); if $g(x) \neq 0$ then there is a projection $p \in \overline{g(x)Eg(x)}$ equivalent to 1. By definition of $p$, $\|px\| < \varepsilon/3$, and so $\|x - p^{\perp}xp^{\perp}\| < 2\varepsilon/3$. There is some $v \in E$ with $v^*v = 1$ and $vv^* = p$; let $s = vp^{\perp} + p^{\perp}v^* + p - vp^{\perp}v^*$. This is a symmetry ($s = s^* = s^{-1}$) and furthermore $p^{\perp}sp^{\perp} = 0$. The self-adjoint element

$$y = p^{\perp}xp^{\perp} + \frac{\varepsilon}{3}s = \frac{\varepsilon}{3}s(\frac{3}{\varepsilon}sp^{\perp}xp^{\perp} + 1)$$

is invertible, since $(sp^{\perp}xp^{\perp})^2 = 0$, and $\|y - x\| \leq \|x - p^{\perp}xp^{\perp}\| + \frac{\varepsilon}{3}\|s\| < \varepsilon$. ∎

PROPOSITION 4.3. *The ideal $J$ has prpoperty FS.*

PROOF. By Proposition 3.6 it suffices to show that for each finite trace projection $P$, the $C^*$-algebra $PJP$ has property $FS$. But this follows from the fact that $PJP/PIP$ is purely infinite (Corollary 3.4), and the fact that $PIP$ is $AF$, so that it has property $FS$ and projections lift (see [4]). ∎

THEOREM 4.4. *The $C^*$-algebra $\mathcal{M}(I)$ has property FS.*

PROOF. This follows immediately from the above results, Corollary 3.4 and Proposition 3.7. ∎

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