ON CONNECTIONS BETWEEN GROWTH AND DISTRIBUTION OF ZEROS OF INTEGRAL FUNCTIONS

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1. The following theorem was proved by Paley and Wiener (**4**, p. 70; **1**, p. 136).

THEOREM 1. *If $f(z)$ is a canonical product of order* 1 *with real zeros, and* $f(0) = 1$, *the conditions*

$$\lim_{r\to\infty} \int_{-r}^{r} x^{-2} \log|f(x)|dx = -\pi^2 A, \tag{1}$$

and

$$\lim_{r\to\infty} r^{-1} n(r) = 2A, \tag{2}$$

are equivalent. $n(r)$ *denotes the number of zeros of absolute value not exceeding* r.

Instead of assuming the zeros to be all real Pfluger assumed that the zeros are close to the real axis and proved the following theorem (**5** or **1**, p. 143).

THEOREM 2. *Let*

$$f(z) = e^{cz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right)$$

be an entire function of exponential type, with $f(0) = 1$. *Then the conditions*

$$\lim_{r\to\infty} r^{-1} n(r) = D, \qquad \sum_{n=1}^{\infty} r_n^{-1} |\sin \theta_n| = \pi C < \infty \tag{3}$$

and

$$\lim_{r\to\infty} \int_{-r}^{r} x^{-2} \log|f(x)|dx = -\pi^2 I \neq \pm \infty \tag{4}$$

are equivalent, and $D = 2C + 2I$.

For a general order $\rho (0 < \rho < 1)$, the following theorem was proved by Boas (**2**).

THEOREM 3. *If $f(z)$ is of order less than* 1, *all its zeros are real and negative and* $f(0) = 1$, *the conditions*

$$\lim_{r\to\infty} r^{-\rho} n(r) = A, \tag{5}$$

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and

$$(6)\quad \int_0^r x^{-1-\sigma}\{\log|f(-x)| - \pi \cot \pi\sigma \cdot n(x)\}dx \sim \pi A(\rho - \sigma)^{-1}(\cot \pi\rho - \cot \pi\sigma)r^{\rho-\sigma}$$

(for any σ, $0 < \sigma < 1$) *are equivalent. When* $\sigma = \rho$, (6) *is to be interpreted as*

$$(6')\qquad \int_0^\infty x^{-1-\rho}\{\log|f(-x)| - \pi \cot \pi\rho \cdot n(x)\}dx = -\pi^2 A \operatorname{cosec}^2 \pi\rho.$$

In Theorem 4 of this note we extend the result of Boas to the case where the zeros do not necessarily lie on the negative real axis but are close to certain lines.

THEOREM 4. *Let*

$$(7)\qquad f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)$$

be an entire function of order less than 1. *If*

$$(8)\qquad \sum_{n=1}^{\infty} r_n^{-\sigma}\{2\sin(\theta_n + \pi)\sigma + \sin 2\pi\sigma\} = C \neq \pm\infty \qquad (0 < \sigma < 1)$$

then the conditions (5) *and*

$$(9)\quad \int_0^r x^{-1-\sigma}\{\log|f(-x)| - \pi \cot \pi\sigma \cdot n(x)\}dx$$

$$\sim \pi A(\rho - \sigma)^{-1}(\cot \pi\rho - \cot \pi\sigma)r^{\rho-\sigma} + \frac{\pi C}{\sigma(1 - \cos 2\pi\sigma)}$$

are equivalent. When $\rho = \sigma$, (9) *is to be interpreted as*

$$(9')\quad \int_0^\infty x^{-1-\rho}\{\log|f(-x)| - \pi \cot \pi\rho \cdot n(x)\}dx =$$

$$-\pi^2 A \operatorname{cosec}^2 \pi\rho + \frac{\pi C}{\rho(1 - \cos 2\pi\rho)}.$$

On putting $\rho = \sigma = \frac{1}{2}$ in the above theorem and interpreting the result in terms of functions of order 1, we get Pfluger's theorem. Since (8) is satisfied *a priori* for every $\sigma > \rho$ we have the following

COROLLARY. *If*

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)$$

is an entire function of order less than 1, *and* $\rho < \sigma < 1$ *then the conditions* (5) *and* (9) *are equivalent.*

Proof of Theorem 4. We prove the theorem by comparing $f(z)$ with another function which has real negative zeros. Let

$$F(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{r_n}\right),$$

where $r_n = |z_n|$. We have

$$\log\left|\frac{f(-x)}{F(-x)}\right| = \sum_{n=1}^{\infty} \log\left|\frac{z_n + x}{r_n - x}\right|.$$

The series on the right has non-negative terms and so

$$(10)\quad \int_0^r x^{-1-\sigma}\log|f(-x)|dx = \int_0^r x^{-1-\sigma}\log|F(-x)|dx + \sum_{n=1}^{\infty}\int_0^r x^{-1-\sigma}\log\left|\frac{z_n + x}{r_n - x}\right| dx.$$

The number of zeros of $F(z)$ in $|z| \leqslant x$ is $n(x, F) \equiv n(x, f) \equiv n(x)$. Subtracting

$$\int_0^r x^{-1-\sigma}\pi \cot \pi\sigma\cdot n(x)\, dx$$

from both sides of (10), we get

$$(11)\quad \int_0^r x^{-1-\sigma}\{\log|f(-x)| - \pi \cot \pi\sigma\cdot n(x)\}\, dx$$
$$= \int_0^r x^{-1-\sigma}\{\log|F(-x)| - \pi \cot \pi\sigma\cdot n(x)\}\, dx + \sum_{n=1}^{\infty}\int_0^r x^{-1-\sigma}\log\left|\frac{z_n + x}{r_n - x}\right| dx.$$

We now show that, as $r \to \infty$, the limit (finite or infinite) of the sum on the right is

$$(12)\quad \frac{\pi}{\sigma(1 - \cos 2\pi\sigma)} \sum_{n=1}^{\infty} r_n^{-\sigma}\{2 \sin (\theta_n + \pi)\sigma + \sin 2\pi\sigma\}.$$

To do this put

$$\phi(z) = \log\frac{z_n + z}{r_n - z},$$

where it is assumed that z_n is not real and negative. If the value of z^σ is the principal value and we integrate $z^{-1-\sigma}\phi(z)$ around the contour consisting of the circle $|z| = r$ with a cut from r to 0 and back again having indentations to avoid r_n's and the origin, then $\phi(z)$ increases by $2\pi i$ as we traverse the contour starting at $z = r$. On integration by parts

$$\int z^{-1-\sigma}\phi(z)\, dz = -\frac{2\pi i}{\sigma r^\sigma} + \frac{1}{\sigma}\int z^{-\sigma}\phi'(z)\, dz$$
$$= -\frac{2\pi i}{\sigma r^\sigma} + \frac{1}{\sigma}\int z^{-\sigma}\left(\frac{1}{z_n + z} + \frac{1}{r_n - z}\right) dz$$
$$= -\frac{2\pi i}{\sigma r^\sigma} + \frac{1}{\sigma}2\pi i\,(-z_n)^{-\sigma} - \frac{1}{\sigma}\,\pi i r_n^{-\sigma}(1 + e^{-2\pi i\sigma})$$

$$= -\frac{2\pi i}{\sigma r^{\sigma}} + \frac{2\pi i}{\sigma}\{r_n e^{i(\theta_n+\pi)}\}^{-\sigma} - \frac{1}{\sigma}\pi i r_n^{-\sigma}(1 + e^{-2\pi i\sigma})$$

$$= -\frac{2\pi i}{\sigma r^{\sigma}} + \frac{2\pi i}{\sigma} r_n^{-\sigma} e^{-i(\theta_n+\pi)\sigma} - \frac{1}{\sigma}\pi i r_n^{-\sigma}(1 + e^{-2\pi i\sigma}).$$

As $r \to \infty$, the integral along $|z| = r$ tends to zero, so we have (combining the integrals along the two sides of the cut and equating real parts in the limiting form of the equation)

$$(1 - \cos 2\pi\sigma)\int_0^\infty x^{-1-\sigma}\log\left|\frac{z_n + x}{r_n - x}\right| dx = \frac{\pi}{\sigma} r_n^{-\sigma}\{2\sin(\theta_n + \pi)\sigma + \sin 2\pi\sigma)\}$$

or

$$\int_0^\infty x^{-1-\sigma}\log\left|\frac{z_n + x}{r_n - x}\right| dx = \frac{\pi}{\sigma(1 - \cos 2\pi\sigma)} r_n^{-\sigma}\{2\sin(\theta_n + \pi)\sigma + \sin 2\pi\sigma\}.$$

$F(z)$ has only real negative zeros and $n(r, F) \equiv n(r, f) \sim Ar^{\rho}$. Therefore (Theorem 3 above) the integral on the right-hand side of (11) is

$$\sim \pi A\,(\rho - \sigma)^{-1}(\cot \pi\rho - \cot \pi\sigma)\, r^{\rho-\sigma}$$

which is to be interpreted for $\sigma = \rho$ as $-\pi^2 A \operatorname{cosec}^2\pi\rho$. Hence if we suppose that (8) and (5) hold, then (9) will hold. The fact that (8) and (9) imply (5) is immediate.

2. The following theorem of the same general nature has been proved by Clunie **(3)**.

THEOREM 5. *Let $f(z)$ be an integral function of genus zero and lower order λf $0 < \lambda < 1$, which has all but a finite number of its zeros, z_n, in the upper hal, plane. If $Rz_n = o(|z_n|)$ as $n \to \infty$, then the conditions*

(3) $$\lim_{x\to\infty} x^{-\rho}\log|f(x)| = \tfrac{1}{2}\pi A \operatorname{cosec} \tfrac{1}{2}\pi\rho$$

and

(5) $$\lim_{r\to\infty} r^{-\rho} n(r) = A$$

are equivalent.

Let $n_+(r)$ and $n_-(r)$ count, respectively, the zeros in $Im\, z > 0$ and $Im\, z < 0$. Following the method of Clunie we prove the following extension of Theorem 5.

THEOREM 6. *Let $f(z)$ be an integral function of genus zero and lower order λ, $0 < \lambda < 1$. If at least one of the two numbers $n_+(r)$ and $n_-(r)$ is $O(r^{\rho})$ as $r \to \infty$ and $Rz_n = o(|z_n|)$ as $n \to \infty$, then* (13) *implies* (5).

Proof. Let us suppose for definiteness that $n_-(r) = O(r^{\rho})$. Without loss of generality we may assume that $f(0) = 1$. Let, consequently,

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right) = \prod_{m=1}^{\infty}\left(1 - \frac{z}{b_m}\right)\prod_{n=1}^{\infty}\left(1 - \frac{z}{c_n}\right) = P(z)\cdot Q(z),$$

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where b_m and c_n denote respectively, the zeros lying in the upper half plane and the lower half plane. If δ is fixed, $0 < \delta < \pi$, and $z = r\,e^{i(\pi-\delta)}$ then (**3**, p. 139) for $m > m_0(\delta)$

$$\left|\frac{b_m - z}{b_m - r}\right| < 1.$$

Hence

$$\left|\frac{P(z)}{P(r)}\right| < \prod_{m=1}^{m_0} \left|\frac{b_m - z}{b_m - r}\right| \to 1$$

as $r \to \infty$, and thus as $r \to \infty$,

$$\log|P(z)| < \log|P(r)| + o(1).$$

Further, if we take δ to be sufficiently small, then, for $n > n_0(\delta)$, we will have

$$\left|\frac{c_n - z}{c_n - r}\right| < 2.$$

Hence

$$\left|\frac{Q(z)}{Q(r)}\right| < \prod_{n=1}^{n_0} \left|\frac{c_n - z}{c_n - r}\right| \cdot 2^{n-(r)},$$

and thus as $r \to \infty$

$$\log|Q(z)| < \log|Q(r)| + O(r^\rho).$$

Therefore on the positive real axis and on the radius $\arg z = \pi - \delta$ we find that

$$\log|f(z)| = O(r^\rho).$$

By the Phragmen-Lindelöf principle it follows that $f(z)$ is of order ρ and mean type. The rest of the argument is the same as that of Clunie (**3**, pp. 139–40).

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