Canad. Math. Bull. Vol. **42** (1), 1999 pp. 3–12

How the Roots of a Polynomial Vary with its Coefficients: A Local Quantitative Result

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Abstract. A well-known result, due to Ostrowski, states that if $\|P-Q\|_2 < \varepsilon$, then the roots (x_j) of P and (y_j) of Q satisfy $|x_j - y_j| \le Cn\varepsilon^{1/n}$, where n is the degree of P and Q. Though there are cases where this estimate is sharp, it can still be made more precise in general, in two ways: first by using Bombieri's norm instead of the classical l_1 or l_2 norms, and second by taking into account the multiplicity of each root. For instance, if x is a simple root of P, we show that $|x-y| < C\varepsilon$ instead of $\varepsilon^{1/n}$. The proof uses the properties of Bombieri's scalar product and Walsh Contraction Principle.

1 The General Theory

A well-known result due to Ostrowski [6], [7] can be stated as follows:

(1) Let $P = \sum_0^n a_{n-j} z^j$, $Q = \sum_0^n b_{n-j} z^j$, be two polynomials, satisfying $a_0 = b_0 = 1$, and with respective roots $x_1, \dots, x_n$, $y_1, \dots, y_n$. Let

$$T = \max\{1, |a_1|, |b_1|, \dots, |a_k|^{1/k}, |b_k|^{1/k}, \dots, |a_n|^{1/n}, |b_n|^{1/n}\}.$$

Then, if the y_j's are suitably ordered, we have, for all j,

$$|x_j - y_j| \le 4nT\delta^{1/n}$$

with

$$\delta = \Big(\sum_0^n |a_j - b_j|^2\Big)^{1/2}.$$

(2) Let P, Q be as before; assume moreover that 0 is not a root of P. Assume that, for all j

$$|a_j - b_j| \le \tau |a_j|$$

where τ is small enough, namely

$$\tau \le \left(\frac{1}{4n}\right)^n.$$

Then the zeros y_j's of Q can be ordered in such a way that

$$\left|\frac{y_j}{x_j} - 1\right| < 8n\tau^{1/n}, \quad \text{for all } j.$$

Received by the editors July 27, 1996; revised April 22, 1998.
In application of contract ETCA/CREA/20461/95, Ministry of Defense, France.
AMS subject classification: 30C10.

Here, we will make this result more precise, in two ways. First, in order to measure $P-Q$, we will use Bombieri's norm, and second, we will take into account the multiplicity of the roots, which, of course, may be different from one to the other: this is why we speak of a "local" result.

Let $P=\sum_0^n a_j z^j$ be a polynomial with complex coefficients and degree n. Its Bombieri's norm is defined by

$$[P]=\left(\sum_0^n \frac{|a_j|^2}{\binom{n}{j}}\right)^{1/2}. \tag{1}$$

This definition is better understood in its original frame, that of homogeneous many-variable polynomials: see Beauzamy-Bombieri-Enflo-Montgomery [3] and Beauzamy-Dégot [4].

Let $x_1,\ldots,x_n$ be the roots of P.

Let Q be another polynomial, with same degree, satisfying

$$[P-Q]\le\varepsilon. \tag{2}$$

Theorem 1 *If x is any zero of P, there exists a zero y of Q, with*

$$|x-y|\le\frac{n(1+|x|^2)^{n/2}}{|Q'(x)|}\varepsilon. \tag{3}$$

If ε is small enough, namely

$$\varepsilon\le\frac{1}{2}\frac{|P'(x)|}{n(1+|x|^2)^{\frac{n-1}{2}}} \tag{4}$$

then (3) implies

$$|x-y|\le\frac{2n(1+|x|^2)^{n/2}}{|P'(x)|}\varepsilon. \tag{5}$$

Before we turn to the proof, let us make some comments about these results.

– Estimates (3) and (5) are homogeneous (which is already an improvement upon Ostrowski's result). Indeed, if all coefficients of P and Q are multiplied by λ, so is ε, and $\varepsilon/|Q'(x)|$ or $\varepsilon/|P'(x)|$ are not modified.

– Theorem 1 is empty if x is not a simple root, either for P or for Q (note that Q can have all roots simple, and P have only one root, as the example of z^n and $z^n+\alpha$ shows).

– The term $(1+|x|^2)^{1/2}$ can itself be bound by a quantity depending only on the coefficients of the polynomial, for instance by $(1+R^2)^{1/2}$, where R is the radius of the largest disk, centered at 0, containing all the zeros. An estimate for R can be found in Marden [5]; others may be given, using for instance Mahler's measure of P. Here, we will give later (Theorem 4) a bound depending on Bombieri's norm $[P]$.

Proof of Theorem 1 We need a few simple facts about Bombieri's norm, and the corresponding scalar product, which is just

$$[P,Q]=\sum_{j=0}^{n}\frac{a_j\overline{b}_j}{\binom{n}{j}}, \tag{6}$$

if $P=\sum_{j=0}^n a_j z^j$, $Q=\sum_{j=0}^n b_j z^j$.

***Lemma 2* (B. Reznick [8])** *For any* z_0,

$$P(z_0)=\big[P,(\overline{z}_0 z+1)^n\big].$$

(See Reznick [8] or Beauzamy-Dégot [4] for a proof.)

As a consequence, we get

$$|P(z_0)|\le[P](1+|z_0|^2)^{n/2}. \tag{7}$$

Indeed

$$|P(z_0)|=|\big[P,(\overline{z}_0 z+1)^n\big]|\le[P]\big[(\overline{z}_0 z+1)^n\big],$$

and an immediate computation shows that

$$\big[(\alpha z+1)^n\big]=(1+|\alpha|^2)^{n/2}.$$

Another property of the scalar product is

$$[P',R]=n[P,zR] \tag{8}$$

if $\deg P=n$, $\deg R=n-1$ (see [4] for a proof).

Lemma 3 *If $f(z)=az+b$ $(a\neq 0)$ satisfies $|f(z_0)|\le\varepsilon$, there exists z_1, with $|z_1-z_0|\le\varepsilon/|a|$, such that $f(z_1)=0$. More generally, if $f(z)=a(z-z_1)\cdots(z-z_k)$ satisfies $|f(z_0)|\le\varepsilon$, one of the roots, say z_1, satisfies*

$$|z_1-z_0|\le(\varepsilon/|a|)^{1/k}.$$

Proof of Lemma 3 This is obvious: if

$$|z_0-z_1|\cdots|z_0-z_k|\le\frac{\varepsilon}{|a|},$$

one of the $|z_0-z_j|$ must be at most equal to $(\varepsilon/|a|)^{1/k}$.

Let us now prove the theorem. Since x is a root of P, we have, by (7):

$$|Q(x)|=|(Q-P)(x)|\le\varepsilon(1+|x|^2)^{n/2}. \tag{9}$$

Set

$$\varepsilon'=\varepsilon(1+|x|^2)^{n/2}.$$

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We know by Lemma 2 that

$$Q(x) = \big[Q, (\bar{x}z + 1)^n\big]. \tag{10}$$

Let us consider

$$f(\zeta) = \big[Q, (\bar{x}z + 1)^{n-1}(\bar{\zeta}z + 1)\big],$$

which is an affine function of ζ, satisfying

$$|f(x)| \le \varepsilon'. \tag{11}$$

By Lemma 3, there is a point x', $|x' - x| \le \varepsilon'/|a|$ (where a is the coefficient of ζ in f), such that $f(x') = 0$. Let's compute a. By definition:

$$\begin{aligned} a &= \big[Q, (\bar{x}z + 1)^{n-1}z\big] \\ &= \frac{1}{n}\big[Q', (\bar{x}z + 1)^{n-1}\big] \quad \text{by (8)} \\ &= \frac{1}{n}Q'(x), \end{aligned}$$

by Lemma 2. So we see that a zero x' of f satisfies

$$|x' - x| \le \frac{n\varepsilon'}{|Q'(x)|}. \tag{12}$$

Let us now apply Walsh Contraction Principle (Walsh [9], see Beauzamy [1] for a detailed study and proof). Consider

$$\varphi(u_1, \dots, u_n) = \big[Q, (\bar{u}_1 z + 1) \cdots (\bar{u}_n z + 1)\big]. \tag{13}$$

This is a symmetric function of $u_1, \dots, u_n$, affine with respect to each of them. It satisfies $\varphi(x, \dots, x, x') = 0$. Therefore, in each disk containing both x and x', and in particular in the disk of diameter xx', there is a point y such that

$$\varphi(y, \dots, y) = 0. \tag{14}$$

Coming back to the definition of φ, we get

$$\varphi(y, \dots, y) = \big[Q, (\bar{y}z + 1)^n\big] = Q(y).$$

So y is a zero of Q. Since it is in the disk of diameter xx', we have also by (12):

$$|x - y| \le \frac{n\varepsilon'}{|Q'(x)|},$$

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and the first part of Theorem 1 is proved. To get the second part, we write simply:

$$\begin{aligned} |P'(x) - Q'(x)| &= |\big[P' - Q', (\overline{x}z + 1)^{n-1}\big]| \\ &= n|\big[P - Q, z(\overline{x}z + 1)^{n-1}\big]| \\ &\le n[P - Q]\big[z(\overline{x}z + 1)^{n-1}\big] \\ &\le n\varepsilon(1 + |x|^2)^{\frac{n-1}{2}}. \end{aligned}$$

So $|Q'(x)| \ge |P'(x)| - n\varepsilon(1 + |x|^2)^{\frac{n-1}{2}}$. If ε is taken as indicated, we get $|Q'(x)| \ge \frac{1}{2}|P'(x)|$; the result follows.

Let us now give a more general version of Theorem 1, valid if x has multiplicity k, empty if it has multiplicity $k + 1$:

Theorem 4 *Let $k \ge 1$ be an integer, P and Q be two polynomials of degree n, with $[P - Q] \le \varepsilon$. If x is any zero of P, there exists a zero y of Q, with*

$$|x - y| \le \left(\frac{n!}{(n-k)!}\frac{(1 + |x|^2)^{n/2}}{|Q^{(k)}(x)|}\right)^{1/k} \varepsilon^{1/k}. \tag{15}$$

If ε is small enough, namely

$$\varepsilon \le \frac{(n-k)!}{2n!}\frac{|P^{(k)}(x)|}{(1 + |x|^2)^{\frac{n-k}{2}}} \tag{16}$$

then (15) implies

$$|x - y| \le \left(\frac{2n!}{(n-k)!}\frac{(1 + |x|^2)^{n/2}}{|P^{(k)}(x)|}\right)^{1/k} \varepsilon^{1/k}. \tag{17}$$

Proof of Theorem 4 It follows the same lines, so we only indicate the minor changes. We now set

$$f(\zeta) = \big[Q, (\overline{x}z + 1)^{n-k}(\overline{\zeta}z + 1)^k\big] \tag{18}$$

which is a polynomial in ζ of degree k, satisfying

$$|f(x)| = |Q(x)| \le \varepsilon'.$$

By Lemma 3, there is a point x', with $f(x') = 0$, such that $|x' - x| \le (\varepsilon'/|a|)^{1/k}$, where a is the coefficient of ζ^k in (18), that is

$$a = \big[Q, (\overline{x}z + 1)^{n-k}z^k\big] = \frac{(n-k)!}{n!}Q^{(k)}(x).$$

So we get

$$|x' - x| \le \left(\frac{n!}{(n-k)!}\frac{\varepsilon'}{|Q^{(k)}(x)|}\right)^{1/k}. \tag{19}$$

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Let $\varphi(u_1,\dots,u_n)$ be defined as before. We now get

$$\varphi(\underbrace{x,\dots,x}_{n-k\text{ times}},\underbrace{x',\dots,x'}_{k\text{ times}})=0,$$

so by Walsh's principle, there is a point y, with $\varphi(y,\dots,y)=0$, satisfying

$$|x-y|\le\left(\frac{n!}{(n-k)!}\frac{\varepsilon'}{|Q^{(k)}(x)|}\right)^{1/k}.$$

This proves the first part of the Theorem. Now:

$$\begin{aligned}|P^{(k)}(x)-Q^{(k)}(x)|&=\big|\big[P^{(k)}-Q^{(k)},(\overline{x}z+1)^{n-k}\big]\big|\\&=\frac{n!}{(n-k)!}\big|\big[P-Q,z^k(\overline{x}z+1)^{n-k}\big]\big|\\&\le\frac{n!}{(n-k)!}\varepsilon(1+|x|^2)^{\frac{n-k}{2}},\end{aligned}$$

and the second part follows.

How sharp is the coefficient of ε in estimates (3) or (5)? We do not know exactly, but the order of magnitude is almost best possible. Indeed take $P=z^n-1$, with $x=1$, and $Q=z^n+\varepsilon\sqrt{\binom{n}{n/2}}\,z^{n/2}-1$ (for n even). Then $[P-Q]=\varepsilon$. The roots of Q are the $n/2$ roots of

$$-\frac{\varepsilon}{2}\sqrt{\binom{n}{n/2}}\pm\sqrt{1+\frac{\varepsilon^2}{4}\binom{n}{n/2}}$$

and if y is the real zero

$$\left(\sqrt{1+\frac{\varepsilon^2}{4}\binom{n}{n/2}}-\frac{\varepsilon}{2}\sqrt{\binom{n}{n/2}}\right)^{2/n}.$$

We find

$$|x-y|\sim\frac{\varepsilon}{n}\sqrt{\binom{n}{n/2}}\sim\frac{\varepsilon}{n}2^{n/2}\left(\frac{2}{\pi n}\right)^{1/4},$$

whereas estimates (3) gave $2^{n/2}\varepsilon$.

2 A Bound for the Largest Zero

We now give an estimate for the largest root of P, in terms of Bombieri's norm. This estimate may be substituted in the term $1+|x|^2$, in Theorems 1 and 2 above. Of course, now, some normalization is necessary. We choose the usual one, that is $a_n=1$.

Theorem 5 *If $P=\sum_0^n a_jz^j$ is a polynomial with $a_n=1$, its roots $x_1,\dots,x_n$ satisfy the estimate*

(20) $$\max_j|x_j|\le\sqrt{n[P]^2-1}.$$

This estimate is best possible.

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Proof Let us order the roots so that $|x_1| \geq |x_2| \geq \cdots \geq |x_n|$.

Applying Bombieri's inequality (see [2]) to the pair $z - x_1$, $(z - x_2)\cdots(z - x_n)$ yields:

$$\begin{aligned}[P] &\geq \sqrt{\frac{1!(n-1)!}{n!}}\,[z - x_1]\big[(z - x_2)\cdots(z - x_n)\big]\\ &\geq \frac{1}{\sqrt{n}}(1 + |x_1|^2)^{1/2},\end{aligned}$$

which gives (20).

The estimate (20) is best possible in the sense that, for every n and every $\varepsilon > 0$, there is a polynomial P which has a root satisfying

$$|x| \geq (1 - \varepsilon)\sqrt{n[P]^2 - 1}. \tag{21}$$

Indeed, with $x > 0$, consider $P = (z - x)(z + \frac{1}{x})^{n-1}$. Since the pair $z - x$, $(z + \frac{1}{x})^{n-1}$ is extremal for the product (see Beauzamy [2]), we get

$$[P] = \frac{1}{\sqrt{n}}[z - x]\left[z + \frac{1}{x}\right]^{n-1} = \frac{1}{\sqrt{n}}(1 + x^2)^{1/2}\left(1 + \frac{1}{x^2}\right)^{\frac{n-1}{2}},$$

so

$$n[P]^2 - 1 = (1 + x^2)\left(1 + \frac{1}{x^2}\right)^{n-1} - 1,$$

and the inequality

$$x^2 \geq (1 - \varepsilon)^2\left((1 + x^2)\left(1 + \frac{1}{x^2}\right)^{n-1} - 1\right),$$

is satisfied, for fixed n and ε, if x is large enough.

3 Blowing Up a Multiple Zero

Theorem 4 indicates that, if you start with a multiple zero x of P, of order k, and if you move P to Q with $[P - Q] \leq \varepsilon$, then x will be moved into y, with $|x - y| \leq C\varepsilon^{1/k}$. But when is such an estimate obtained? Are there cases where a better one holds? The answer is: if the multiple zero stays multiple, stronger estimates can be obtained; the worst case comes if the multiple zero "blows up" into single ones. We will describe this phenomenon in detail in the case of $P = (z - a)^n$.

– Case 1: Q has itself a multiple zero of order n, $Q = (z - b)^n$. Then the condition $[P - Q] \leq \varepsilon$ implies $|b - a| \leq \varepsilon$.

This is clear, from the formula $[P']_{(n-1)} \leq n[P]_{(n)}$, which itself is obtained by elementary manipulations of the binomial coefficients. Here we indicate by a suffix (n) or $(n - 1)$ which norm is used, so as to avoid any confusion.

– Case 2: all roots of Q are simple (or we have no information on Q). Then (17), with $k = n$, gives for $Q = (z - b_1)\cdots(z - b_n)$:

$$|b_j - a| \leq 2^{1/n}(1 + |a|^2)^{1/2}\varepsilon^{1/n}. \tag{22}$$

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This estimate is best possible in general: if $Q = (z-a)^n - \varepsilon$, then $[P - Q] = \varepsilon$, and $|b_j - a| = \varepsilon^{1/n}$ for all j.

– Case 3: mixed case $Q = (z-b)^k (z-b_1)\cdots(z-b_{n-k})$. Then, first, the estimate $|b-a| \le \varepsilon^{1/n}$ can be improved, and we get

$$|b-a| \le \varepsilon^{1/n-k+1} 2^{1/n-k+1} (1+|a|^2)^{1/2}. \tag{23}$$

Indeed, we consider $P^{(k-1)}$ and $Q^{(k-1)}$ (which both have a and b respectively as zeros) and apply (22).

Then, also, we can obtain an estimate of the same form for $b_1, \ldots, b_{n-k}$, namely

$$|b_j - a| \le C(a,n)\varepsilon^{1/n-k+1}, \quad j = 1, \ldots, n-k. \tag{24}$$

In order to prove (24), we first assume $a = 0$, that is

$$\left[z^n - (z-b)^k (z-b_1)\cdots(z-b_{n-k})\right] \le \varepsilon, \tag{25}$$

and we know by (23) that

$$|b| = 0(\varepsilon^{1/n-k+1}). \tag{26}$$

We write $\varepsilon' = \varepsilon^{1/n-k+1}$. Let's also write

$$z^n - (z-b)^k (z-b_1)\cdots(z-b_{n-k}) = c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$$
$$(z-b_1)\cdots(z-b_{n-k}) = c'_1 z^{n-k} + c'_2 z^{n-k-1} + \cdots + c'_{n-k}.$$

Then:

$$|c_1| = |kb + b_1 + \cdots + b_{n-k}| \le \sqrt{\binom{n}{1}}\,\varepsilon.$$

Also, we have:

$$|c_{j+1}| = \left| \binom{k}{j+1} b^{j+1} + \binom{k}{j} b^j c'_1 + \cdots + \binom{k}{l} b^l c'_{j-l+1} + \cdots + \binom{k}{1} b c'_j + c'_{j+1} \right|$$
$$\le \sqrt{\binom{n}{j+1}}\,\varepsilon.$$

If we assume $|c'_l| = 0(\varepsilon'^l)$, $l = 1, \ldots, j$, we deduce from this formula that $|c'_{j+1}| = 0(\varepsilon'^{j+1})$, and so we have shown by induction that

$$|c'_j| = 0(\varepsilon'^j), \quad j = 1, \ldots, n-k. \tag{27}$$

We need a lemma.

Lemma 6 *Let $R = z^m + a_{m-1}z^{m-1} + \cdots + a_0$ be a polynomial where the coefficients $a_{m-1}, \ldots, a_0$ depend on some parameter α and satisfy*

$$|a_{m-1}| = 0(\alpha), |a_{m-1}| = 0(\alpha^2), \ldots, |a_{m-k}| = 0(\alpha^k), \ldots, a_0 = 0(\alpha^m),$$

when $\alpha \to 0$. Then all zeros of R are $0(\alpha)$, $\alpha \to 0$.

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This lemma is well-known and follows from estimates found for instance in Marden [5]. Let's give a quick proof. We have $|a_{m-k}| \le C\alpha^k$, $k = 1, \ldots, m$. Let z be a zero of R. Then:

$$1 = -\frac{a_{m-1}}{z} \cdots - \frac{a_k}{z^{m-k}} \cdots - \frac{a_0}{z^m},$$

and so

$$1 \le C \sum_1^{\infty} \left(\frac{|\alpha|}{|z|}\right)^k,$$

which implies $|z| \le (1+C)|\alpha|$. So the lemma is proved, and (24) follows from (27).

Let us now consider the general case, $a \ne 0$.

We define $\tau_a P = P(z-a)$. Our estimate will follow from the estimate in the case $a = 0$ and the following.

Lemma 7 *For all P, Q, of degree n,*

$$[\tau_a P - \tau_a Q] \le C(a,n)[P-Q],$$

where

$$C(a,n) = \max_{0 \le l \le n} \left\{ \binom{n}{l} (1+|a|^2)^l \right\}^{1/2}.$$

Proof of Lemma 7 We have

$$\begin{aligned}
[\tau_a P]^2 &= \sum_{k=0}^{n} \frac{1}{\binom{n}{k} k!^2} \left| \sum_{j=0}^{n-k} P^{(k+j)}(0) \frac{a^j}{j!} \right|^2 \\
&\le \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(n-k)!\,|a|^{2j}}{n!\,k!\,j!^2} |P^{(k+j)}(0)|^2 \\
&= \sum_{l=0}^{n} \sum_{j=0}^{l} \frac{(n-l+j)!\,l!\,|a|^{2j}}{(l-j)!\,j!^2\,(n-l)!} \frac{|P^{(l)}(0)|^2}{l!^2 \binom{n}{l}} \\
&\le \left(\max_{0 \le l \le n} \sum_{j=0}^{l} \frac{(n-l+j)!\,l!\,|a|^{2j}}{(l-j)!\,j!^2\,(n-l)!} \right) [P]^2.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{j=0}^{l} \frac{(n-l+j)!\,l!}{(l-j)!\,j!^2\,(n-l)!} |a|^{2j} &= \sum_{j=0}^{l} \binom{n-l+j}{j} \binom{l}{j} |a|^{2j} \\
&\le \binom{n}{l} \sum_{j=0}^{l} \binom{l}{j} |a|^{2j} \\
&= \binom{n}{l} (1+|a|^2)^l,
\end{aligned}$$

and the lemma follows.

Remark We do not think that the above constant $C(a,n)$ is sharp. One might think that $(1+|a|^2)^{n/2}$ is the right constant.

So we see that, starting with $P=(z-a)^n$ and moving it to Q with $[P-Q]\leq\varepsilon$, the estimate $|x-y|\leq\varepsilon^{1/n}$ can always be improved if one of the zeros of Q is multiple. The only case where it is sharp is the case where the multiple zero of P has blown up into n distinct simple zeros for Q.

As we already mentioned in [1], the combination of Bombieri's scalar product and Walsh Contraction Principle provides very efficient tools for the study of quantitative properties of polynomials: the proofs are simpler than the existing ones and the results are sharper. Other results on these lines will be published elsewhere.

References

[1] B. Beauzamy, *Finding the roots of polynomial equations: an algorithm with linear command*. To appear.
[2] ———, *Product of many-variable polynomials: pairs that are maximal in Bombieri's norm*. J. Number Theory (1) **55**(1995), 129–143.
[3] B. Beauzamy, E. Bombieri, P. Enflo, and H. Montgomery, *Products of polynomials in many variables*. J. Number Theory (2) **36**(1990), 219–245.
[4] B. Beauzamy and J. Dégot, *Differential Identities*. Trans. Amer. Math. Soc. (7) **347**(1995), 2607–2619.
[5] M. Marden, *Geometry of polynomials*. Mathematical Surveys, A.M.S., 1966.
[6] A. Ostrowski, *Recherches sur la méthode de Graeffe*. Acta Math. **72**, 1940.
[7] ———, *Solutions of equations and systems of equations (Appendix B)*. Academic Press, New-York, 1960.
[8] Bruce Reznick, *An inequality for products of polynomials*. Proc. Amer. Math. Soc. (4) **117**(1993), 1063–1073.
[9] J. L. Walsh, *On the location of the roots of certain type of polynomials*. Trans. Amer. Math. Soc. **12**(1922) 163–180.

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