ON A POINTWISE CONSTRUCTION OF THE LEMNISCATE

Richard Blum

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The writer of the following lines is aware of the possibility that the property discussed below is not a new discovery. He will, of course, be grateful to any reader who will provide him with bibliographical references.

It is well known that the nine point circle of a triangle is tangent to the 4 contact (i.e. the inscribed and the escribed) circles. Given two tangent circles $\mathcal{N}$ and $\mathcal{T}$ we propose to find the locus of vertices of the triangles (Δ) which admit $\mathcal{N}$ as their nine point circle and $\mathcal{T}$ as one of their contact circles.

Let η be the common tangent of $\mathcal{N}$ and $\mathcal{T}$ where they touch, and A the point of contact. Let ξ be one side of a triangle Δ, intersecting $\mathcal{N}$ in B, B' and η in Ω (see Fig. 1). In the variable oblique cartesian coordinate system $\xi \Omega \eta$ we have $A(0, \alpha)$, $B(\beta, 0)$, $B'(\alpha^2/\beta, 0)$ and $\angle \xi \Omega \eta = \theta$. It can be seen that the parameters α, β, θ satisfy the following conditions:

$$(1) \quad \alpha = r \cot \tfrac{1}{2}\theta$$

$$(2) \quad \alpha^2 - 2\alpha\beta\cos\theta + \beta^2 = 2R\beta\sin\theta$$

where r and R are the radii of $\mathcal{T}$ and $\mathcal{N}$, respectively.

In the triangle Δ, one of the points B, B', say B, is the foot of the altitude on ξ and the other, say B', the mid-point. Let $V_0(\xi_0, \eta_0)$ be the vertex of Δ opposite to ξ. Then V_0 must be located on the perpendicular through B to ξ, which gives

$$(3) \quad \xi_0 + \eta_0 \cos\theta = \beta$$

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The equation of $\mathcal{T}$ is

$$\mathcal{T}(\xi,\eta) \equiv \xi^2 + 2\xi\eta\cos\theta + \eta^2 - 2\alpha\xi - 2\alpha\eta + \alpha^2 = 0$$

The equation of the pair of tangents through V_0 to $\mathcal{T}$ (which are, of course, the other two sides of the triangle Δ), is

$$\mathcal{T}(\xi,\eta)\,\mathcal{T}(\xi_0,\eta_0) - \mathcal{T}^2(\xi,\eta\,|\,\xi_0,\eta_0) = 0,$$

where $\mathcal{T}(\xi,\eta\,|\,\xi_0,\eta_0)$ is the polar form of $\mathcal{T}(\xi,\eta)$. The vertices $V_1(\xi_1, 0)$ and $V_2(\xi_2, 0)$ of Δ which are located on ξ, will be obtained by intersecting the last equation with $\eta = 0$. Since B' is the mid-point of V_1V_2, we have

$$(\xi_1 + \xi_2)/2 = \alpha^2/\beta .$$

When (3) is taken into account, this condition leads to

$$(4) \qquad \eta_0 - \xi_0 = 2\alpha .$$

This straight line is parallel to the line Ω F, where F is the center of $\mathcal{T}$, and passes through the fixed point 0 symmetrically situated to F with respect to A . Equations (3) and (4) provide us, therefore, with a pointwise construction, using compass and straight edge, of the required locus.

We choose now the fixed orthogonal cartesian coordinate system x0y, such that 0x is 0F, and 0y is parallel to η, as indicated in Fig. 1. The transformation formulae leading from $\xi\Omega\eta$ to x0y are

$$(5) \qquad \xi_0 = (x - r)\csc\theta ,$$

$$(6) \qquad \eta_0 = -(x - r)\cot\theta + (y + \alpha).$$

Eliminating $\alpha, \beta, \theta, \xi_0, \eta_0$ from the equations (1) to (6) we obtain

$$(7) \quad (x^2+y^2)^2 - 4(r+R)\,x\,(x^2+y^2) + 4r\left[(r+3R)x^2 + (r-R)y^2\right] = 0.$$

The locus is, therefore, a rational bicircular quartic having a double point at 0. It should be noted that, if we choose always $r > 0$, then $\mathcal{T}$ and $\mathcal{N}$ will be tangent internally if $R > 0$ and externally if $R < 0$.

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If we put in (7) $R = -r$, we obtain

$$(8) \quad (x^2+y^2)^2 - 8r^2(x^2-y^2) = 0$$

i.e. the lemniscate with center 0 and a focus F.

We have therefore the result: The locus of the vertices of the triangles which admit the two tangent circles $\mathcal{N}$ and $\mathcal{T}$ of equal radius as the nine point circle and a contact circle, respectively, is the lemniscate having its center at the center of $\mathcal{N}$ and one of its foci at the center of $\mathcal{T}$.

This and the remarks made in connection with equations (3) and (4) lead to the following pointwise construction, by compass and straight edge, of the lemniscate (see Fig. 2):

Let 0 be the center and F one focus of the lemniscate to be constructed. Draw two circles $\mathcal{N}$ and $\mathcal{T}$ of equal radius and tangent externally having their centers at 0 and F, respectively. Let η be their common tangent and ξ a (variable) tangent to $\mathcal{T}$, intersecting η in Ω and $\mathcal{N}$ in B. The straight line through 0 parallel to Ω F intersects the perpendicular through b to ξ in a point V of the lemniscate.

University of Saskatchewan

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Fig. 1.

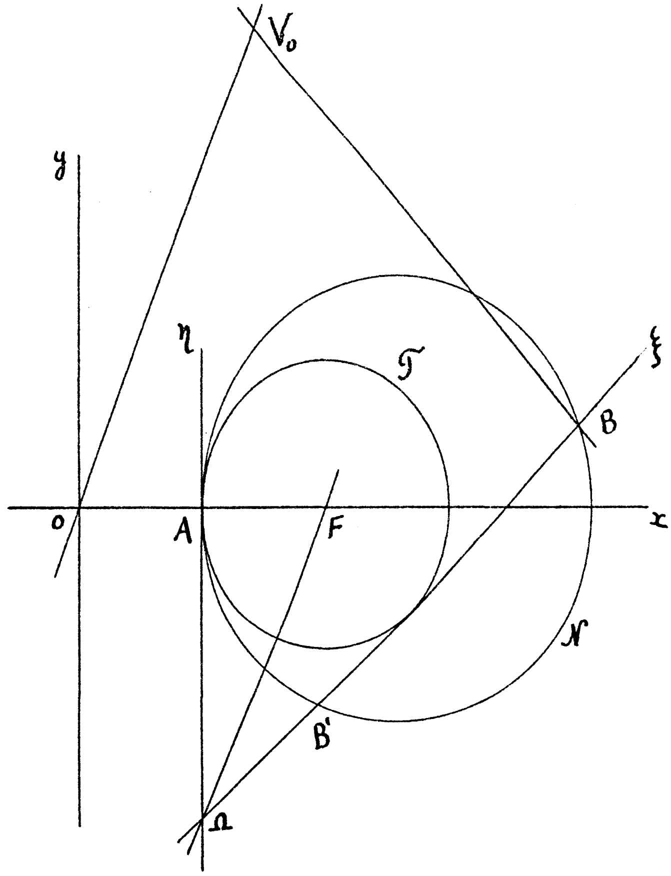
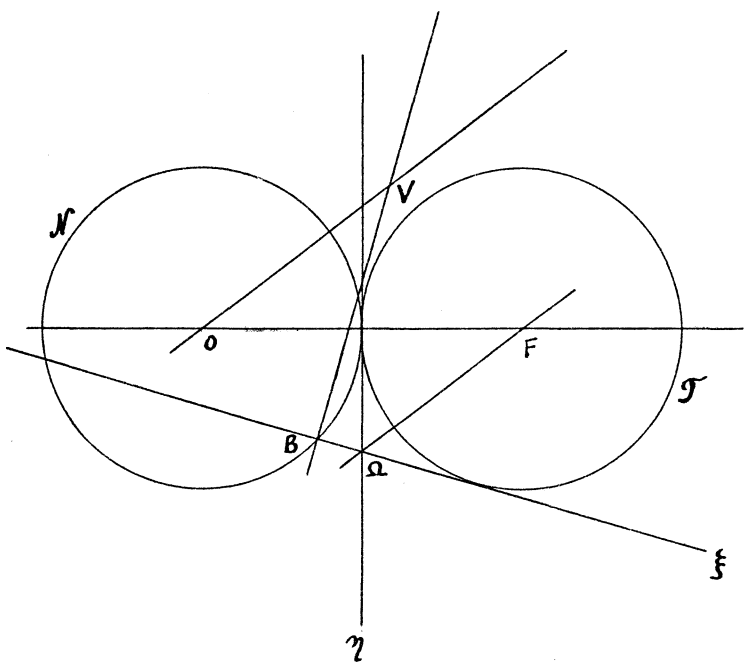


Fig. 2



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