1

Basic notions

In this chapter we shall deal with semigroups $\{V_t,\ t \in \mathbb{R}^+ = [0, +\infty)\}$ of continuous operators $V_t : X \to X$ acting on a complete metric space X. We shall denote them $\{V_t,\ t \in \mathbb{R}^+,\ X\}$ or simply $\{V_t\}$.

In what follows, the term *semigroup* refers to any family of continuous operators $V_t : X \to X$ depending on a parameter $t \in \mathbb{R}^+$ and enjoying the semigroup property: $V_{t_1}(V_{t_2}(x)) = V_{t_1+t_2}(x)$ for all $t_1, t_2 \in \mathbb{R}^+$ and $x \in X$.

A semigroup $\{V_t\}$ is called *pointwise continuous* if the mapping $t \to V_t(x)$ from $\mathbb{R}^+$ to X is continuous for each $x \in X$. A semigroup is called *continuous* if the mapping $(t, x) \to V_t(x)$ from $\mathbb{R}^+ \times X$ to X is continuous.

Given a semigroup $\{V_t\}$ the following notation will be frequently used:

$$\begin{aligned}
\gamma^+(x) &:= \{y \in X \mid y = V_t(x),\ t \in \mathbb{R}^+\} \equiv \{V_t(x),\ t \in \mathbb{R}^+\};\\
\gamma^+_{[t_1,\, t_2]}(x) &:= \{V_t(x),\ t \in [t_1, t_2]\};\\
\gamma^+_t(x) &:= \gamma^+_{[t,\, \infty)}(x) \equiv \{V_\tau(x),\ \tau \in [t, \infty)\};\\
\gamma^+(A) &:= \bigcup_{x \in A} \gamma^+(x);\\
\gamma^+_{[t_1,\, t_2]}(A) &:= \bigcup_{x \in A} \gamma^+_{[t_1,\, t_2]}(x);\\
\gamma^+_t(A) &:= \bigcup_{x \in A} \gamma^+_t(x).
\end{aligned}$$

It is easy to verify that $V_t(\gamma^+(A)) = \gamma^+_t(A)$.
The curve $\gamma^+(x)$ is called the positive semi-trajectory of x.
The collection of all bounded subsets of X is denoted by $\mathcal{B}$.

We use the letter B (with or without indices) to denote the elements of $\mathcal{B}$, i.e. the bounded subsets of X.

https://doi.org/10.1017/9781009229814.004 Published online by Cambridge University Press

A semigroup $\{V_t\}$ is called *locally bounded* if $\gamma^+_{[0,\,t]}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$ and all $t \in \mathbb{R}^+$. $\{V_t\}$ is a *bounded semigroup* if $\gamma^+(B) \in \mathcal{B}$ for each $B \in \mathcal{B}$.

Let A and M be subsets of X. We say that A *attracts* M or M *is attracted* to A by semigroup $\{V_t\}$ if for every $\epsilon > 0$ there exists a $t_1(\epsilon, M) \in \mathbb{R}^+$ such that $V_t(M) \subset \mathcal{O}_\epsilon(A)$ for all $t \geq t_1(\epsilon, M)$. Here $\mathcal{O}_\epsilon(A)$ is the ϵ-neighbourhood of A (i.e. the union of all open balls of radii ϵ centered at the points of A). We say that the set $A \subset X$ *attracts the point* $x \in X$ if A attracts the one-point set $\{x\}$.

If A attracts each point x of X then A is called a *global attractor* (for the semigroup). A is called a *global B-attractor* if A attracts each bounded set $B \in \mathcal{B}$.

A semigroup is called *pointwise dissipative* (respectively, *B-dissipative*) if it has a bounded global attractor (respectively a bounded global B-attractor).

Our main purpose here is to find those semigroups for which there is a *minimal closed global B-attractor* and investigate properties of such attractors. These attractors will be denoted by $\mathcal{M}$. We shall examine also the existence of a *minimal closed global attractor* $\widehat{\mathcal{M}}$. It is clear that $\widehat{\mathcal{M}} \subset \mathcal{M}$ and later on we will also verify that $\widehat{\mathcal{M}}$ might be just a small part of $\mathcal{M}$.

The concept of invariant sets is closely related to these subjects. We call a set $A \subset X$ *invariant* (relative to semigroup $\{V_t\}$) if $V_t(A) = A$ for all $t \in \mathbb{R}^+$.

A set $A \subset X$ is called *absorbing* if for every $x \in X$ there exists a $t_1(x) \in \mathbb{R}^+$ such that $V_t(x) \in A$ for all $t \geq t_1(x)$. A set A is called *B-absorbing* if for every $B \in \mathcal{B}$ there exists a $t_1(B) \in \mathbb{R}^+$ such that $V_t(B) \subset A$ for all $t \geq t_1(B)$.

In our investigation of the problems concerning the attractors $\mathcal{M}$ and $\widehat{\mathcal{M}}$ the concept of ω-limit sets will play a fundamental role. For $x \in X$ the *ω-limit set* $\omega(x)$ is, by definition, the set of all $y \in X$ such that $y = \lim_{k\to\infty} V_{t_k}(x)$ for a sequence $t_k \nearrow +\infty$.

The *ω-limit set* $\omega(A)$ for a set $A \subset X$ is the set of the limits of all converging sequences of the form $V_{t_k}(x_k)$, where $x_K \in A$ and $t_k \nearrow +\infty$.

An equivalent description of the ω-limit sets is given by

Lemma 1.1

$$\omega(x) = \bigcap_{t\geq 0}[\gamma_t^+(x)]_X; \quad \omega(A) = \bigcap_{t\geq 0}[\gamma_t^+(A)]_X. \tag{1.1}$$

Here the symbol $[\ ]_X$ means the closure in the topology of the metric space X.

The proof of Lemma 1.1 is traditional and so is omitted. Since, $\gamma_{t_2}^+(A) \subset \gamma_{t_1}^+(A)$ whenever $t_2 > t_1$, the intersection over all $t \in \mathbb{R}^+$ in (1.1) may be replaced by $\bigcap_{t\geq T}$ with any $T \in \mathbb{R}^+$.

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It is necessary to have in mind that for locally non-compact spaces X the use of the concept of limit sets requires some caution since the intersection $A_0 = \bigcap_{k=1}^{\infty} A_k$ of $A_k = [A_k]_X \supset A_{k+1} = [A_{k+1}]_X$ in them may be empty (and therefore unhelpful).