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# A Real Holomorphy Ring without the Schmüdgen Property

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*Abstract.* A preordering $T$ is constructed in the polynomial ring $A = \mathbb{R}[t_1, t_2, \dots]$ (countably many variables) with the following two properties: (1) For each $f \in A$ there exists an integer $N$ such that $-N \le f(P) \le N$ holds for all $P \in \mathrm{Sper}_T(A)$. (2) For all $f \in A$, if $N + f, N - f \in T$ for some integer $N$, then $f \in \mathbb{R}$. This is in sharp contrast with the Schmüdgen-Wörmann result that for any preordering $T$ in a finitely generated $\mathbb{R}$-algebra $A$, if property (1) holds, then for any $f \in A$, $f > 0$ on $\mathrm{Sper}_T(A) \Rightarrow f \in T$. Also, adjoining to $A$ the square roots of the generators of $T$ yields a larger ring $C$ with these same two properties but with $\Sigma C^2$ (the set of sums of squares) as the preordering.

## 1 Introduction

For any finite subset $S = \{f_1, \dots, f_m\}$ of the polynomial ring $\mathbb{R}[t_1, \dots, t_n]$, let $K_S = \{a \in \mathbb{R}^n \mid f_i(a) \ge 0, i = 1, \dots, m\}$ and let $T_S$ denote the preordering of $\mathbb{R}[t_1, \dots, t_n]$ generated by $S$, *i.e.*, the set of all finite sums of terms of the form $f_1^{e_1} \dots f_m^{e_m} g^2$, $g \in \mathbb{R}[t_1, \dots, t_n]$, $e_1, \dots, e_m \in \{0, 1\}$. We have the following result:

***Theorem 1*** **(Schmüdgen [7, Cor. 3])** *Let $S$ be a finite subset of $\mathbb{R}[t_1, \dots, t_n]$ with $K_S$ compact. Then, for any $f \in \mathbb{R}[t_1, \dots, t_n]$, if $f > 0$ on $K_S$ then $f \in T_S$.*

***Notes***

(1) According to the Positivstellensatz (see the version in [5] for example), if $f > 0$ on $K_S$ then $(1 + s)f = 1 + t$ holds for some $s, t \in T_S$. The conclusion in Theorem 1 is stronger. (If $f > 0$ on $K_S$ then, by compactness of $K_S$, $f > \frac{1}{n}$ on $K_S$ for some integer $n \ge 1$, so, by Theorem 1, $nf = 1 + n(f - \frac{1}{n}) \in 1 + T$.) The hypothesis of Theorem 1 is also stronger. In the Positivstellensatz, $K_S$ is not required to be compact.

(2) As one might expect, the Positivstellensatz is a major ingredient in the proof of Theorem 1.

(3) See [7] for the connection of Theorem 1 to the $K$-moment problem for Borel measures on compact semi-algebraic sets.

We introduce some terminology and notation. Let $A$ be any commutative ring with 1. If $T \subseteq A$ is any preordering, *i.e.*, any additively and multiplicatively closed subset of $A$ containing the squares, $\mathrm{Sper}_T(A)$ denotes the subspace of the real spectrum $\mathrm{Sper}(A)$ [3], [5] consisting of all orderings $P$ of $A$ such that $P \supseteq T$. We will say $f \in A$ is $T$-*bounded* if there exists an integer $N \ge 0$ such that $-N \le f(P) \le N$ holds for all $P \in \mathrm{Sper}_T(A)$. The elements of $A$ which are $T$-bounded form a subring of $A$ which we will denote by $\mathrm{B}_T(A)$.

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In the case where $T = \Sigma A^2$, the set of sums of squares, $\mathrm{B}_T(A)$ is what is called the real holomorphy ring of $A$; see [1]. We will say $f \in A$ is *strongly $T$-bounded* if there exists an integer $N \geq 0$ such that $N - f, N + f \in T$. The elements of $A$ which are strongly $T$-bounded also form a subring of $A$ which we denote by $\mathrm{SB}_T(A)$. If $T = \Sigma A^2$, we denote $\mathrm{B}_T(A)$ (resp., $\mathrm{SB}_T(A)$) simply by $\mathrm{B}(A)$ (resp., $\mathrm{SB}(A)$). Clearly $\mathrm{SB}_T(A) \subseteq \mathrm{B}_T(A)$.

***Theorem 2*** *Suppose $\mathbb{Q} \subseteq A$ and $B_T(A) = A$. Then the following are equivalent:*

*(1)* $\mathrm{SB}_T(A) = A$.
*(2)* *For any $f \in A$, if $f > 0$ on $\mathrm{Sper}_T(A)$, then $f \in T$.*

**Proof** The fact that (1) implies (2) follows from the Kadison-Dubois Theorem; see [2], [8]. The other implication is obvious. ■

In view of Theorem 1, it is natural to consider rings $A$ with the property that (1) and (2) hold for every preordering $T$ of $A$ such that $\mathrm{B}_T(A) = A$. We will refer to this property of rings as the *Schmüdgen property*. In [8] Wörmann proves that every finitely generated $\mathbb{R}$-algebra has the Schmüdgen property. Applying Wörmann's result to $A = \mathbb{R}[t_1, \dots, t_n]$, $T = T_S$ (using the well-known correspondence between semi-algebraic sets and constructible sets [3], [5]) yields another proof of Theorem 1. In [6], Monnier shows that the Schmüdgen property holds for certain more general types of $\mathbb{R}$-algebra of finite transcendence degree.

The object of the present paper is to show that, without some special assumption on $A$, the Schmüdgen property can fail badly. We do this by producing an $\mathbb{R}$-algebra $A$ of infinite transcendence degree and a preordering $T$ in $A$ such that $\mathrm{B}_T(A) = A$, but $\mathrm{SB}_T(A) = \mathbb{R}$; see Proposition 1 below. Also, replacing $A$ by a suitable extension obtained by adjoining square roots of the generators of $T$, we can even assume $T = \Sigma A^2$ if we want; see Proposition 2 below.

***Notes***

(1) $\mathrm{Sper}_T(A) = \mathrm{Sper}_{\tilde{T}}(A)$ and $\mathrm{B}_T(A) = \mathrm{B}_{\tilde{T}}(A) = \mathrm{SB}_{\tilde{T}}(A)$ where $\tilde{T} = \cap\{P \mid P \in \mathrm{Sper}_T(A)\} = \{f \in A \mid f \geq 0 \text{ on } \mathrm{Sper}_T(A)\}$. Thus our question is related to the question of how $T$ "sits" inside the bigger preordering $\tilde{T}$.
(2) By the Positivstellensatz (*e.g.*, see [5]), $f > 0$ on $\mathrm{Sper}_T(A)$ iff there exist $s, t \in T$ such that $f(1 + s) = 1 + t$. This is valid for any $A$ and any preordering $T \subseteq A$. Also, going to the localization $A \to A[1/f]$, $f \geq 0$ on $\mathrm{Sper}_T(A)$ iff $f(f^{2k} + s) = f^{2k} + t$ for some $s, t \in T$ and some integer $k \geq 0$.
(3) Suppose that $\mathrm{B}_T(A) = A$ holds and let $t \in T$ be given. Then, using the Positivstellensatz, there exists a sequence of elements $t_i$ in $T$ and integers $N_i \geq 1$, such that $t_1 = t$ and $(N_i - t_i)(1 + t_{i+1}) \in T$, for $i = 1, 2, \dots$. This is clear and, moreover, it is the motivation for our construction.

## 2 The Example

Take $A = \mathbb{R}[t_1, t_2, \dots]$, the polynomial algebra over $\mathbb{R}$ in countably many variables, and let $T$ be the preordering in $A$ generated by the elements $t_i$ and the elements $(1 - t_i)(1 + t_{i+1})$,

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$i \geq 1$. Clearly $T = \bigcup_{n\geq 1} T_n$ where $T_n$ denotes the preordering in $\mathbb{R}[t_1, \dots, t_n]$ generated by $t_1, \dots, t_n$ and the elements $(1 - t_i)(1 + t_{i+1})$, $i = 1, \dots, n - 1$. Also, $T_n + T_n(1 - t_n) \subseteq S_n$, where $S_n$ denotes the preordering in $\mathbb{R}[t_1, \dots, t_n]$ generated by the elements $t_i$, $1 - t_i$, $i = 1, \dots, n$.

***Lemma 1*** $S_n \cap -S_n = \{0\}$.

**Proof** This is well-known, *e.g.*, by [3, Prop. 7.5.6], there exists a support $\{0\}$ ordering on $\mathbb{R}[t_1, \dots, t_n]$ containing $S_n$. Here is an elementary proof. We want to show that any sum of non-zero elements of $S_n$ is non-zero. Suppose $f \in S_n$ is a sum of non-zero terms of the form a square in $\mathbb{R}[t_1, \dots, t_n]$ times some product of the generators $t_1, \dots, t_n, 1 - t_1, \dots, 1 - t_n$ of $S_n$. Expanding as a polynomial in $t_n$ with coefficients in $\mathbb{R}[t_1, \dots, t_{n-1}]$, we see that, in each term, the coefficient of the lowest power of $t_n$ appearing is a square times a product of generators of $S_{n-1}$. Adding, we see that the coefficient of the lowest power of $t_n$ appearing in $f$ is a sum of non-zero elements of $S_{n-1}$ so, by induction on $n$, it is not zero. This implies $f \neq 0$. ■

***Lemma 2*** *Suppose $f \in T_n$, $f \notin T_{n-1}$. Then, as a polynomial in $t_n$ with coefficients in $\mathbb{R}[t_1, \dots, t_{n-1}]$, $f$ has degree $\geq 1$ and the leading coefficient of $f$ is in $S_{n-1}$.*

**Proof** Since $f \in T_n$, $f$ is a sum of non-zero terms of the form a square in $\mathbb{R}[t_1, \dots, t_n]$ times some product of elements from the set $\{t_1, \dots, t_n, (1 - t_1)(1 + t_2), \dots, (1 - t_{n-1})(1 + t_n)\}$. Expanding as a polynomial in $t_n$ with coefficients in $\mathbb{R}[t_1, \dots, t_{n-1}]$, we see that, in each term, the coefficient of the highest power of $t_n$ appearing is a square times a product of generators of $T_{n-1}$ times $(1 - t_{n-1})^\delta$, $\delta = 0$ or 1. Adding and using Lemma 1, $f$ has degree $\geq 1$ and the coefficient of the highest power of $t_n$ appearing is an element of $T_{n-1} + T_{n-1}(1 - t_{n-1}) \subseteq S_{n-1}$. ■

***Proposition 1*** $\mathrm{B}_T(A) = A$, $\mathrm{SB}_T(A) = \mathbb{R}$.

**Proof** If $P \in \mathrm{Sper}_T(A)$ then $t_i \in P$ and $(1 - t_i)(1 + t_{i+1}) \in P$, so $0 \leq t_i(P) \leq 1$. Since the elements $t_i$ generate $A$ as an $\mathbb{R}$-algebra, it follows that $\mathrm{B}_T(A) = A$. Let $f \in \mathrm{SB}_T(A)$, so $N - f^2 \in T$ for some integer $N \geq 0$. If $f \notin \mathbb{R}$, then $N - f^2 \in T_n \backslash T_{n-1}$, $n \geq 1$. By Lemma 2, the leading coefficient of $N - f^2$ is $-g^2$, where $g$ is the leading coefficient of $f$, and $-g^2 \in S_{n-1}$. Since $S_{n-1} \cap -S_{n-1} = \{0\}$, this is impossible. ■

***Note*** Since $\mathrm{B}_T(A) = A$, it follows (for example by applying the Kadison-Dubois Theorem [2] to the preordering $\tilde{T} = \cap\{P \mid P \in \mathrm{Sper}_T(A)\}$) that the maximal elements in $\mathrm{Sper}_T(A)$ all arise from $\mathbb{R}$-algebra homomorphisms $\alpha \colon A \to \mathbb{R}$ such that $\alpha(T) \geq 0$. In this way, $\mathrm{SperMax}_T(A)$ is identified with the Hilbert Cube $[0, 1]^\infty$. If $a = (a_1, a_2, \dots) \in [0, 1]^\infty$, the evaluation mapping $f \mapsto f(a)$ is an $\mathbb{R}$-algebra homomorphism from $A$ to $\mathbb{R}$ with $f(a) \geq 0$ for all $f \in T$. The corresponding element of $\mathrm{SperMax}_T(A)$ is $P_a = \{f \in A \mid f(a) \geq 0\}$. The identification $\mathrm{SperMax}_T(A) \cong [0, 1]^\infty$ is a homeomorphism of topological spaces.

Formally adjoining to $A$ the square roots of the generators of $T$, we get the big ring

$$C = A\big[\sqrt{t_i}, \sqrt{(1-t_i)(1+t_{i+1})} \mid i \geq 1\big].$$

The ring $C$ with the preordering $\Sigma C^2$ has the same two properties as the ring $A$ with the preordering $T$. That is:

***Proposition 2*** $\mathrm{B}(C) = C$, $\mathrm{SB}(C) = \mathbb{R}$.

**Proof** Let $u_i = (1-t_i)(1+t_{i+1})$. Since the inequalities $-1 \leq \sqrt{t_i} \leq 1$ and $-\sqrt{2} \leq \sqrt{u_i} \leq \sqrt{2}$ hold on $\mathrm{Sper}(C)$ and since the elements $\sqrt{t_i}$, $\sqrt{u_i}$ generate $C$ as an $\mathbb{R}$-algebra, it follows that $\mathrm{B}(C) = C$. Formally adjoining square roots of finitely many of the elements $t_i$ and then of finitely many of the elements $u_i$ we obtain a finite tower of subrings $A = D_0 \subseteq \cdots \subseteq D_s$ of $C$ where, at each stage, $D_k = D_{k-1}[\sqrt{p_k}]$. Any finite subset of $C$ belongs to such a tower. One verifies that $C$ is an integral domain by verifying $D_s$ is an integral domain, by induction on $s$, for any such tower. This just amounts to checking, in each case ($p_s$ is equal to $t_i$ or $u_i$ for some $i$), that the polynomial $X^2 - p_s$ is irreducible over the field of fractions of $D_{s-1}$. Let $T_k$ be the preordering in $D_k$ generated by the elements $t_i$, $u_i$. To show $\mathrm{SB}(C) = \mathbb{R}$, it suffices to prove, by induction on $s$, that if $f \in D_s$ and $N - f^2 \in T_s$ for some integer $N \geq 1$, then $f \in \mathbb{R}$. By assumption, $N - f^2 = g_1 + \cdots + g_r$ where each $g_i$ is a square in $D_s$ times a product of generators of $T$. Thus $f = f_1 + f_2\sqrt{p_s}$, $g_i = g_{i1} + g_{i2}\sqrt{p_s}$, with $f_j, g_{ij} \in D_{s-1}$, $j = 1, 2$ and $N - (f_1^2 + f_2^2 p_s) = g_{11} + \cdots + g_{r1}$. Also, the elements $g_{i1}$ belong to $T_{s-1}$, so $N - f_1^2 \in T_{s-1}$ and $N^2 - (f_2^2 p_s)^2 = (N - f_2^2 p_s)(N + f_2^2 p_s) \in T_{s-1}T_{s-1} \subseteq T_{s-1}$. Thus, by induction on $s$, $f_1, f_2^2 p_s \in \mathbb{R}$. Since $f_2^2 p_s$ has a square root in $C$, we must have $f_2^2 p_s \geq 0$, and since $X^2 - p_s$ is irreducible over the field of fractions of $D_{s-1}$, this implies $f_2 = 0$. Thus $f = f_1 \in \mathbb{R}$. ∎

***Note*** The functorial mapping $\mathrm{Sper}(i)\colon \mathrm{Sper}(C) \to \mathrm{Sper}(A)$, where $i\colon A \to C$ is the inclusion, is continuous with image $\mathrm{Sper}_T(A)$ and, by [4, Th. 6.2], the mapping $\mathrm{Sper}(i)$ is closed. $\mathrm{SperMax}(C)$ is homeomorphic to the infinite torus $(\mathbb{S}^1)^\infty$ where $\mathbb{S}^1$ is the 1-sphere. If $(b_1, c_1, b_2, c_2, \ldots) \in (\mathbb{S}^1)^\infty$ (so $b_i^2 + c_i^2 = 1$) the associated $\mathbb{R}$-algebra homomorphism from $C$ to $\mathbb{R}$ is given by $\sqrt{t_i} \mapsto b_i$, $\sqrt{u_i} \mapsto c_i\sqrt{1 + b_{i+1}^2}$. The surjection $(\mathbb{S}^1)^\infty \to [0,1]^\infty$ corresponding to the mapping $\mathrm{SperMax}(C) \to \mathrm{SperMax}_T(A)$ induced by $\mathrm{Sper}(i)$ is given by $(b_1, c_1, b_2, c_2, \ldots) \mapsto (b_1^2, b_2^2, \ldots)$.

## References


[1] E. Becker and V. Powers, *Sums of powers in rings and the real holomorphy ring.* J. Reine Angew. Math. **480**(1996), 71–103.

[2] E. Becker and N. Schwartz, *Zum Darstellungssatz von Kadison-Dubois.* Arch. Math. **39**(1983), 421–428.

[3] J. Bochnak, M. Coste and M.-F. Roy, *Géométrie Algébrique Réelle.* Ergeb. Math. Grenzgeb., Springer, Berlin-Heidelberg-New York, 1987.

[4] M. Coste and M.-F. Roy, *La topologie du spectre réel.* In: Ordered fields and real algebraic geometry, Contemp. Math. **8**, Amer. Math. Soc., 1981, 27–59.

[5] T.-Y. Lam, *An introduction to real algebra.* Rocky Mtn. J. Math. **14**(1984), 767–814.

[6] J.-P. Monnier, *Schmüdgen Positivstellensatz.* Manuscripta Math., to appear


https://doi.org/10.4153/CMB-1999-041-2 Published online by Cambridge University Press

[7] K. Schmüdgen, *The $K$-moment problem for compact semi-algebraic sets.* Math. Ann. **289**(1991), 203–206.
[8] T. Wörmann, *Strikt positive Polynome in der semialgebraischen Geometrie.* PhD Thesis, Dortmund, 1998.

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