BULL. AUSTRAL. MATH. SOC. 52A10
VOL. 57 (1998) [289–294]

AN ISODIAMETRIC PROBLEM WITH LATTICE-POINT CONSTRAINTS

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The isodiametric problem in the Euclidean plane is solved for bounded convex sets, which are symmetric about the origin, and which contain no interior non-zero point of an arbitrary lattice L.

1. INTRODUCTION

Let L denote an arbitrary lattice in the Euclidean plane E^2 with $\det L \neq 0$. Let $\mathcal{K}_A$ denote the family of bounded convex sets K symmetric with respect to the origin O which contain no interior non-zero points of L, and which have area A. Let $A(K)$ and $D(K)$ respectively denote the area and the diameter of K.

A classical theorem of Minkowski states that if $K \in \mathcal{K}_A$, then the area $A = A(K) \leqslant 4 \det L$ [**2**].

In 1979, Scott and Arkinstall [**3**] solved a lattice-constrained isoperimetric problem in the Euclidean plane: for each allowable value of A find the set in $\mathcal{K}_A$ of minimal perimeter, in the case where L is the integer lattice $\mathbf{Z}^2$. In this paper, we solve the corresponding isodiametric problem: for each allowable value of A find the set in $\mathcal{K}_A$ of minimal diameter, but allowing L to be an arbitrary lattice.

Let H be the hexagon, centrally symmetric with respect to O and having no non-zero lattice-points in its interior, which is constructed as follows.

Let $G_1 \in L$ be a nearest lattice-point from the origin, and $G_1^S \in L$ its symmetral with respect to O. Let r_1 and r_1^S be the straight lines through G_1 and G_1^S respectively, which are orthogonal to $\overrightarrow{G_1^S G_1}$, and let $S(r_1)$ denote the slab bounded by those lines (see Figure 1).

Now, let $G_2 \in L$ be a nearest lattice-point from the origin contained in the interior of $S(r_1)$, and $G_2^S \in L$ its symmetral with respect to O, and r_2, r_2^S the lines through G_2 and G_2^S respectively which are orthogonal to $\overrightarrow{G_2^S G_2}$.

The parallelogram P bounded by the lines r_1, r_1^S, r_2, r_2^S may (or may not) contain two lattice-points in its interior. If G_3, G_3^S are lattice points interior to P, then let r_3 and r_3^S denote the lines through G_3 and G_3^S respectively which are orthogonal to $\overrightarrow{G_3^S G_3}$.

Received 27th August, 1997

https://doi.org/10.1017/S000497270003166X Published online by Cambridge University Press

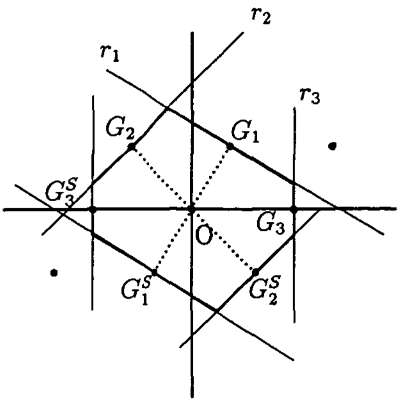

Figure 1: Hexagon H

The lines r_i, r_i^S $(i = 1, 2, 3)$ determine the above hexagon H, which may possibly degenerate to a parallelogram.

We next define a special set K^*. Let C_r denote the circular disk centred at the origin and having radius r, and let $K^* = H \cap C_r$. The set K^* is bounded, convex, and contains the origin O but no interior non-zero points of L. We shall show shortly (Lemma 1, Corollary 1) that $A(H) = 4 \det L$. It is clear that while K^* is a proper subset of H, the area $A(K^*)$ increases monotonically as r increases. It will follow that for each A, $0 < A \leqslant 4 \det L$, there is a unique value of r such that $K^* \in \mathcal{K}_A$.

We now state the main result of this paper:

Theorem 1. *For each $K \in \mathcal{K}_A$, there exists $K^* \in \mathcal{K}_A$ with $D(K^*) \leqslant D(K)$. Equality holds here when and only when $K = K^*$.*

The proof of the theorem will be established by the following lemmas:

Lemma 1. *The vertices of the hexagon H lie on a circle centred at the origin.*

Corollary 1. *The area of the hexagon H is $A(H) = 4 \det L$. So, H is an optimal set for Minkowski's Theorem.*

Lemma 2. *If $K, K^* \in \mathcal{K}_A$ then $D(K^*) \leqslant D(K)$.*

Lemma 3. *If $K, K^* \in \mathcal{K}_A$, then $D(K) = D(K^*)$ if and only if $K = K^*$.*

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2. Proofs of the Lemmas

Proof of Lemma 1:

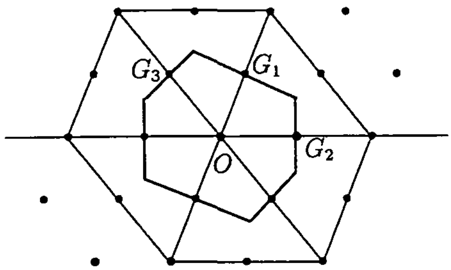

Figure 2.

The six points $\pm 2\overrightarrow{OG_i}$ $(1 \leqslant i \leqslant 3)$ form the vertices of a lattice hexagon X. This hexagon is partitioned into six congruent triangles by the three diagonals through the origin. By construction, each vertex V_i of H is the circumcentre of one of these triangles. Since the triangles are congruent, it follows that the six vertices are equidistant from O, and the result follows. ▯

From Lemma 1, $D(H) = 2d(V_i, O)$ for all $i = 1, \dots, 6$; thus, all the vertices of the hexagon H lie on the circle $x^2 + y^2 = [D(H)/2]^2$. We observe in passing that each of the points G_i, G_i^S $(1 \leqslant i \leqslant 3)$ is the midpoint of the edge of the hexagon passing through it.

Corollary 1 follows by noting that the three diagonals of hexagon X partition H into six quadrilaterals which are easily rearranged to fit into a fundamental parallelogram of lattice $2L$.

Proof of Lemma 2: From Minkowski's Theorem and Corollary 1, $A(K) \leqslant 4 \det L = A(H)$. Since for any such value of A there is a set $K^* \in \mathcal{K}_A$ with $D(K^*) \leqslant D(H)$, we may asume that K lies within the disk $C_{D(H)/2} : x^2 + y^2 \leqslant [D(H)/2]^2$.

If K has any point in common with the circle $C_{D(H)/2}$, then by the symmetry of K, $D(K) = D(H)$. As $A(K) \leqslant A(H)$, we can clearly take a new convex set $K^* \in \mathcal{K}_A$, with $D(K^*) \leqslant D(K)$.

Hence, we may assume that K lies strictly in the interior of $C_{D(H)/2}$.

Let $r < D(H)/2$ be the smallest positive real number such that $C_r \supset K$.

As $G_i, G_i^S (i = 1, 2, 3)$ are not interior to K, there exist two parallel lines g_1, g_1^S passing through G_1 and G_1^S respectively, which bound a slab $S(g_1)$ containing K.

Analogously, we can take parallel straight lines g_2, g_2^S through G_2 and G_2^S, and g_3, g_3^S through G_3 and G_3^S respectively, defining slabs $S(g_2)$, $S(g_3)$ which contain K. We notice that:

(i) If $r \leqslant d(G_1, O)$, then $C_r \subset H$, and hence, the classic isodiametric inequality gives the disk as the optimal set for the inequality $A \leqslant \pi D^2/4$.

(ii) If $d(G_1, O) < r \leqslant d(G_2, O)$, we only take parallel lines through G_1 and G_1^S; in this case points G_2, G_3, G_2^S, G_3^S are not interior to C_r.

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(iii) Finally, if $d(G_2, O) < r \leqslant d(G_3, O)$, we only take lines through G_1, G_1^S, G_2, G_2^S; in this case points G_3, G_3^S are not interior to C_r.

To allow for these cases, and to ensure that three slabs are always defined, for $i = 1, \dots, 3$ we define l_i to be the edge of the hexagon H through point G_i. If G_j is not interior to C_r, then we take $S(g_j) = S(l_j)$ – the slab determined by the line along l_j and its symmetral in O. In this way $\bigcap_{i=1}^{3} S(g_i)$ is well defined and never equal to the empty set.

Then

$$K \subset C_r \cap \left(\bigcap_{i=1}^{3} S(g_i) \right)$$

(see Figure 3).

Let γ denote the circle which is the reflection of C_r in the line containing the edge l_1 of H (see Figure 3). Then γ and C_r meet on l_1 at points E and F say. As $OG_1 \perp EF$, G_1 is the midpoint of segment EF.

Henceforth, we shall use the shorthand notation $\left\{ A_1 \frown A_2 A_3 \dots A_n \right\}$ to denote the set determined by a circular arc with endpoints A_1, A_2, and the line segments $A_2A_3, \dots, A_{n-1}A_n, A_nA_1$.

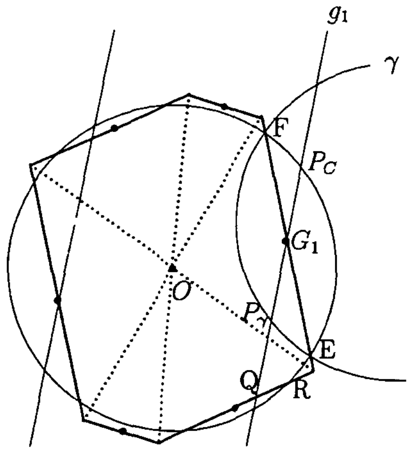

Figure 3.

Using the notation of Figure 3, the set $\left\{ P_\gamma \frown E G_1 \right\}$ is the symmetric reflection of $\left\{ P_C \frown F G_1 \right\}$ with respect to G_1 (because G_1 is the midpoint of the segment determined by $l_1 \cap C_r$).

On the other hand, it is clear that $\left\{ P_\gamma \frown E G_1 \right\} \subset \left\{ R \frown E G_1 Q \right\}$. Hence, we have

$$A\left(\left\{ P_C \frown F G_1 \right\}\right) = A\left(\left\{ P_\gamma \frown E G_1 \right\}\right) \leqslant A\left(\left\{ R \frown E G_1 Q \right\}\right)$$

and equality holds if and only if line g_1 contains the edge l_1 of H, in which case the above areas are equal to zero.

Carrying this out for each G_i (and G_i^S) for which this is possible (and taking into account the previous notes (i), (ii) and (iii)), we obtain the inequality

$$A\left(C_r \cap \left(\bigcap_{i=1}^{3} S(g_i)\right)\right) \leqslant A\,(H \cap C_r)\,.$$

Equality is attained here when and only when $g_i \supset l_i$ for each i, that is, when $\left(\bigcap_{i=1}^{3} S(g_i)\right) = H$.

As $K \subset C_r \cap \left(\bigcap_{i=1}^{3} S(g_i)\right)$, then

$$A(K) \leqslant A\,(H \cap C_r)\,.$$

Thus, there is an $r^* \leqslant r$ and $K^* = H \cap C_{r^*}$ for which $K^* \in \mathcal{K}_A$ and $D(K^*) \leqslant D(K)$.

This completes the proof of Lemma 2. □

PROOF OF LEMMA 3: By Lemma 2, we may suppose that $K \subset H$.

We have $K,\ K^* \in \mathcal{K}_A$. Let r_* be the radius of the disk C_{r_*} defining $K^* = H \cap C_{r_*}$. If $K \neq K^*$, then K intersects at least two arcs of the circular boundary of disk C_{r_*}.

Since K is centrally symmetric, $D(K)$ is the distance from some point which is beyond the arc of the circle C_{r_*} (the farthest point from origin), to its symmetral with respect to O. Then, $D(K) > D(K^*) = 2r_*$. Thus, the set K^* is the domain with minimum diameter when the area is fixed in the range $(0, 4 \det L]$.

This completes the proof of Lemma 3, and establishes the Theorem. □

3. Final Comments

If we define $f_A(r) := A\,(H \cap C_r)$, then as mentioned previously, $f_A(r)$ is an increasing function of r. It follows that

$$A(K) \leqslant f_A\left(\frac{D(K)}{2}\right),$$

and equality is attained precisely when $K = K^*$. This means that for each value of $A(K)$ we can theoretically obtain an inequality giving an upper bound for $A(K)$ in terms of $D(K)$. However, in practice such inequalities are rather unpleasant. For example, in the case of the integer lattice we have:

$$\text{If } \pi \leqslant A \leqslant 4, \text{ then } A \leqslant \pi\frac{D^2}{4} - D^2 \arccos\frac{2}{D} + 2\sqrt{D^2-4}.$$

As the domain K is centrally symmetric with respect to O, the circumradius $R(K)$ of K satisfies $R(K) = D(K)/2$. So, we can state the following corollary:

COROLLARY 2. *Let K be a bounded convex set in the Euclidean plane E^2, which is symmetric about the origin O, but contains no non-zero point of an arbitrary given lattice L in its interior. Let K have area $A(K)$ and circumradius $R(K)$. Then,*

$$A(K) \leqslant f_A\left(R(K)\right),$$

and equality holds if and only if $K = K^*$, where $K^* \in \mathcal{K}_A$.

The following corollary is a consequence of the Theorem and Corollary 1.

COROLLARY 3. *The hexagon $H/2$ is the tile with least diameter that permits a lattice-tiling of the plane E^2.*

PROOF: It is well known [**1**] that the only convex polygons which admit a lattice-tiling in the plane are the parallelograms and the centrally symmetric hexagons with area equal to $\det L$.

If we consider the domain $H/2$, we have from Corollary 1 that $A\left(H/2\right) = \det L$. Hence, the family $\left\{H/2 + u,\ u \in L\right\}$ forms a lattice-tiling. Using the Theorem, we inmediately obtain the result. ∎

REFERENCES

[1] E.S. Fedorov, 'Elements of the study of figures', *Zap. Mineral. Imper. S. Petersburgskogo Obsc.* **21** (1885), 1–279; *Izdat Akad Nauk SSSR* Moscow (1953).

[2] H. Minkowski, *Geometrie der Zahlen* (Leipzig, Berlin, 1896 and 1910; Chelsea, New York 1953).

[3] P.R. Scott and J. Arkinstall, 'An isoperimetric problem with lattice point constraints', *J. Austral. Math. Soc.* **27** (1979), 27–36.

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