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A NOTE ON REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

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Abstract

We study real hypersurfaces of a complex projection space and show that there are no such hypersurfaces with harmonic curvature on which the structure vector is principal.

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Introduction

Let P^nC be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. Let M be a real hypersurface of P^nC and (P, E, ω, g) be an almost contact metric structure induced from the complex structure of P^nC. Kimura [2] proved recently the following

THEOREM A. *There are no real hypersurfaces with parallel Ricci tensor of P^nC on which E is principal.*

The hypersurface M is said to be with *harmonic curvature,* if the Ricci tensor S satisfies

$$\nabla_X S(Y) = \nabla_Y S(X) \tag{0.1}$$

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for any vector fields X and Y, where ∇ denotes the Riemannian connection of M. The purpose of this note is to prove the following

THEOREM. *There are no real hypersurfaces with harmonic curvature of P^nC on which E is principal.*

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1. Preliminaries

Let M be a real hypersurface of P^nC $(n \geq 2)$. On a neighborhood of each point, by ξ is denoted a local unit normal vector field of M in P^nC. As is well known, M admits an almost contact metric structure induced from the complex structure J of P^nC (see Yano and Kon [5]). Namely, for the Riemannian metric g of M induced form the Fubini-Study metric g' of P^nC, we define a tensor field P of type $(1, 1)$, a vector field E and a 1-form ω on M by

$$g(PX, Y) = g'(JX, Y), \qquad g(E, Y) = \omega(Y) = g'(J\xi, Y)$$

for any vector fields X and Y on M. Then we have

$$(1.1) \qquad P^2X = -X + \omega(X)E, \qquad PE = 0, \qquad g(E, E) = 1.$$

Moreover we have

$$(1.2) \qquad g(PX, PY) = g(X, Y) - \omega(X)\omega(Y).$$

By ∇ and ∇' are denoted the Riemannian connections of M and P^nC respectively. The Gauss and Weingarten formulas are given by

$$(1.3) \qquad \nabla'_X Y = \nabla_X Y + g(AX, Y)\xi,$$

and

$$(1.4) \qquad \nabla'_X \xi = -AX,$$

respectively, where A is the shape operator of M in P^nC derived from the unit vector ξ. From (1.3) it follows easily that we have

$$(1.5) \qquad \nabla_X P(Y) = \omega(Y)AX - g(AX, Y)E,$$

and

$$(1.6) \qquad \nabla_X E = PAX.$$

Let R be the Riemannian curvature tensor of M. Since P^nC is of constant holomorphic sectional curvature 4, we have the following Gauss and Codazzi equations

$$(1.7)\quad \begin{aligned} R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY \\ - 2g(PX,Y)PZ + g(AY,Z)AX - g(AX,Z)AY; \end{aligned}$$

$$(1.8)\quad \nabla_X A(Y) - \nabla_Y A(X) = \omega(X)PY - \omega(Y)PX - 2g(PX,Y)E.$$

By (1.1), (1.6), (1.7) and (1.8) we get

$$(1.9)\quad SX = (2n+1)X - 3\omega(X)E + hAX - A^2X,$$

and

$$(1.10)\quad \begin{aligned} \nabla_X S(Y) = & -3\{g(PAX,Y)E + \omega(Y)PAX\} \\ & + dh(X)AY + (h-A)\nabla_X A(Y) - \nabla_X A(AY), \end{aligned}$$

where $h = \operatorname{Tr} A$ and S denotes the Ricci tensor of M.

An eigenvector X of the shape operator A is called a *principal vector* and an eigenvalue λ is called a *principal curvature*. We assume that structure vector E is principal. By α is denoted the principal curvature associated with E, that is, it satisfies $AE = \alpha E$. Then it is seen that α is constant (see [5]) and hence (1.6) implies $\nabla_X A(E) = \alpha PAX - APAX$, from which, together with the Codazzi equation (1.8), it follows that

$$(1.11)\quad \begin{aligned} & 2APA = \alpha(PA + AP) + 2P, \\ & \nabla_X A(E) = \alpha(PA - AP)X/2 - PX, \text{ and} \\ & \nabla_E A(Y) = \alpha(PA - AP)Y/2. \end{aligned}$$

2. Proof of theorem

First of all, we define a tensor field T of type $(0,3)$ by

$$(2.1)\quad T(X,Y,Z) = g(\nabla_X S(Y) - \nabla_Y S(X), Z)$$

for any vector fields X, Y and Z on M. According the hypersurface M has harmonic curvature if and only if the tensor field T vanishes identically. By means of (1.10), we have

$$(2.2)\quad \begin{aligned} T(X,Y,Z) = & \ \omega(X)\{hg(PY,Z) - g((AP - 3PA)Y,Z)\} \\ & - \omega(Y)\{hg(PX,Z) - g((AP - 3PA)X,Z)\} \\ & - \omega(Z)\{2(h-\alpha)g(PX,Z) + 3g((PA + AP)X,Y)\} \\ & + dh(X)g(AY,Z) - dh(Y)g(AX,Z) \\ & + g(AX, \nabla_Y A(Z)) - g(AY, \nabla_X A(Z)). \end{aligned}$$

Assume that M has harmonic curvature. Taking account of the second equation of (1.11) and (2.1) with $Z = E$, we have

$$(2.3)\quad -2g((PA+AP)X,Y) - \alpha g(APAX,Y) + \alpha g((PA^2+A^2P)X,Y)/2 \\ + 2(\alpha - h)g(PX,Y) + \alpha\{dh(X)\omega(Y) - dh(Y)\omega(X)\} = 0.$$

Similarly, putting $X = E$ in (2.2), we obtain

$$(2.4)\quad g((3PA-AP)Y,Z) + (h-\alpha)g(PY,Z) + \alpha^2 g((PA-AP)Y,Z)/2 \\ - \alpha g((PA-AP)AY,Z)/2 + dh(E)g(AY,Z) - \alpha dh(Y)\omega(Z) = 0,$$

and then putting $Z = E$, we have

$$(2.5)\qquad \alpha\{dh(E)\omega - dh\} = 0.$$

Accordingly, from (2.3) and (2.5) it follows that

$$T(X,Y,E) + \omega(Y)T(E,X,E) - \omega(X)T(E,Y,E) \\ = \alpha g((PA^2+A^2P)X,Y)/2 - 2g((PA+AP)X,Y) - \alpha g(APAX,Y) \\ + 2(\alpha - h)g(PX,Y) = 0.$$

Therefore (2.4) and the above equation mean that if M has harmonic curvature, then we have

$$(2.6)\quad 3PA - AP + (h-\alpha)P + \alpha(PA-AP)(\alpha - A)/2 + \beta A - \alpha\,\mathrm{grad}\,h \otimes \omega = 0,$$

and

$$(2.7)\qquad \alpha(PA^2+A^2P)/2 - 2(PA+AP) - \alpha APA + 2(\alpha - h)P = 0,$$

where $\beta = dh(E)$.

We prove here that the principal curvature α is a non-zero constant. Suppose that $\alpha = 0$. Then (2.6) and (2.7) are reduced to $3PA - AP + hP + \beta A = 0$, $PA + AP + hP = 0$, and hence we have $4PA + 2hP + \beta A = 0$. let X be a principal vector with principal curvature λ which is orthogonal to E. Then, by means of the above equation, we have $(4\lambda + 2h)PX + \beta\lambda X = 0$, which implies that $4\lambda + 2h = 0$ and $\beta\lambda = 0$, because X and PX are mutually orthogonal. This yields that the trace of A satisfies $h = \alpha + (2n-2)\lambda = -(n-1)h$, which means that $\lambda = h = 0$, and hence M is totally geodesic, a contradiction.

Next, the fact that h is constant is proved. Since α is non-zero constant, (2.5) yields $\mathrm{grad}\,h = \beta E$ or $dh = \beta\omega$, from which we have $d\beta(X)\omega(Y) - d\beta(Y)\omega(X) = -\beta g((PA+AP)X,Y)$, because of the fact that

$$g(\nabla_X \mathrm{grad}\,h, Y) = g(\nabla_Y \mathrm{grad}\,h, X).$$

Suppose that there exist points x at which $\beta(x) \neq 0$. Putting $Y = E$ in the above equation we have $d\beta = d\beta(E)\omega$ and hence this implies that

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$\beta(PA + AP) = 0$, which contradicts the first equation of (1.11). Thus β vanishes identically and by (2.5), h must be constant.

For a principal vector X with principal curvature λ which is orthogonal to E, $Y = PX$ is also a principal vector with principal curvature $\mu = (\alpha\lambda + 2)/(2\lambda - \alpha)$, by the first equation of (1.11). Hence (2.6) gives rise to

$$3\lambda - \mu + h - \alpha + \alpha(\lambda - \mu)(\alpha - \lambda)/2 = 0, \tag{2.8}$$

because h is constant. Accordingly the principal curvature λ is the root of the following cubic equation with constant coefficients

$$\alpha x^3 - 2(\alpha^2 + 3)x^2 + (\alpha^3 + 5\alpha - 2h)x + (\alpha h + 2) = 0.$$

Thus M has at most four distinct constant principal curvatures. By Kimura's theorem [1], M is congruent to an open subset of a homogeneous real hypersurface of type A_1, A_2 or B of P^nC.

On the other hand, for a principal vector $Y = PX$ with principal curvature μ, $PY = -X$ is also a principal vector with principal curvature λ and hence we can change λ and μ in (2.8). Thus we have $3\mu - \lambda + h - \alpha + \alpha(\mu - \lambda)(\alpha - \mu)/2 = 0$, which together with (2.8) yield $(\lambda - \mu)\{\alpha(\lambda + \mu) - 2(\alpha^2 + 4)\} = 0$. this is equivalent to $(\lambda^2 - \alpha\lambda - 1)\{\alpha\lambda^2 - 2(\alpha^2 + 4)\lambda + \alpha(\alpha^2 + 5)\} = 0$.

Suppose that M is congruent to an open subset of a homogeneous real hypersurface of type B. Then the distinct principal curvatures at three, say $\alpha = 2\cot 2t$, $\lambda_1 = \cot(t - \pi/4)$ and $\lambda_2 = -\tan(t - \pi/4)$ (for details, see [4, page 47, Table]). By the way, λ_1 and λ_2 have to satisfy $\lambda^2 - 2(\cot 2t)\lambda - 1 = 0$, which leads to a contradiction. Thus M is congruent to an open subset of a homogeneous hypersurface of type A_1 or A_2. By a theorem in [3], the Ricci tensor S is cyclic-parallel, namely it satisfies

$$g(\nabla_X S(Y), Z) + g(\nabla_Y S(Z), X) + g(\nabla_Z S(X), Y) = 0.$$

hence it is parallel and we can apply Theorem A to our situation, which concludes the proof.

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