

LOCAL BOUNDEDNESS OF MONOTONE OPERATORS
UNDER MINIMAL HYPOTHESES

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We give a short proof the local boundedness of a monotone operator as an easy consequence of the continuity of an associated convex function.

Let X be a real normed space with normed dual X^* . Recall that a set-valued mapping $T: X \rightarrow X^*$ is *monotone* if

$$\langle x^* - y^*, x - y \rangle \geq 0$$

for any pairs $x^* \in T(x)$, $y^* \in T(y)$. The *domain* of T , $D(T)$, is defined by $D(T) := \{x \in X: T(x) \neq \emptyset\}$. The mapping is *locally bounded* at $x \in D(T)$ if for some $\varepsilon > 0$ the set $T(\{y \in X: \|y - x\| < \varepsilon\})$ is bounded. For any set S in X , the cone generated by S at x is $S_x = \{y \in X \mid y = t(s - x), s \in S, t > 0\}$. We will (a little non-standardly) call a point x of S an *absorbing point* of S if $S_x = X$. Clearly every interior point is an absorbing point. If S is convex then absorbing points and core points coincide.

A function $f: X \rightarrow]-\infty, \infty]$ is *convex* if $\text{epi}(f) = \{(x, r) \in X \times R \mid f(x) \leq r\}$ is a convex set, and *lower semicontinuous* if $\{x \in X \mid f(x) \leq r\}$ is closed for each real r . [Here X may be any real topological vector space.] The *essential domain* of f is $\text{dom}(f) = \{x \in X: f(x) < \infty\}$. Finally, recall that a real locally convex space X is *barrelled* if each closed convex set C in X for which 0 is an absorbing point [that is $C_0 = X$] is a neighbourhood of zero. It follows from the Baire category theorem that every Banach space is barrelled, as indeed is every Baire normed space. Moreover, a Banach space may possess many non-Baire but barrelled normed subspaces including some dense hyperplanes [1]. The following result is basic to much convex analysis [2]. We include a proof of a somewhat more general version.

THEOREM 1. *Let X be a barrelled locally convex space. Let $f: X \rightarrow]-\infty, \infty]$ be convex and lower semicontinuous and suppose that x is an absorbing point of $\text{dom}(f)$. Then f is continuous at x .*

PROOF: We may assume by translation that $f(0) = 0$ and that 0 is an absorbing point of $\text{dom}(f)$. Let $\varepsilon > 0$ be given and set $C = \{x \in X \mid f(x) \leq \varepsilon\}$. Since f is

Received 15 August 1988

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convex and lower semicontinuous, C is convex and closed. We verify that $U = C \cap -C$ is a symmetric neighbourhood of zero. As X is barrelled it suffices to show that 0 is an absorbing point of C . Let $x \in X$. By hypothesis $f(rx) < \infty$ for some $r > 0$. Select $\lambda > 0$ such that $\lambda f(rx) < \varepsilon$ and $\lambda < 1$. Then λrx lies in C because

$$f(\lambda rx) \leq \lambda f(rx) + (1 - \lambda)f(0) = \lambda f(rx) < \varepsilon.$$

Finally, since $f(-x) + f(x) \geq 0$, we have $|f(x) - f(0)| \leq \varepsilon$ for all x in U . ■

We may now establish our main result.

THEOREM 2. *Let X be a barrelled normed space. Let $T: X \rightarrow X^*$ be monotone and suppose that x is an absorbing point of $D(T)$. Then T is locally bounded at x .*

PROOF: We may assume by translation that $0 \in T(0)$ and that 0 is an absorbing point of $D(T)$ (which need not be convex). Consider the function $f: X \rightarrow]-\infty, \infty]$ given by

$$f(x) = \sup\{\langle y^*, x - y \rangle : y^* \in T(y), \|y\| \leq 1\}.$$

Indeed, f is always nonnegative since $0 \in T(0)$. Then f is convex and lower semicontinuous as a supremum of continuous affine functionals. Let $x \in X$. As 0 is an absorbing point of $D(T)$, we can find $t > 0$ with $T(tx) \neq \emptyset$. Choose any $u^*(t) \in T(tx)$. For each $y^* \in T(y)$

$$\langle y^*, tx - y \rangle \leq \langle u^*(t), tx - y \rangle$$

because T is monotone. Since $0 \in T(0)$ this also shows $f(0) = 0$. Hence

$$f(tx) \leq \sup\{\langle u^*(t), tx - y \rangle : \|y\| \leq 1\} < \infty.$$

This shows that 0 is an absorbing point of $\text{dom}(f)$. Thus Theorem 1 ensures that for some $\delta > 0$, $f(x) \leq 1$ whenever $\|x\| \leq 2\delta < 1$.

Equivalently, if $y^* \in T(y)$, $\|y\| \leq 1$, $\|x\| \leq 2\delta$, then

$$\langle y^*, x \rangle \leq \langle y^*, y \rangle + 1.$$

Hence if $y^* \in T(y)$ with $\|y\| \leq \delta$, then

$$2\delta \|y^*\| = \sup\{\langle y^*, x \rangle : \|x\| \leq 2\delta\} \leq \|y^*\| \|y\| + 1 \leq \delta \|y^*\| + 1$$

and $\|y^*\| \leq 1/\delta$. This establishes that T is locally bounded at 0 . ■

Remarks. 1. The usual result is established under the stronger assumptions that x is an interior point of $D(T)$, and that X is Banach or Baire [4, 3]. In our setting this is not presumed. For instance, $D = \{(x, y) : x^2 + y^2 = 2|x|\} \cup \{(0, y) : |y| = 1\}$ has 0 as an absorbing point and so is a possible domain in \mathbb{R}^2 .

2. We could establish directly that $\{x \mid f(x) \leq 1\}$ is a barrel and so omit Theorem 1. This would also omit the pleasant parallel between the statements of the two results.

3. This technique also works of $x \in \text{core conv } D(T)$, as in [4].

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