# Geometrical Interpretation of a few concomitants of the cubic in the Argand Plane.

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§ 1. Let $f = a_z^3 = a_z'^3 = a_0 z^3 + 3a_1 z^2 + 3a_2 z + a_3 = 0$ ..... ..........(1)
be the binary cubic whose coefficients $a_i$ are complex numbers represented on the Argand Plane. Then if its roots are $z_1$, $z_2$, $z_3$, the three corresponding points form the vertices of a triangle $A_1 A_2 A_3$. Let this triad of points be said to represent the cubic. Then its Hessian

$$H = (a\,a')^2 a_z a_z' \quad \text{..........................} \quad (2)$$

is represented by a certain pair of other points; likewise every first polar

$$\left(y\frac{\partial}{\partial z}\right) f = a_z^2\, a_y \quad \text{..........................} \quad (3)$$

associates a definite pair of points $(z)$ with any given point $(y)$. It is understood that $y$, $z$ ... are each complex numbers.

In particular the first polar

$$\frac{df}{dz} \text{ i.e. } a_0 z^2 + 2a_1 z + a_2 \quad \text{..........................} \quad (4)$$

of the point at infinity gives, as is well known, the real foci of the conic, which touches at their mid points the lines joining $A_1 A_2 A_3$. The centre of the conic is the point $-\dfrac{a_1}{a_0}$.

Since $(f, H)^2 = 0$ identically* then $H$ is apolar to the first polar of any point $y$.

Now the mid point $w$ of the two points $z$ represented by (3) whose equation—non-symbolically—is

$$(a_0 y + a_1) z^2 + 2(a_1 y + a_2) z + a_2 y + a_3 = 0$$

is given by

$$w = -\frac{a_1 y + a_2}{a_0 y + a_1} \text{ i.e. } a_0 yw + a_1 (y + w) + a_2 = 0.$$

* *Cf.* Grace and Young, *Algebra of Invariants* (190) p.



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Thus $y$, $w$, form an involution whose double points are

$$a_0 z^2 + 2a_1 z + a_2 = 0.$$

Hence *the points $y$, $w$, are harmonic to the focal points* (4) .........I.

§ 2. This gives us a Geometrical construction for the roots of the first polar of the point $y$.

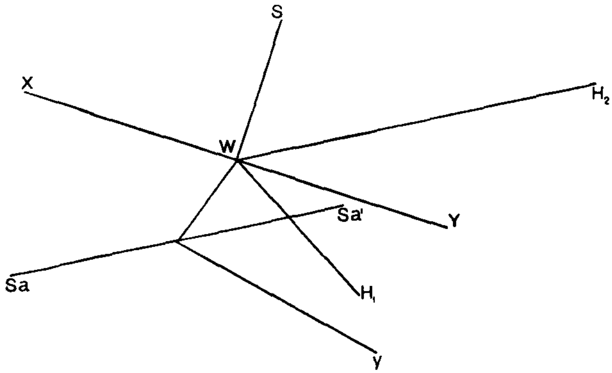


**Fig. 1**

(The figure is not drawn accurately to scale.)

Let $S_a$ and $S_{a'}$ be the focal points and $y$ the given point. Determine $W$ so that $(S_a S_{a'},\ W y)$ are apolar; in fact $W$ is the harmonic conjugate of $y$ with regard to $S_a$, $S_{a'}$ on the circle $S_a S_{a'} y$. Let $H_1 H_2$ be the Hessian points.* The Geometrical determination of these points will be explained below. Let $XY$ bisect the angle $H_1 W H_2$. Take $S$ the point where the perpendicular to $XY$ at $W$ and the perpendicular to $H_1 H_2$ at its mid point meet: then the circle whose centre is $S$ and radius $SH_1$ will meet the line $XY$ at the points $XY$ which represent the first polar of $y$.

For $WX \,.\, WY = WH_2 \,.\, WH_1$ and the line $XY$ bisects the angle $H_2 WH_1$.

The point $y$ together with the second polar of $y$ with regard to $f$, *i.e.* $a_y^2 a_z = 0$ are apolar to the first polar of $y$ since

$$(a\,a')\, a_y^2\, a_y'^2 = 0.$$

Hence we can also construct geometrically the *second* polar of $y$, and incidentally the apolar triad $a_x a_y a_z = 0$.

* See Grace and Young, *loc. cit.*, pp. 209, 211.

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§ 3. If $A_1 A_2 A_3$ represent roots of $a_z^3 = 0$ the first polar of $A_1$ is $A_1$ itself together with the point $P_1$ which is the fourth harmonic with regard to $A_2 A_3$ on the circle $A_1 A_2 A_3$: *i.e.* $P_1$ is one of the roots of the cubic covariant of $a_z^3 = 0$.

This gives us immediately the algebraical expression for the roots of the cubic covariant in terms of those of the equation $a_z^3 = 0$.

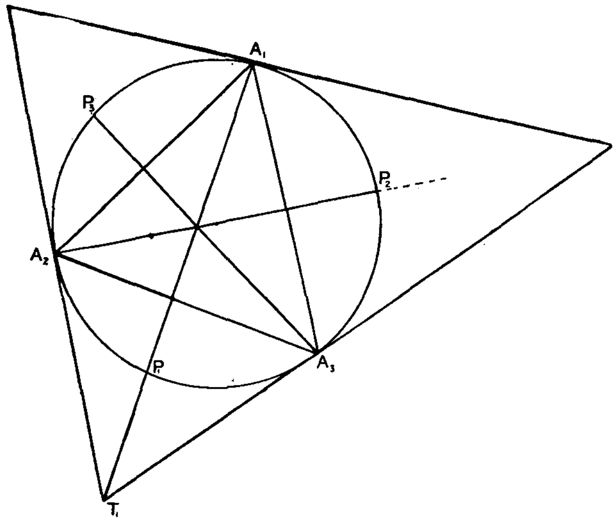


**Fig. 2**

Tangents are drawn to the circle circumscribing $A_1 A_2 A_3$. $A_1$ is joined to $T_1$ where the tangents at $A_2$ and $A_3$ meet. The line $A_1 T_1$ cuts the circle in $P_1$.

The Hessian points * are got by drawing a circle through $A_1$ so that $A_2 A_3$ are inverse points with regard to this circle and a second circle through $A_2$ so that $A_1 A_3$ are inverse points with respect to it: these two circles intersect in the Hessian points. These circles also pass respectively through $P_1$ and $P_2$ since $A_1 P_1$ being the first polar of the point $A_1$ are apolar to the Hessian points.

The point $A_1$ and the mid point of $A_1 P_1$ are, by result I. harmonic to the focal points.

* *Cf.* Grace and Young, *loc. cit.*

Similarly by considering the cubic covariant in place of the original cubic we can prove that the point $P_1$ and the mid point of $A_1 P_1$ are harmonic to the focal points of a similar conic inscribed in the triangle $P_1 P_2 P_3$.

So the centres of these two conics are harmonic to the points $H_1 H_2$, and also the pairs of real foci of these conics are also harmonically related as may be easily verified.

§ 4. *Two Cubics.* The apolar invariant of the two cubics $f = a_z^3 = 0$ and $f' = b_z^3 = 0$ is $(ab)^3 = 0$. The foci being the first polar of the point at infinity: let this point be $\rho$. Then $a_\rho a_z^2 = 0$, $b_\rho b_z^2 = 0$ give the four foci $S_a, S_{a'}; S_b, S_{b'}$.

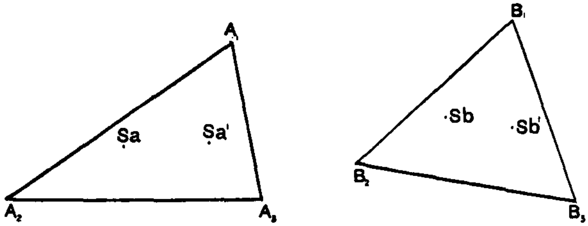


Fig. 3

Choose $y$ so that its first polar with respect to $f' = 0$, *i.e.* $b_y b_z^2 = 0$ is apolar to $S_a S_{a'}$, therefore $b_y a_\rho (ab)^2 = 0$ ................(5).
This is a homography between $y$ and $\rho$. The double points are points $KK'$ given by $(ab)^2 a_K b_K = 0$, *i.e.* by the vanishing of the second transvectant of the two cubics.

If (5) is an involution $y$ and $\rho$ can be interchanged

therefore $\qquad (ab)^2 a_\rho b_y - (ab)^2 a_y b_\rho = 0,$

or $\qquad (ab)^3 (y\rho) = 0.$

Thus unless $y = \rho$ which only happens when these two points coincide at $K$ or $K'$, (5) is an involution if $(ab)^3$ vanishes and if so (5) holds for all points of the plane.

§ 5. The last result may be shown non-symbolically.

For the first polar of $y$ with regard to $f$ is

$$(a_0 y + a_1) z^2 + 2 (a_1 y + a_2) z + (a_2 y + a_3) = 0$$

and of $w$ with regard to $f'$ is

$$(b_0 w + b_1) z^2 + 2 (b_1 w + b_2) z + (b_2 w + b_3) = 0.$$

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These are apolar if

$$yw\,(a_0 b_2 + a_2 b_0 - 2a_1 b_1) + y\,(b_1 a_2 + a_0 b_3 - 2a_1 b_2)$$
$$+ w\,(a_1 b_2 + a_3 b_0 - 2a_2 b_1) + a_3 b_1 + a_1 b_3 - 2a_2 b_2 = 0.$$

This is an involution whose double points are the roots of the second transvectant

$$(a_0 b_2 + a_2 b_0 - 2a_1 b_1)\, z^2 + (a_0 b_3 + a_3 b_0 - a_1 b_2 - a_2 b_1)\, z$$
$$+ (a_1 b_3 + a_3 b_1 - 2a_2 b_2) = 0$$

$$\text{if} \quad a_0 b_3 - 3a_1 b_2 + 3a_2 b_1 - a_3 b_0 = 0,$$
$$i.e. \quad (ab)^3 = 0.$$

§ 6. The first polars of the origin with respect to $f$ and $f'$ are

$$a_1 z^2 + 2a_2 z + a_3 = 0, \qquad \text{.........................(6)}$$
$$b_1 z^2 + 2b_2 z + b_3 = 0. \qquad \text{.........................(7)}$$

Now (6) is apolar to the "focal" points of $f'$, and (7) is apolar to the "focal" points of $f$ if

$$(a_1 b_1 - a_2 b_0)\, z^2 + (a_1 b_2 - a_3 b_0)\, z + (a_2 b_2 - a_3 b_1) = 0, \quad \text{.........(8)}$$
$$(a_1 b_1 - a_0 b_2)\, z^2 + (b_1 a_2 - a_0 b_3)\, z + (a_2 b_2 - a_1 b_3) = 0. \quad \text{.........(9)}$$

But the second transvectant is the sum of (8) and (9). Hence the pair of points apolar to (8) and (9) is apolar to the second transvectant

The centres of the conics for the two cubics, *i.e.* the points $-\dfrac{a_1}{a_0}$, $-\dfrac{b_1}{b_0}$ have for their first polars respectively

$$(b_0 a_1 - a_0 b_1)\, z^2 + 2z\,(b_1 a_1 - b_2 a_0) + b_2 a_1 - b_3 a_0 = 0$$

and
$$(a_0 b_1 - a_1 b_0)\, z^2 + 2z\,(b_1 a_1 - a_2 b_0) + a_2 b_1 - a_3 b_0 = 0.$$

The mid point of the pair of points apolar to these two can be shown to be the mid point of the pair of points which are the roots of the second transvectant, $(a_z^3,\ b_z^3)^2$.

This enables us to construct the points representing this second transvectant.