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HIGH-POWER ANALOGUES OF THE TURÁN-KUBILIUS INEQUALITY, AND AN APPLICATION TO NUMBER THEORY

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1. Statement of results. An arithmetic function $f(n)$ is said to be *additive* if it satisfies $f(ab) = f(a) + f(b)$ whenever a and b are coprime integers. For such a function we define

$$A(x) = \sum_{p^m \leqq x} p^{-m} f(p^m), \quad B(x) = \left(\sum_{p^m \leqq x} p^{-m} |f(p^m)|^2 \right)^{1/2}, \quad x \geqq 2.$$

A standard form of the Turán–Kubilius inequality states that

$$(1) \qquad \sum_{n \leqq x} |f(n) - A(x)|^2 \leqq c_1 x B(x)^2$$

holds for some absolute constant c_1, uniformly for all complex-valued additive arithmetic functions $f(n)$, and real $x \geqq 2$. An inequality of this type was first established by Turán [**11**], [**12**] subject to some side conditions upon the size of $|f(p^m)|$. For the general inequality we refer to [**10**].

This inequality, and more recently its dual, have been applied many times to the study of arithmetic functions. For an overview of some applications we refer to [**2**]; a complete catalogue of the applications of the inequality (1) would already be very large. For some applications of the dual of (1) see [**3**], [**4**], and [**1**].

THEOREM 1. *Let β be a real number. Then there is a constant c_2, depending at most upon β, so that the inequality*

$$(2) \qquad x^{-1} \sum_{n \leqq x} |f(n) - A(x)|^\beta \leqq \begin{cases} c_2 B(x)^\beta + c_2 \sum_{p^m \leqq x} p^{-m} |f(p^m)|^\beta \text{ if } \beta \geqq 2, \\ c_2 B(x)^\beta \text{ if } 0 \leqq \beta \leqq 2, \end{cases}$$

holds uniformly for all additive functions $f(n)$, and real $x \geqq 2$.

Remarks. If $f(n)$ is real, $f(p^m) = f(p)$, $|f(p)| \leqq 1$ for each prime p and positive integer m, and $B(x) \to \infty$ as $x \to \infty$, then

$$x^{-1} \sum_{n \leqq x} |f(n) - A(x)|^\beta \sim c_3(\beta) B(x)^\beta, \quad x \to \infty,$$

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where the constant $c_3(\beta)$ has the value

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u|^{\beta} e^{-u^2/2} du.$$

This result may be deduced from Theorem 1 and the fact that the function $\{f(n) - A(x)\}/B(x)$ is in this case approximately distributed as a Gaussian law with mean zero and variance one. This last is the well-known result of Erdös and Kac [**6**]. The presence of the term $B(x)^{\beta}$ on the right-hand side of the inequality (2) is therefore appropriate.

However, if $f(n)$ is zero on all prime-powers except for those of one prime q, then

$$\sum_{n \leqq x} |f(n) - A(x)|^{\beta} = |f(q)(1 - q^{-1})|^{\beta} \left[\frac{x}{q}\right] \geqq x|f(q)|^{\beta} q^{-1} 2^{-\beta-1}$$

for all $x \geqq 2q$. For $\beta \leqq 2$,

$$|f(q)|^{\beta} q^{-1} \leqq (|f(q)|^2 q^{-1})^{\beta/2} = B(x)^2.$$

For $\beta > 2$ the extra sum in (2) involving the $|f(p^m)|^{\beta}$ is, thus, also appropriate.

By the appropriate dualisation we obtain

THEOREM 2. *Let P be a set of primes. For $x \geqq 2$ define*

$$L = L(x) = \sum_{p \leqq x, p \in P} \frac{1}{p}.$$

Let α be a real number, $1 < \alpha \leqq 2$. Then there is a constant c_4, depending at most upon α, so that

$$(3) \qquad \sum_{p \leqq x, p \in P} p^{\alpha-1} \left| \sum_{\substack{n \leqq x \\ p \| n}} a_n - p^{-1} \sum_{n \leqq x} a_n \right|^{\alpha} \leqq c_4 x^{\alpha-1} (L+1)^{2-\alpha} \sum_{n \leqq x} |a_n|^{\alpha}$$

holds uniformly for all complex numbers a_n, $1 \leqq n \leqq x$, and real $x \geqq 2$.

If $\alpha \geqq 2$ there is a constant c_5 so that

$$(4) \qquad \sum_{p^m \leqq x} p^m \left| \sum_{\substack{n \leqq x \\ p^m \| n}} a_n - p^{-m} \sum_{n \leqq x} a_n \right|^2 \leqq c_5 x^{2-(2/\alpha)} \left(\sum_{n \leqq x} |a_n|^{\alpha} \right)^{2/\alpha}$$

holds with the same uniformities.

Remark. In this theorem $p^m \| n$ means that p^m divides n but p^{m+1} does not.

These results may be supplemented by

THEOREM 3. *For $\alpha > 1$ and a suitable c_6,*

$$(5) \qquad \sum_{\substack{p^m \leqq x \\ p, m \geqq 2}}\sum p^{m(\alpha-1)} \left| \sum_{\substack{n \leqq x \\ p^m \| n}} a_n \right|^{\alpha} \leqq c_6 x^{\alpha-1} \sum_{n \leqq x} |a_n|^{\alpha}$$

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whilst, in the notation of Theorem 2,

$$\sum_{p\leqq x, p\in P} p^{\alpha-1}\left|\sum_{\substack{n\leqq x\\ p\| n}} a_n\right|^{\alpha} \leqq c_7 x^{\alpha-1}(L+1)\sum_{n\leqq x}|a_n|^{\alpha} \tag{6}$$

for all complex numbers a_n, $1 \leqq n \leqq x$, *and real* $x \geqq 2$.

As an application of some of these inequalities we prove

THEOREM 4. *In order that the real-valued additive arithmetic function* $f(n)$ *satisfy*

$$\sum_{n\leqq x}|f(n)|^{\alpha} \leqq cx \tag{7}$$

for a given constant $\alpha > 1$, *some* $c > 0$ *and all* $x \geqq 2$, *it is both necessary and sufficient that the series*

$$\sum_{|f(p)|\leqq 1} p^{-1}|f(p)|^2, \quad \sum_{|f(p^m)|>1} p^{-m}|f(p^m)|^{\alpha} \tag{8}$$

converge, and that the partial sums

$$\sum_{p\leqq x, |f(p)|\leqq 1} p^{-1}f(p)$$

be bounded uniformly for all $x \geqq 2$.

Remarks. As we indicate, in a subsequent paper, the peculiar form of the condition (8), which involves both $|f(p)|^2$ and $|f(p^m)|^{\alpha}$, is typical of problems involving the αth moment of an arithmetic function, $\alpha > 1$.

2. Small values of $f(p)$. In this section we obtain some preliminary results, necessary for the proof of Theorem 1.

LEMMA 1. *Let* $g(m)$ *be a real-valued multiplicative function which satisfies* $0 \leqq g(m) \leqq 1$ *for every integer* $m \geqq 1$. *Then*

$$x^{-1}\sum_{m\leqq x} g(m) \leqq e^{\gamma}\left(1+O\left(\frac{\log\log x}{\log x}\right)\right)\prod_{p\leqq x}\left(1-\frac{1}{p}\right) \times\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\dots\right)$$

holds uniformly for all $x \geqq 2$.

Proof. This result is obtained by Hall [**8**] under the weaker assumption that $g(m)$ be submultiplicative, in the sense that $g(ab) \leqq g(a)g(b)$ whenever $(a, b) = 1$, and that $g(1) = 1$.

LEMMA 2. *Let* $g(m)$ *be a real-valued non-negative multiplicative function which satisfies* $g(p) \geqq 1$ *for each prime* p. *Then*

$$x^{-1}\sum_{m\leqq x} g(m) \leqq \exp\left(\sum_{p\leqq x}\frac{g(p)-1}{p} + \sum_{p\leqq x, m\geqq 2}\sum \frac{g(p^m)}{p^m}\right)$$

holds uniformly for all $x \geqq 1$.

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Proof. Let $h(d)$ be the Möbius inverse to the function $g(n)$, so that

$$\sum_{d|n} h(d) = g(n)$$

holds identically. Then $h(p) = g(p) - 1 \geqq 0$ and $h(d) \geqq 0$ whenever d is square free. Hence

$$\sum_{n \leqq x} \mu^2(n) g(n) = \sum_{n \leqq x} \sum_{d|n} \mu^2(d) h(d) = \sum_{d \leqq x} \mu^2(d) h(d) \left[\frac{x}{d}\right]$$
$$\leqq x \prod_{p \leqq x} \left(1 + \frac{h(p)}{p}\right) \leqq x \exp\left(\sum_{p \leqq x} \frac{g(p) - 1}{p}\right).$$

More generally, each integer m may be uniquely decomposed into the form $m = m_1 m_2$ where m_1 contains only those prime divisors of m which occur to exactly the first power, and m_2 contains the remaining prime powers.

Then

$$\sum_{m \leqq x} g(m) \leqq \sum_{m_2 \leqq x} g(m_2) \sum_{m_1 \leqq x/m_2} g(m_1)$$
$$\leqq x \exp\left(\sum_{p \leqq x} \frac{g(p) - 1}{p}\right) \sum_{m_2 \leqq x} g(m_2) m_2^{-1}.$$

Moreover,

$$\sum_{m_2 \leqq x} \frac{g(m_2)}{m_2} \leqq \prod_{p \leqq x} \left(1 + \frac{g(p^2)}{p^2} + \frac{g(p^3)}{p^3} + \dots\right)$$
$$\leqq \exp\left(\sum_{p \leqq x} \sum_{m=2}^{\infty} p^{-m} g(p^m)\right).$$

This completes the proof of lemma 2.

Let x be a real number, $x \geqq 2$. Let $f(n)$ be a real-valued arithmetic function, whose values may depend upon x. For convenience of notation we write A and B in place of $A(x)$ and $B(x)$ respectively.

For each complex number z we define the multiplicative function

$$g(n) = g(n, z) = e^{zf(n)/B}.$$

Let

$$\varphi(z) = x^{-1} \sum_{n \leqq x} g(n) e^{-zA/B} = x^{-1} \sum_{n \leqq x} \exp\left(z\{f(n) - A\}/B\right).$$

LEMMA 3. *Assume that* $0 \leqq f(p^m) \leqq \delta B$ *holds for some* $\delta > 0$ *and all prime-powers* p^m *not exceeding* x. *Then there is a constant* c_8, *whose value depends at most upon* δ, *so that the bound*

$$|\varphi(z)| \leqq c_8$$

is satisfied on the whole complex disc, $|z| \leqq 1$.

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Proof. Assume first that $z = r$ is real, $r \leqq 0$. Then $0 \leqq g(n) \leqq 1$ for every n not exceeding x, so that by Lemma 1

$$\sum_{n \leqq x} g(n) \leqq c_9 x \prod_{p \leqq x} \left(1 - \frac{1}{p}\right) \prod_{p \leqq x} \left(1 + \frac{g(p)}{p} + \dots\right)$$
$$\leqq c_{10} x \exp\left(\sum_{p \leqq x} \frac{g(p) - 1}{p}\right).$$

From an application of the Cauchy–Schwarz inequality

$$\left|\frac{A}{B} - \sum_{p \leqq r} \frac{f(p)}{pB}\right| \leqq \sum_{\substack{p^m \leqq x \\ m \geqq 2}} \frac{|f(p^m)|}{p^m B} \leqq \left(\sum_{\substack{p^m \leqq x \\ m \geqq 2}} p^{-m}\right)^{1/2} \left(\sum_{p^m \leqq x} \frac{|f(p^m)|^2}{p^m B^2}\right)^{1/2}$$
$$\leqq \left(\sum_{p \leqq x} \frac{1}{p(p-1)}\right)^{1/2} \leqq 1.$$

Hence

$$\varphi(r) \leqq c_{11} \exp\left(\sum_{p \leqq x} \{g(p) - 1 - rf(p)B^{-1}\} p^{-1}\right). \tag{9}$$

For real numbers w the estimate

$$|e^w - 1 - w| \leqq |w|^2 e^{|w|}$$

may be obtained by integrating by parts. Therefore, for each prime p not exceeding x,

$$|g(p) - 1 - rf(p)B^{-1}| \leqq r^2 f(p)^2 B^{-2} \exp\left(|r| f(p) B^{-1}\right) \leqq \lambda f(p)^2 B^{-2}$$

where $\lambda = r^2 \exp(|r|\delta)$; and the exponent on the right-hand side of the inequality (9) does not exceed

$$\lambda B^{-2} \sum_{p \leqq x} p^{-1} f(p)^2 \leqq \lambda.$$

Thus $\varphi(r) \leqq c_{11} \exp(\lambda)$.

Suppose now that $z = r \geqq 0$. Then $g(n)$ is non-negative and $g(p) \geqq 1$. We argue with Lemma 2 in place of Lemma 1 and obtain $\varphi(r) \leqq c_{12} \exp(\lambda)$, say.

In the general case when $r = \operatorname{Re}(z)$, z complex, then

$$|\varphi(z)| \leqq x^{-1} \sum_{n \leqq x} |g(n) \exp(-zA/B)| \leqq x^{-1} \sum_{n \leqq x} \exp(r\{f(n) - A\}/B)$$

so that

$$|\varphi(z)| \leqq \varphi(r) \leqq e^{\lambda} \max(c_{11}, c_{12}).$$

This completes the proof of Lemma 3.

LEMMA 4. *Let the complex-valued additive function $f(n)$ satisfy $|f(p^m)| \leqq \delta B$ for all $p^m \leqq x$. Then for each $\beta > 0$ there is a constant c_{13}, depending*

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at most upon β, δ, so that the inequality

$$(10)\quad \sum_{n\leqq x} |f(n) - A|^{\beta} \leqq c_{13} x B^{\beta}$$

holds for all $x \geqq 2$.

Proof. Since the sum

$$\left([x]^{-1} \sum_{n\leqq x} |f(n) - A|^{\beta}\right)^{1/\beta}$$

is non-decreasing as β increases, it will suffice to establish the inequality (10) for arbitrary large even integer values of β.

By considering real and imaginary parts separately we see that there is no loss in generality in assuming that $f(n)$ assumes only real values, and, indeed, only non-negative values. For example, we can define additive functions $f_j(n)$, $j = 1, 2$, by

$$f_1(p^m) = \begin{cases} f(p^m) & \text{if } f(p^m) \geqq 0, \\ 0 & \text{otherwise} \end{cases} \qquad f_2(p^m) = \begin{cases} -f(p^m) & \text{if } f(p^m) < 0, \\ 0 & \text{otherwise} \end{cases}$$

and corresponding to each function $f_j(n)$ the sum

$$A_j = \sum_{p^m \leqq x} p^{-m} f_j(p^m), \quad j = 1, 2.$$

Then

$$|f(n) - A|^{\beta} = |f_1(n) - A_1 - \{f_2(n) - A_2\}|^{\beta} \leqq 2^{\beta} \sum_{j=1}^{2} |f_j(n) - A_j|^{\beta}.$$

Summing over the n not exceeding x justifies our last assertion.

For every positive integer k

$$x^{-1} \sum_{n\leqq x} (f(n) - A)^k B^{-k} = \varphi^{(k)}(0),$$

the kth derivative of $\varphi(z)$ evaluated at $z = 0$. By Cauchy's integral representation theorem

$$\varphi^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=1} z^{-k-1} \varphi(z) dz$$

and by Lemma 3

$$|\varphi^{k}(0)| \leqq \frac{k!}{2\pi} 2\pi \max_{|z|=1} |z^{-k-1}\varphi(z)| \leqq k! c_8.$$

This proves Lemma 4.

3. Large values of $f(p)$. We begin this section with the remark that those prime-powers $p^m \leqq x$ for which $|f(p^m)| > \delta B$ holds satisfy

$$(11)\quad \sum_{\substack{p^m \leqq x \\ |f(p^m)| > \delta B}} \frac{1}{p^m} \leqq \sum_{p^m \leqq x} \frac{1}{p^m} \left|\frac{f(p^m)}{\delta B}\right|^2 = \delta^{-2},$$

and are in this sense few in number.

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LEMMA 5. *Let P be a set of primes not exceeding x, and define*

$$L = L(x) = \sum_{p \leqq x, p \in P} \frac{1}{p}.$$

Let $\omega(n)$ denote the number of distinct factors of the integer n which belong to the set P, or which occur to some power $m \geqq 2$. Then the inequality

$$(12) \quad \sum_{n \leqq y} \omega(n)^{\beta} \leqq c_{14}(\beta) y (L+1)^{\beta}$$

holds uniformly for all x, y, $1 \leqq y \leqq x$, and $\beta \geqq 0$. Here $c_{14}(\beta)$ is a constant which depends only upon β.

Remark. $\omega(n)$ here is not the standard prime divisors counting function unless P includes all primes not exceeding x.

Proof. The sum

$$\left([x]^{-1} \sum_{n \leqq x} \omega(n)^{\beta} \right)^{1/\beta}$$

is non-decreasing as β increases, and it will therefore suffice to establish the inequality (12) for all integers $k \geqq 0$.

We argue inductively on k.

For $k = 0$ the inequality (12) is trivially valid. Assume that it holds for $k = 0, 1, \ldots, t-1$, $t \geqq 1$. Then

$$\sum_{n \leqq y} \omega(n)^{t} = \sum_{n \leqq y} \omega(n)^{t-1} \sum_{p^m \| n} 1 = \sum_{p^m \leqq y} \sum_{\substack{r \leqq p^{-m} y \\ (r,p)=1}} \omega(p^m r)^{t-1}$$

with the proviso that if $m = 1$ then p must belong to the set P. According to our induction hypothesis the inner sum may be estimated to be not more than

$$\sum_{r \leqq p^{-m} y} (1 + \omega(r))^{t-1} = \sum_{j=0}^{t-1} \binom{t-1}{j} \sum_{r \leqq p^{-m} y} \omega(r)^{j} \leqq \sum_{j=0}^{t-1} \binom{t-1}{j}$$
$$\times c_{14}(j) p^{-m} y (L+1)^{j} \leqq \max_{0 \leqq j \leqq t-1} c_{14}(j) p^{-m} y (L+1+1)^{t-1}$$
$$\leqq c_{15} p^{-m} y (L+1)^{t-1}.$$

Hence

$$\sum_{n \leqq y} \omega(n)^{t} \leqq c_{15} y \sum_{p^m \leqq y} p^{-m} (L+1)^{t-1} \leqq c_{16} y (L+1)^{t},$$

and the desired inequality holds if $c_{14}(\beta) = c_{16}$.

This completes the proof of Lemma 5.

LEMMA 6. *Let the complex-valued additive function $f(n)$ satisfy $|f(p^m)| > \delta B$ for each prime-power $p^m \leqq x$. Then there is a constant c_{17}, depending* *at most upon β, δ, so that the inequality*

$$\sum_{n\leqq x} |f(n) - A|^{\beta} \leqq c_{17}x \sum_{p^m\leqq x} p^{-m}|f(p^m)|^{\beta}$$

holds for all $x \geqq 1$, for each $\beta > 1$, whilst the inequality

$$\sum_{n\leqq x} |f(n)|^{\beta} \leqq c_{17}x \sum_{p^m\leqq x} p^{-m}|f(p^m)|^{\beta}$$

holds for all $x \geqq 1$, for each $\beta > 0$.

Proof. By Hölder's inequality when $\beta \geqq 1$, and by the elementary inequality $(u_1 + u_2 + \ldots + u_k)^{\beta} \leqq u_1^{\beta} + \ldots + u_k^{\beta}$ when $0 \leqq \beta < 1$, (each $u_i \geqq 0$), we see that

$$|f(n)|^{\beta} \leqq \max\,(1, \omega(n)^{\beta-1}) \sum_{p^m\| n} |f(p^m)|^{\beta},$$

where $\omega(n)$ is the function which is defined in the statement of Lemma 5. Hence

$$(13)\quad \sum_{n\leqq x} |f(n)|^{\beta} \leqq \sum_{p^m\leqq x} |f(p^m)|^{\beta} \sum_{\substack{n\leqq x\\ p^m\| n}} \max\,(1, \omega(n)^{\beta-1}).$$

If $p^m\|n$, say $n = p^m v$ where $(p, v) = 1$, then $v \leqq p^{-m}x$ and $\omega(n) \leqq 1 + \omega(v)$. A typical inner sum on the right-hand side of (13) is by Lemma 5 not more than

$$(14)\quad \sum_{v\leqq p^{-m}x} \max\,(1, (1 + \omega(v))^{\beta-1}) = O(p^{-m}x(L + 1)^{\beta-1}) \leqq c_{18}p^{-m}x$$

since

$$L = \sum_{\substack{p^m\leqq x\\ |f(p^m)|>\delta B}} \frac{1}{p^m} \leqq \delta^{-2}$$

from our remark (11).

The inequalities in (13) and (14) show that

$$(15)\quad \sum_{n\leqq x} |f(n)|^{\beta} \leqq c_{18}x \sum_{p^m\leqq x} p^{-m}|f(p^m)|^{\beta}.$$

Moreover, for $\beta > 1$, Hölder's inequality shows that

$$|A|^{\beta} \leqq L_1^{1/\alpha} \sum_{p^m\leqq x} p^{-m}|f(p^m)|^{\beta}$$

where $\alpha^{-1} + \beta^{-1} = 1$, and

$$L_1 = \sum_{p\leqq x, p\in P} \frac{1}{p} + \sum_{\substack{p^m\leqq x\\ m\geqq 2}} \frac{1}{p^m}.$$

Once again applying our remark (11) we see that L_1 is bounded in terms of δ, and

$$\sum_{n \leqq x} |A|^\beta \leqq c_{19} x \sum_{p^m \leqq x} p^{-m} |f(p^m)|^\beta.$$

The result of Lemma 6 is now clearly true.

4. Proof of theorem 1. We define additive function $h_j(n)$, $j = 1, 2$ by

$$h_1(p^m) = \begin{cases} f(p^m) & \text{if } |f(p^m)| \leqq B, \\ 0 & \text{otherwise,} \end{cases} \quad h_2(p^m) = \begin{cases} f(p^m) & \text{if } |f(p^m)| > B \\ 0 & \text{otherwise.} \end{cases}$$

Correspondingly we define

$$H_j = \sum_{p^m \leqq x} p^{-m} h_j(p^m).$$

Since

$$|f(n) - A|^\beta \leqq 2^\beta \sum_{j=1}^{2} |h_j(n) - H_j|^\beta$$

when $\beta \geqq 1$, the first of the desired inequalities of Theorem 1 follows from Lemma 4 applied to the function $h_1(n)$, with $\delta = 1$, together with Lemma 6 applied to the function $h_2(n)$ with $\delta = 1$.

The second of the desired inequalities of Theorem 1, valid when $0 \leqq \delta \leqq 2$, follows from the fact that the value of the expression

$$\left([x]^{-1} \sum_{n \leqq x} |f(n) - A|^\beta \right)^{1/\beta}$$

is no larger than that of the similar expression with β replaced by 2, which in turn is at most $c_{20} B$ for some positive absolute constant c_{20}. Indeed, the case $\beta = 2$ is the standard Turán–Kubilius inequality.

Theorem 1 is proved.

5. Proof of theorem 2. Let $\alpha \geqq 2$ hold. Define β by $\beta^{-1} + \alpha^{-1} = 1$. Hence

$$\beta = \alpha(\alpha - 1)^{-1} \leqq 2.$$

Define

$$\epsilon(p^m, n) = \begin{cases} p^{m/2}(1 - p^{-m}) & \text{if } p^m \| n, \\ -p^{-m/2} & \text{otherwise.} \end{cases}$$

Then the second inequality in the statement of Theorem 1 may be written in the form

$$\left(\sum_{n \leqq x} \left| \sum_{p^m \leqq x} \epsilon(p^m, n) f(p^m) \right|^\beta \right)^{1/\beta} \leqq \mu \left(\sum_{p^m \leqq x} |f(p^m)|^2 \right)^{1/2}$$

with

$$\mu = (c_2 x)^{1/\beta},$$

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and is valid for all complex numbers $f(p^m)$. Regarding this as an inequality between norms (see [**9**] Theorem 286; [**7**]) we deduce immediately that

$$\left(\sum_{p^m \leqq x} \left| \sum_{n \leqq x} \epsilon(p^m, n) a_n \right|^2 \right)^{1/2} \leqq \mu \left(\sum_{n \leqq x} |a_n|^\alpha \right)^{1/\alpha}$$

holds for all complex numbers a_n, $1 \leqq n \leqq x$, and this is (4) of Theorem 2.

If $1 < \alpha \leqq 2$, and β is defined as before, then $\beta \geqq 2$. By Hölder's inequality with exponents $\rho, \beta/2, \rho^{-1} + 2\beta^{-1} = 1$,

$$\left(\sum_{p \leqq x, p \in P} p^{-1} |f(p)|^2 \right)^\beta \leqq L^{\beta - 2} \sum_{p \leqq x, p \in P} p^{-1} |f(p)|^\beta$$

so that the first inequality of Theorem 1 has the corollary

$$(16) \quad \sum_{n \leqq x} \left| \sum_{p \| n} f(p) - \sum_{p \leqq x} p^{-1} f(p) \right|^\beta \leqq c_{21} x (L + 1)^{\beta - 2} \sum_{p \leqq x} p^{-1} |f(p)|^\beta,$$

where the prime p belongs to the (special) set P.

Define

$$v(p, n) = \begin{cases} p^{1/\beta}(1 - p^{-1}) & \text{if } p \| n, \\ -p^{-1 + (1/\beta)} & \text{otherwise.} \end{cases}$$

Then the inequality (16) may be written in the form

$$\left(\sum_{n \leqq x} \left| \sum_{p \leqq x} v(p, n) f(p) \right|^\beta \right)^{1/\beta} \leqq \{c_{21} x (L + 1)^{\beta - 2}\}^{1/\beta} \left(\sum_{p \leqq x} |f(p)|^\beta \right)^{1/\beta}.$$

Dualising we obtain

$$\left(\sum_{p \leqq x} \left| \sum_{n \leqq x} v(p, n) a_n \right|^\alpha \right)^{1/\alpha} \leqq \{c_{21} x (L + 1)^{\beta - 2}\}^{1/\beta} \left(\sum_{n \leqq x} |a_n|^\alpha \right)^{1/\alpha}$$

which gives the inequality (3) of Theorem 2.

6. Proof of theorem 3. We prove inequality (6); the proof of (5) proceeds in a similar manner.

Let $\omega(n)$ denote the number of prime divisors of n which belong to the set P. Then if p belongs to P we have

$$\begin{aligned} \sum_{\substack{n \leqq x \\ p \| n}} |a_n| &= \sum_{\substack{n \leqq x \\ p \| n}} |a_n| \omega(n)^{-1/\alpha} \omega(n)^{1/\alpha} \\ &\leqq \left(\sum_{\substack{n \leqq x \\ p \| n}} \omega(n)^{\beta/\alpha} \right)^{1/\beta} \left(\sum_{\substack{n \leqq x \\ p \| n}} |a_n|^\alpha \omega(n)^{-1} \right)^{1/\alpha}, \end{aligned}$$

where, as usual, $\beta^{-1} + \alpha^{-1} = 1$. We see from Lemma 5 that

$$\sum_{\substack{n \leqq x \\ p \| n}} \omega(n)^{\beta/\alpha} \leqq \sum_{m \leqq p^{-1} x} (1 + \omega(m))^{\beta/\alpha} \leqq c_{22} p^{-1} x (L + 1)^{\beta/\alpha}.$$

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Hence

$$\sum_{p \leqq x, p \in P} p^{\alpha-1} \left| \sum_{\substack{n \leqq x \\ p \| n}} a_n \right|^\alpha \leqq c_{23} \sum_{p \leqq x, p \in P} p^{\alpha-1}(p^{-1}x)^{\alpha-1}(L+1)$$
$$\times \sum_{\substack{n \leqq x \\ p \| n}} |a_n|^\alpha \omega(n)^{-1} = c_{23}x^{\alpha-1}(L+1) \sum_{n \leqq x} |a_n|^\alpha \omega(n)^{-1} \sum_{p \| n, p \in P} 1$$
$$= c_{23}x^{\alpha-1}(L+1) \sum_{n \leqq x} |a_n|^\alpha,$$

which gives (6).

7. Proof of theorem 4. *Sufficiency.* Define the additive functions

$$t_1(p^m) = \begin{cases} f(p^m) & \text{if } |f(p^m)| > 1, \\ 0 & \text{otherwise} \end{cases} \quad t_2(p^m) = \begin{cases} f(p^m) & \text{if } |f(p^m)| \leqq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It will clearly suffice to prove that

$$x^{-1} \sum_{n \leqq x} |t_i(n)|^\alpha$$

is bounded uniformly for all $x \geqq 2$, $i = 1, 2$.

Consider the function $t_1(n)$ first. If $t_1(n)$ is identically zero there is nothing to prove. Otherwise let

$$b = \Sigma p^{-m}|t_1(p^m)|^2 > 0.$$

Then, by Lemma 6 with $\delta = b^{-1}$

$$\sum_{n \leqq x} |t_1(n)|^\alpha \leqq c_{17}(\alpha) x \sum p^{-m}|t_1(p^m)|^\alpha \leqq c_{24}x, \quad x \geqq 1.$$

For the function $t_2(n)$ we have

$$|A| = \left| \sum_{p^m \leqq x} p^{-m} t_2(p^m) \right| \leqq \left| \sum_{p \leqq x, |f(p)| \leqq 1} p^{-1} f(p) \right| + \sum_{p, m \geqq 2}\sum p^{-m} < c_{25}$$
$$B^2 = \sum_{p^m \leqq x} p^{-m}|t_2(p^m)|^2 \leqq \sum_{|f(p)| \leqq 1} p^{-1}|f(p)|^2 + \sum_{p, m \geqq 2}\sum p^{-m} < c_{26}$$

from the hypotheses (8) and following, of Theorem 4. By Lemma 4, once again with $\delta = b^{-1}$. we deduce the uniform boundedness of

$$x^{-1} \sum_{n \leqq x} |t_1(n) - A|^\alpha$$

and then of

$$x^{-1} \sum_{n \leqq x} |t_1(n)|^\alpha.$$

This completes the proof of the sufficiency of the conditions (8).

8. Proof of theorem 4. *Necessity.* From Theorem 3, (5), with $a_n = f(n)$, and the hypothesis that

$$x^{-1} \sum_{n \leqq x} |f(n)|^\alpha$$

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is bounded uniformly for $x \geqq 1$, we see that

$$\sum_{p,m\geqq 2} p^{m(\alpha-1)} \left| \sum_{n\leqq x, p^m \| n} f(n) \right|^\alpha \leqq c_{27} x^\alpha.$$

Typically

$$\sum_{n\leqq x, p^m\| n} f(n) = f(p^m)\left\{\left[\frac{x}{p^m}\right] - \left[\frac{x}{p^{m+1}}\right]\right\} + \sum_{\substack{u\leqq p^{-m}x \\ (u,p)=1}} f(u).$$

By Hölder's inequality

$$\left| \sum_{\substack{u\leqq p^{-m}x \\ (u,p)=1}} f(u) \right| \leqq (p^{-m}x)^{\alpha-1} \sum_{u\leqq p^{-m}x} |f(u)|^\alpha = O(p^{-m}x).$$

Thus, if the constant c is chosen sufficiently large, $c > 1$,

$$\sum_{\substack{p,m\geqq 2 \\ |f(p^m)|>c}} p^{-m}|f(p^m)|^\alpha \leqq x^{-\alpha} \sum_{p,m\geqq 2} p^{m(\alpha-1)} \left| \sum_{n\leqq x, p^m\| n} f(n) \right|^\alpha \leqq c_{27}.$$

Moreover,

$$\sum_{\substack{p,m\geqq 2 \\ 1<|f(p^m)|\leqq c}} p^{-m}|f(p^m)|^\alpha \leqq c^\alpha \sum_p p^{-1}(p-1)^{-1} = c_{28},$$

which gives the convergence of the second of the two series at (8) in so far as it pertains to prime-powers p^m with $m \geqq 2$.

For $1 < \alpha \leqq 2$ one may continue by an application of Theorem 2, (3), to obtain the convergence of the series $\Sigma p^{-1}|f(p)|^\alpha$, $|f(p)| > 1$. However, an application of the following lemma will enable us to treat every case $\alpha > 0$ at once.

LEMMA 7. *Let $f(n)$ be a real-valued additive arithmetic function. Let $w(x)$ be a real-valued non-decreasing function of $x \geqq 2$, positive for all sufficiently large values of x. Assume that on a sequence of integers $b_1 < b_2 < \ldots$ with*

$$\liminf_{x\to\infty} x^{-1} \sum_{b_i\leqq x} 1 > 0$$

we have

$$|f(n)| \leqq c_1 w(x)$$

for some constant c_1.

Then there is a constant c so that for all large enough values of x

$$\sum_{p\leqq x} \frac{1}{p} \left\| \frac{f(p)}{w(x^c)} \right\|^2 \leqq c_2,$$

where

$$||y|| = \begin{cases} y & \text{if } |y| \leqq 1, \\ 1 & \text{if } |y| > 1. \end{cases}$$

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Proof. This result is proved by Elliott and Erdös [**5**] using the methods of probabilistic number theory.

We continue with our proof of Theorem 4 by noting that if the constant K is fixed at a large enough value

$$x^{-1}\sum_{n\leqq x,\,|f(n)|>K} 1 \leqq K^{-\alpha}x^{-1}\sum_{n\leqq x}|f(n)|^{\alpha} < 1/4$$

so that the hypotheses of Lemma 7 are satisfied with the function $w(x) \equiv 1$ identically. Hence the series

$$\sum_{|f(p)|>1}\frac{1}{p}, \quad \sum_{|f(p)|\leqq 1}\frac{|f(p)|^2}{p}$$

converge.

From Theorem 3, (6), taking for P the set of those primes p such that $|f(p)| > 1$, we deduce that

$$\sum_{p\leqq x,\,p\in P} p^{\alpha-1}\left|\sum_{n\leqq x,\,p\| n} f(n)\right|^{\alpha} \leqq c_{29}x^{\alpha}.$$

For in this case L is uniformly bounded. Arguing as we did for the values $f(p^m)$ with $m \geqq 2$ we deduce the convergence of

$$\sum_{|f(p)|>c} p^{-1}|f(p)|^{\alpha}$$

for some $c > 1$, and then the convergence of

$$\sum_{|f(p)|>1} p^{-1}|f(p)|^{\alpha}.$$

This gives the convergence of both the series at (8). We may now deduce from Lemma 4 that

$$(17)\quad \sum_{n\leqq x}\left|\sum_{p|n,\,|f(p)|\leqq 1} f(p) - F\right|^{\alpha} \leqq c_{13}x\left(\sum_{|f(p)|\leqq 1} p^{-1}|f(p)|^2\right)^{\alpha} \leqq c_{30}x,$$

where

$$F = \sum_{p\leqq x,\,|f(p)|\leqq 1} p^{-1}f(p).$$

Moreover,

$$(18)\quad \sum_{n\leqq x}\left|\sum_{p|n,\,|f(p)|\leqq 1} f(p) - f(n)\right|^{\alpha} \leqq c_{31}x,$$

from an application of Lemma 6. From (17) and (18) we deduce that F is uniformly bounded, and the proof of Theorem 4 is complete.

Remark. Consider the additive function $f(n)$ which is defined by

$$f(p) = (\log\log p)^{-1/2-\epsilon}, \quad 0 < \epsilon < 1/2, \text{ and}$$
$$f(p^m) = 0, \qquad m \geqq 2.$$

For any fixed $\epsilon > 0$, Theorem 4 allows us to assert that

$$\sum_{n\leqq x} |f(n)|^\alpha = O(x), \quad x \geqq 1.$$

In this case

$$\sum_{u\leqq y} f(u) = \sum_{p\leqq y} (\log\log p)^{-1/2-\epsilon}\left[\frac{y}{p}\right]$$
$$= y\left(\frac{(\log\log y)^{1/2-\epsilon}}{1/2-\epsilon} + c_0 + o(1)\right), \quad y\to\infty,$$

for some constant c_0. Hence, for each (fixed) prime p

$$\sum_{\substack{m\leqq p^{-1}x\\(m,p)=1}} f(m) - p^{-1}\sum_{n\leqq x} f(n) = \sum_{m\leqq p^{-1}x} f(m) - p^{-1}\sum_{n\leqq x} f(n) - \sum_{r\leqq p^{-2}x} f(pr)$$
$$= o(p^{-1}x) + O\left((p^{-2}x)^{\alpha-1}\sum_{r\leqq p^{-2}x}|f(r)|^\alpha\right) + O(p^{-2}xf(p))$$
$$= O(p^{-2}x\{1+|f(p)|\}).$$

Suppose now that an inequality of the form (3) holds without the factor $(L+1)^{2-\alpha}$. Setting $a_n = f(n)$ in our hypothetical form of (3) we could deduce that with a suitably chosen positive constant p_0 the sum

$$\sum_{p_0<p\leqq D} p^{-1}|f(p)|^\alpha$$

is bounded uniformly for all $D \geqq 2$. We would obtain in this way the convergence of the series

$$\sum p^{-1}|f(p)|^\alpha = \sum p^{-1}(\log\log p)^{-\alpha(1/2+\epsilon)}.$$

Since $1 < \alpha < 2$ we may fix ϵ at a value so small that $\alpha(1/2+\epsilon) < 1$ and obtain a contradiction.

This argument shows that the factor $(L+1)^{2-\alpha}$ in the inequality (3) cannot be entirely removed.

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