ON ORDER IN A PLANE

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To Professor H. S. M. Coxeter
on the occasion of his sixtieth birthday

When a set of axioms is laid down as the basis of any mathematical doctrine, it must be proved that this set never leads to a contradiction. In this note we turn the question around. A set of axioms is given and we wish to adjoin an axiom of a specified type. How far does the demand of non-contradiction limit the choice of the new axiom?

1. As an example of this question we take the axioms of order in a plane. The set of axioms already laid down shall be such that the following hold. There is just one line which contains two given distinct points; if A, B, C are distinct points on a line, and $[ABC]$ holds (this means that B is between A and C), then $[CBA]$ is true, but $[CAB]$, $[BAC]$, $[ACB]$, $[BCA]$ are all false; finally, for any three distinct points on a line two of these relations hold.

We do not assume any theorem on the orders of four points on a line; for example, we do not assume that $[ABC]$ and $[BCD]$ imply $[ACD]$.

We now wish to adjoin an axiom about the order in a plane. Veblen's axiom runs as follows:

If ABC is a triangle, and $[BCD]$, $[CEA]$ hold, then there is a point F on the line DE such that $[AFB]$ holds.

The figure with these three-point relations we call Veblen's figure. We could weaken this axiom and assume only: (A) There is at least one figure consisting of a triangle and a transversal meeting all three side-lines, at least two sides internally.

We make no demands on the order of the points on the transversal. We call this the "weak" form of the axiom, and the axiom as formulated above the "strong" form.

If no such figure exists, then:

If ABC is any triangle, and E is on the side-line AC distinct from A, C, and F is on the side-line AB distinct from A, B, then EF, BC never meet.

2. Take a triangle and a line not through a vertex meeting the side-lines of the triangle. Name the points of meeting D, E, F so that $[DEF]$ holds, and name the vertices of the triangle A, B, C so that D, E, F are respectively on the

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side-lines BC, CA, AB. Thus, in the absence of any axiom, such as Veblen's, we might have $[BCD]$, $[CEA]$, and $[ABF]$.

Take the vertices so named of the triangle, in order A, B, C and traverse the sides in directions BC, CA, AB. Each side so traversed is divided, apart from the vertices, into three parts; number these 1, 2, 3. Thus, for example, the parts 1, 2, 3 on AB will be the set of points X which satisfy respectively $[XAB]$, $[AXB]$, $[ABX]$. Then Veblen's axiom acquires the symbol 322, and the example 323. If we traverse the sides in the contrary direction, these "numeric" symbols are replaced by 221 and 121 respectively. We call the symbols 322 and 221 "equivalent," and write $322 = 221, 323 = 121$.

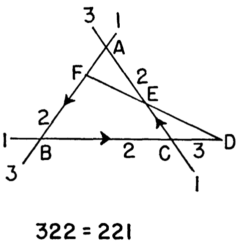

FIGURE 1

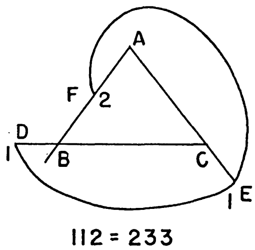

FIGURE 2

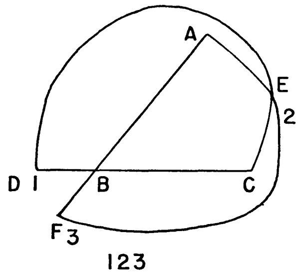

FIGURE 3

Taking the weak form of Veblen's axiom and the example, we also denote the figures, less completely, by the "literal" symbols IIE, IEE respectively, the last meaning that the transversal cuts the first side internally and the other two externally. We can use such numeric and literal symbols for any triangle and transversal.

3. We now restrict the possible cases. Two literal symbols are "contradictory" if they agree in two letters in the same position and disagree in the third. Thus IIE and III are contradictory, while IIE does not contradict any of EII, EEE, IEI. Two non-equivalent numeric symbols are "contradictory" if they agree in two numbers in the same position and disagree in the third.

We note that 123, 121 are contradictory though both are EIE, while 123, 322 are non-contradictory though the corresponding EIE, EII are contradictory.

4. Since two points define a line, it is natural to demand a postulate of uniformity. Two suggestions are:

(U) No two figures consisting of a triangle and a transversal shall have contradictory literal symbols.

(U_1) The same postulate for numeric symbols.

We first discuss a single figure made up of a triangle ABC and a transversal DEF with letters as in §2. We apply (U_1) to this figure and indicate when conflict occurs. If instead we take (U), we get the same cases of conflict.

The census of such figures is given in the following table. The first column gives the symbols for the triangle ABC and the transversal DEF, the other columns for the triangles at the head and their transversal.

ABC	DEC	EFA	FDB	ABC	DEC	EFA	FDB	ABC	DEC	EFA	FDB
111*	211*	132	212	123	123	123	123	221	131	131	221
112	112	112	112	131	221	131	221	222	311	311*	321*
113	121*	122*	231	132	212*	111	211*	231	121*	113	122*
121*	231	113	122*	211*	111	132	212*	311*	311	321*	222
122*	231	113	121*	212*	111	211*	321	321*	311*	311	222

The asterisks denote the contradictory figures. Thus the only figures which survive (U_1) are 131, 221, both Euclidean, and the exceptional figures 112 and 123. They are also the only figures which survive (U).

If we take a strong form of 111—say if $[DBC]$, $[ECA]$, then there is a point F satisfying $[DEF]$, $[FAB]$—we get an immediate contradiction, since then $[ECA]$, $[FAB]$ would imply $[EFD]$. Similarly, the strong form of 222—if $[BDC]$, $[CEA]$, then there is an F satisfying $[AFB]$—gives an immediate contradiction. Thus, taking a single figure, the only survivors besides the Euclidean cases are $112 = 233$ and 123.

5. Assume now (A) of §1, the weak form of Veblen's axiom. The transversal cuts the two sides internally. The possible cases are 122, 212, 222, all rejected, and 221; *IIE*. This is the case of Veblen's figure $[BCD]$, $[CEA]$, $[AFB]$, $[DEF]$.

Now assume (U) in full strength, and apply it to two figures instead of to only one. This rules out the non-contradictory cases in §4 apart from the Euclidean, since 112 and 123 are of types *EEI*, *EIE*, which contradict *EII*.

6. But *EEE* does not contradict *EII*. We consider how it arises. Let $[CBD]$, $[CAE]$ and suppose the line DE meets the side-line AB in F. Such a figure exists by (A). We must now allow all orders of D, E, F on DE. If $[DEF]$ holds, then $[ABF]$, $[AFB]$ yield respectively 113 and 132, and both are impossible. Hence $[DEF]$ implies $[FAB]$. If $[FDE]$ holds, then $[FAB]$ and also $[AFB]$ yield the same impossible cases. Hence $[FDE]$ implies $[ABF]$. If $[DFE]$ holds, then $[ABF]$, $[AFB]$, $[FAB]$ imply respectively 311, 321, 331, all impossible. This rules out $[DFE]$ and leaves only the cases $[CBD]$, $[CAE]$ with $[DEF]$, $[FAB]$ or $[FDE]$, $[ABF]$, which are Veblen's figure apart from notation.

7. Consider now that part of Veblen's axiom in its strong form which asserts that:

(a) If $[BCD]$, $[CEA]$, then DE meets AB in F, say. Then if we assume (U), this gives $[AFB]$, $[DEF]$ and hence Veblen's axiom.

Suppose we replace (a) by:

(b) If [BCD], [AFB], then DF meets AC in E, say.

It is known **(1)** that from Veblen's axiom follow the usual theorems on orders of four collinear points and also (b). Thus (a) implies (b), and also [CEA] if (U) is assumed.

If we assume (b), we must consider all orders of D, E, F on DF. Then, according to the position of E on the line AC, we have the cases: If [DEF], 231, 221, 211; if [DFE], 123, 122, 121; if [EDF], 112, 212, 112. Thus apart from the Euclidean case, 221, we could also have 123 and 112. Hence unless we assume (U) in full strength we cannot replace (a) by (b).

If we replace (a) by:

(c) if [AFB], [CEA], then FE and BC meet in D, say,

then, as [BDC] is impossible, we have [BCD] or [CBD]. Interchange of B, C makes these equivalent, though incompatible.

Thus Veblen's axiom in its strong form (a) cannot be replaced by (b) or (c).

Similarly, in the EEE figure of §6, the only possible existence postulate is: If D, F are given, then E exists. For if we assume that if E, F are given then D exists, we interchange B, C; and if we assume that if D, E are given then F exists, we interchange A, B. These lead to a contradiction.

Our final result is: If we assume (U) or (U_1) in its full strength and (A), then Veblen's axiom is the only possible axiom of its type.

This investigation was begun long ago, laid aside, and completed for this note. A preliminary account of some results was given in **(1)**. Some similar questions were treated in **(2)**.

References

1. H. G. Forder, *The foundations of Euclidean geometry* (1927) (Dover reprint, 1958).
2. M. Geiger, *Systematische Axiomatik* (1924).

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