CORRESPONDENCE.

To the Editor of the *Mathematical Gazette.*

LOGARITHMS.

In the last two numbers of the *Mathematical Gazette* Dr. Henderson and Mr. Fletcher have traversed some of the ground so ably covered by Dr. Hutton in his *History of the Calculation of Mathematical Tables* which forms an introduction of 125 pages to his famous tables.

When I got these tables a good many years ago I was much attracted by the ingenuity of Briggs's method of calculating the logarithms to base 10 of prime numbers described by Dr. Henderson on p. 254 of this journal. Hutton, p. 63, says . . . " only a few of the first figures of the powers in the first column are retained, those being sufficient to determine the number of places in them " ; . . . " Indeed these multiplications might be performed in the same manner, retaining only the first three figures and those to the nearest unit in the third place ; which would make this a very easy way indeed of finding the logarithms of a few prime numbers."

When I read this I asked myself what becomes of the errors which, + or −, may amount to 5 in the 6th place. The processes of squaring and multiplying will cause these to grow at first slowly but later with terrible acceleration, and soon the number of figures will be wrong. Happily Hutton had committed himself to the 3 and 4 digit fallacy, and the first of these was quickly disposed of with a 50 cm. slide rule, which gives the third digit with certainty even though the first digit is 9. Barlow's tables provided the means for disposing of the 4-digit fallacy nearly as easily. It was obvious then that with 5 or even more digits only retained in the operations the errors would invade the first digit in time.

In the case of $\log_{10} 2$ the whole disturbance hangs on to the square of 1024, which is the first figure cut down to 5 digits. The true figure is 1048576, so 10486 is taken, which is 0·24 too much. This is the quantity which will increase like leaven, and it is quite immaterial if it is succeeded by negative errors. This gets the start and prevails.

I have calculated the values of the really true first *five* digits for the powers of 2 on p. 63 of Hutton's introduction. These are only carried up to $2^{1,000,000,000}$, sufficient to give the logarithm of 2 to nine places only. In nearly every case Hutton's figure for the first 5 digits of the power is exact, but is in three cases in error by −1. If the process is carried out with 5 figures only, taking always the nearest fifth digit in each square or product, the growth of error increases with increasing rapidity. Thus taking only the powers of 2, which are the 10th, 100th, 1000th, 10000th and 100000th, the errors are respectively 0, $2\frac{1}{2}$, 22, 409, and one digit too many, so at this point the process has failed.

It is certain that Briggs himself, who seemed in no way dismayed by the prospect of doing innumerable long multiplication or division sums with numbers of 20 or 30 digits, did use far more than five digits for his calculations, but in explaining the process I expect he only gave the first few digits merely as an indication, and that Hutton jumped to the conclusion that these only were needed and went further and proclaimed that four or three digits only were sufficient.

However, the matter may be looked at in a far more fundamental or general way. Suppose it is desired to find $\log_{10} 2$ to thirty places. The 100th power of 2 has 31 digits, and its logarithm = 100 times log 2. So its logarithm must be found to 28 places. The Hutton fallacy asserts that in order to do this it is immaterial whether all the digits in 2^{100} are utilised in the calculation or all are discarded except the first 5 (or 4 or 3) ! The fact is that at the beginning of the process the quantities must be known to at least the number of places desired in the logarithm, and then at each 10-fold increase in the index, when

Log 1·3712 88574 23862 35368 61362 10629 96899 58844 85848 46566 95033 93510 0
=·13712 88574 23862 35368 61362 10629 96899 58844 85848 46566 95033 93509 9

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another figure has been established and those remaining are one less in number, one digit can be dropped, until at the end of the process only 3, 4 or 5 remain. I have no access to Briggs's original papers, but I suggest as a possibility that Briggs indicated this, and that Hutton did not distinguish between the tail end of the process and the whole process. I would add that the process is no more convenient for " small " primes than for any primes.

Having thus found that Hutton's statement that three or four digits would be sufficient, or even that five would be, was wrong, I made a note in my copy to that effect, but I did not publish the matter, as a 120-year old error in an obsolete process did not seem worth raking up, but now that Dr. Henderson is using your valued columns to perpetuate this error and not one of your mathematical readers has corrected him, I think it well to put the matter right.

While on the subject of logarithms I should like, if this letter is not already too long, to refer to some correspondence in the *English Mechanic* which began in November 1917. Mr. E. M. Nelson gave the Oliver Byrne number 13712 ... to 16 digits. I was interested in this sufficiently to verify the accuracy of the statement that this number and its logarithm to base 10 had the same digits. I found only the first 13 correct, and so I showed how, knowing the gradient of the curve $y=\log_{10}x$ and of $y=x/10$ for any value of x, the difference of the gradients is known, and this is the convergence or divergence, as the case may be, between the two curves. If, then, at any point x the number and its supposed logarithm differ at all, this divergence gives the point where they agree. If the error is accurately determined, each new application of the process will about double the number of correct digits. I accordingly calculated this number to 20 places. I then noticed that the first eight digits were exactly equal to $277 \times 1000001 \times 5 \div 101$, all of which have their logarithms given in Hutton's table to 61 figures. So I then calculated the number to 40 figures by two methods, one using Hutton's tables and interpolation, and the other and much shorter by finding the Naperian logarithm after dividing by the number expressed by the first eight digits. This was so rapidly convergent that the 40 places were found, and then being multiplied by ·434 ..., to 40 places, the logarithm to base 10 was found. It was satisfactory to find that the logarithm and the number differed only by 1 in the 40th place. I then said that the same process repeated would with the aid of Hutton's 61-figure table give the first 61 figures. In the following March I received from Mr. W. O. Murdoch of Aberdeen the result to 61 places, again the divergence is 1 in the 61st place. These have never been published, and it is possible you may care to put them on permanent record. I quite realise that these very accurate determinations are not of any " use ", and that such calculations are as futile and as fascinating as crossword puzzles or games of patience with cards, and it must be remembered that people deemed otherwise sane are guilty of these futilities. C. V. BOYS.

I am exceedingly grateful to Prof. E. H. Neville for obtaining the loan for me of Oliver Byrne's Method of Calculating Logarithms (1849). The book is very interesting in revealing Byrne as a clever arithmetician and crank at enmity with the orthodox and unsparing in his observations. He shows how he calculated his series of 'Oliver Byrne' numbers beginning with the 1·371... number, which he says he gives correctly to 16 places (p. 68). Of these the last three are wrong if my result confirmed by Mr. Murdoch is right! So apparently Mr. Nelson correctly gave his incorrect result.

ERRATA.

Vol. XV. p. 300, l. 2 up. *Delete final* d.

p. 303, l. 5. *For* one-twelfth *read* twelve.

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