# COMPLETE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE IN A SPHERE

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**1. Introduction.** Let $M^n$ be an $n$-dimensional manifold immersed in an $(n+p)$-dimensional unit sphere $S^{n+p}$, with mean curvature $H$ and second fundamental form $B$. We put $\phi(X, Y) = B(X, Y) - \langle X, Y\rangle H$ where $X$ and $Y$ are tangent vector fields on $M^n$. Assume that the mean curvature is parallel in the normal bundle of $M^n$ in $S^{n+p}$. Following Alencar and do Carmo [**1**] we denote by $B_H$ the square of the positive root of

$$t^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\,t - n(|H|^2+1) = 0.$$

Alencar and do Carmo [**1**] proved that if $M^n$ is compact, $p=1$ and $|\phi|^2 \le B_H$, then either $|\phi|^2 = 0$ (and $M^n$ is totally umbilic) or $|\phi|^2 = B_H$ and $M^n$ is a Clifford torus or an $H(r)$-torus of appropriate radii. The case of compact submanifolds of codimension $p>1$ was considered by Xu [**3**].

The purpose of this paper is to generalize the results due to Alencar, do Carmo and Xu to complete submanifolds.

THEOREM 1.1. *Let $M^n$ be a complete submanifold with parallel mean curvature $(H \neq 0)$ in $S^{n+p}$. Then either $M^n$ is pseudoumbilical, or* $\sup|\phi|^2 \ge B_H$.

THEOREM 1.2. *Let $M^n$ be a complete submanifold with parallel mean curvature $(H \neq 0)$ in $S^{n+p} (p \ge 2)$. Then either $M^n$ is a totally umbilical sphere, or* $\sup|\phi|^2 \ge C_H$, *where* $C_H = \min\left\{B_H, \dfrac{n(|H|^2+1)}{1+\frac{1}{2}\mathrm{sgn}(p-2)}\right\}$.

As a consequence of the above results we have the following partial answers to a question of Alencar and do Carmo [**1**].

COROLLARY 1.3. *Let $M^n$ be a complete submanifold with parallel mean curvature $H \neq 0$ in $S^{n+p}$. If $|\phi|^2 =$ constant and $M^n$ is not pseudoumbilical, then $|\phi|^2 \ge B_H$.*

COROLLARY 1.4. *Let $M^n$ be a complete submanifold with parallel mean curvature $H \neq 0$ in $S^{n+p}(p\ge 2)$. If $|\phi|^2 =$ constant, then either $M^n$ is a totally umbilical sphere, or $|\phi|^2 \ge C_H$.*

**2. Preliminaries.** Let $M^n$ be a submanifold in $S^{n+p}$. Choose a local orthonormal frame field $\{e_1, \ldots, e_{n+p}\}$ in $S^{n+p}$ such that when restricting on $M^n$, $\{e_1, \ldots, e_n\}$ are tangent to $M^n$ and $\{e_{n+1}, \ldots, e_{n+p}\}$ are normal to $M^n$. The mean curvature $H$ is defined by

$$H = \frac{1}{n}\sum_{i=1}^{n} B(e_i, e_i),$$

and the square length of the second fundamental form $B$ is defined by

$$S = \sum_{i,j=1}^{n} |B(e_i, e_j)|^2.$$

It is easy to see that the square length of the tensor $\phi$ is given by

$$|\phi|^2 = S - n\,|H|^2. \tag{2.1}$$

The Weingarten map associated with $e_\alpha (\alpha \geq n+1)$ is denoted by $A_\alpha$.

Now we assume that $M^n$ is a submanifold with parallel mean curvature $H \neq 0$, that is $H$ is parallel in the normal bundle. We choose $e_{n+1}$ such that $H \parallel e_{n+1}$. We consider the linear transformation $\phi_{n+1}: T_PM^n \to T_PM^n$ of the tangent space $T_PM^n$ at the point $P$, given by $\phi_{n+1} = A_{n+1} - |H|\,I$, where $I$ denotes the identity map. It is easily verified that

$$|\phi_{n+1}|^2 = |A_{n+1}|^2 - n\,|H|^2. \tag{2.2}$$

Obviously, $M^n$ is pseudoumbilical in $S^{n+p}$ if and only if $|\phi_{n+1}|^2 = 0$.

Following the computation in [3] and taking into account (2.1) and (2.2), we have

$$\tfrac{1}{2}\Delta\,|\phi_{n+1}|^2 \geq \sum_{i,j,k} (h_{ijk}^{n+1})^2 + |\phi_{n+1}|^2\left(-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\,|\phi_{n+1}| + n(|H|^2+1)\right). \tag{2.3}$$

Moreover we have [3, p. 494]

$$\begin{aligned}\tfrac{1}{2}\Delta(|\phi|^2 - |\phi_{n+1}|^2) \geq & \sum_{\substack{i,j,k\\ \alpha\neq n+1}} (h_{ijk}^\alpha)^2 + n\,|H| \sum_{\alpha\neq n+1} \operatorname{tr}(A_{n+1}A_\alpha^2)\\ & - \sum_{\alpha \neq n+1} (\operatorname{tr}(A_{n+1}A_\alpha))^2 + n(|\phi|^2 - |\phi_{n+1}|^2)\\ & - (1 + \tfrac{1}{2}\operatorname{sgn}(p-2))(|\phi|^2 - |\phi_{n+1}|^2)^2, \quad p \geq 2,\end{aligned} \tag{2.4}$$

where $h_{ij}^\alpha$ are the components of the second fundamental form and $h_{ijk}^\alpha$ are the covariant derivatives of $h_{ij}^\alpha$.

## 3. Proofs of the theorems.

First we state two lemmas.

Lemma 3.1 ([2]). *Let $M^n$ be a submanifold in $S^{n+p}$, and let* Ric *denote the minimum Ricci curvature at each point. Then*

$$\mathrm{Ric} \geq \frac{n-1}{n}\left(-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\,|\phi| + n(|H|^2+1)\right).$$

*Proof.* It follows immediately from the main theorem in [2] and (2.1).

Lemma 3.2 ([4]). *Let $M^n$ be a complete Riemannian manifold with Ricci curvature bounded from below. Let $f$ be a $C^2$-function bounded from above on $M^n$. Then there exists a sequence $\{P_m\}$ of points in $M^n$ such that*

$$\lim f(P_m) = \sup f,\ \lim |\operatorname{grad} f|\,(P_m) = 0 \quad \textit{and} \quad \limsup \Delta f(P_m) \leq 0.$$

*Proof of Theorem* 1.1. Assume that $M^n$ is not pseudoumbilical and $\sup|\phi|^2 < B_H$. Then, from Lemma 3.1 we conclude that the Ricci curvature of $M^n$ is bounded from below. Since $|\phi_{n+1}|^2 \le |\phi|^2$, $|\phi_{n+1}|^2$ is bounded from above. Hence, from Lemma 3.2 there exists a sequence $\{P_m\}$ in $M^n$ such that

$$\lim |\phi_{n+1}|^2(P_m) = \sup |\phi_{n+1}|^2 \tag{3.1}$$

and

$$\limsup \Delta |\phi_{n+1}|^2(P_m) \le 0. \tag{3.2}$$

Since $|\phi|^2$ is bounded, $|\phi|^2(P_m)$ is a bounded sequence. Therefore, there exists a subsequence $\{P_{m'}\}$ of $\{P_m\}$ such that

$$\lim |\phi|^2(P_{m'}) = l^2 \tag{3.3}$$

for some $l \ge 0$. From (2.3) we get

$$\tfrac{1}{2}\Delta |\phi_{n+1}|^2 \ge |\phi_{n+1}|^2\Big(-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\,|\phi| + n(|H|^2+1)\Big). \tag{3.4}$$

Taking into account (3.1), (3.2) and (3.3), the inequality (3.4) gives rise to the inequality

$$\sup |\phi_{n+1}|^2\Big(-l^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\,l + n(|H|^2+1)\Big) \le 0.$$

Since $M^n$ is not pseudoumbilical, we get $l^2 \ge B_H$. Thus, we have $\sup |\phi|^2 \ge l^2 \ge B_H$ which contradicts our assumption. This completes the proof.

*Proof of Theorem* 1.2. Assume that $\sup |\phi|^2 < C_H$. Then, Theorem 1.1 implies that $M^n$ is pseudoumbilical. By virtue of $\operatorname{tr} A_\alpha = 0$, $(\alpha \ge n+2)$, and (2.1), the inequality (2.4) yields the inequality

$$\tfrac{1}{2}\Delta |\phi|^2 \ge |\phi|^2(n(|H|^2+1) - (1 + \tfrac{1}{2}\operatorname{sgn}(p-2))\,|\phi|^2). \tag{3.5}$$

By our assumption on $\sup |\phi|^2$ and applying Lemma 3.1, we deduce that the Ricci curvature of $M^n$ is bounded from below. Since $|\phi|^2$ is bounded from above, according to Lemma 3.2, there exists a sequence $\{P_m\}$ in $M^n$ such that

$$\lim |\phi|^2(P_m) = \sup |\phi|^2 \tag{3.6}$$

and

$$\limsup \Delta |\phi|^2(P_m) \le 0. \tag{3.7}$$

From (3.5), (3.6) and (3.7) we get

$$\sup |\phi|^2(n(|H|^2+1) - (1 + \tfrac{1}{2}\operatorname{sgn}(p-2))\sup |\phi|^2) \le 0.$$

Hence $|\phi|^2 = 0$, which says that $M^n$ lies in a totally geodesic sphere $S^{n+1}$ and $M^n$ is a totally umbilical sphere.

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