FINITE GROUPS WHICH CONTAIN A SELF-CENTRALIZING SUBGROUP OF ORDER 3

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Dedicated to RICHARD BRAUER on his sixtieth birthday

§ 1. Introduction

The polyhedral group (l, m, n) is defined in [3] by the presentation

$$(l, m, n) = \langle x, y, z \mid x^l = y^m = z^n = xyz = 1 \rangle.$$

It is known ([3] page 68) that (l, m, n) is finite if and only if

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1.$$

The groups $(2, 2, n)$ and $(1, n, n)$ are respectively the dihedral group of order $2n$ and the cyclic group of order n. Using the above mentioned criterion it can be shown that the list of finite polyhedral groups is completed by including

$$\mathfrak{A}_4 = (2, 3, 3), \ \mathfrak{S}_4 = (2, 3, 4) \text{ and } \mathfrak{A}_5 = (2, 3, 5).$$

Let G be a finite group. If C_1, C_2, C_3 are three conjugate classes of G which contain elements of order l, m, n respectively and if K_1, K_2, K_3 are the corresponding class sums in the group ring of G, a moment's reflection reveals that in order to compute the multiplicity of K_3 in K_1K_2 by group theoretic methods as distinct from character theoretic methods it is necessary to deal with factor groups of (l, m, n). R. Brauer and K. A. Fowler [1] first realized the importance of this idea for studying finite groups. They were only concerned with the groups $(2, 2, n)$ but these were sufficient to prove some powerful results about groups of even order. Using the groups $(2, 2, n)$ this idea has been used by many authors in recent years and has proved very fruitful for

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the study of groups of even order. It is unlikely that knowledge about the other polythedral groups can be utilized as widely as that for the groups (2, 2, n). However, the other polyhedral groups can surely play a role in group theory which is not totally eclipsed by the groups (2, 2, n).

The purpose of this paper is to illustrate how the above mentioned method can be used with the group (3, 3, 3). By the result referred to above the group (3, 3, 3) is infinite. However, it is manageable since, as is shown in section 2, it has an abelian commutator subgroup.

The following result will be proved in this paper.

THEOREM. *Let G be a finite group which contains a self-centralizing subgroup of order* 3. *Then one of the following statements is true.*

(*I*) *G contains a nilpotent normal subgroup N such that G/N is isomorphic to either $\mathfrak{A}_3$ or $\mathfrak{S}_3$.*

(*II*) *G contains a normal subgroup N which is a* 2-*group such that G/N is isomorphic to $\mathfrak{A}_5$.*

(*III*) *G is isomorphic to PSL* (2, 7).

As an immediate consequence of this theorem we get

COROLLARY. *Let G be a non-cyclic simple group which contains a self-centralizing subgroup of order* 3. *Then G is isomorphic to $\mathfrak{A}_5$ or PSL* (2, 7).

If A is a subset of the group G then $C(A)$, $N(A)$, $\langle A \rangle$, $|A|$ will denote respectively the centralizer of A, normalizer of A, group generated by A and the number of elements in A. $H \lhd G$ means that H is a normal subgroup of G. If p is a prime then a S_p subgroup of G is a Sylow p-subgroup of G. Elements of order two are called involutions. For any subgroup H of G, 1_H denotes the principal character of H. If α is a class function of H then α^* denotes the class function of G induced by α.

§2. The Group (3, 3, 3).

THEOREM 1. *The group* (3, 3, 3) *possesses a normal abelian subgroup of index* 3.

Proof. Let

$$(3, 3, 3) = \langle x, y \mid x^3 = y^3 = (xy)^3 = 1 \rangle.$$

The relation $(xy)^3 = 1$ can be rewritten as

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$$xyx = y^{-1}x^{-1}y^{-1}.$$

Since $y^{-2} = y$ and $x^{-2} = x$ this implies that

$$xy^{-1}y^{-1}x = y^{-1}xxy^{-1}.$$

Hence xy^{-1} and $y^{-1}x$ commute. Conjugating this relation by x and x^2 yields

$$y^{-1}x \cdot x^{-1}y^{-1}x^{-1} = x^{-1}y^{-1}x^{-1} \cdot y^{-1}x$$
$$x^{-1}y^{-1}x^{-1} \cdot xy^{-1} = xy^{-1} \cdot x^{-1}y^{-1}x^{-1}.$$

Thus $H = \langle xy^{-1}, y^{-1}x, x^{-1}y^{-1}x^{-1}\rangle$ is abelian. Since x permutes the three elements xy^{-1}, $y^{-1}x$, $x^{-1}y^{-1}x^{-1}$ cyclically x normalizes H. Hence y also normalizes H as $xy^{-1} \in H$. Thus H is a normal subgroup of $(3, 3, 3)$. Since $(3, 3, 3)$ can be mapped homomorphically onto a non abelian group of order 27, H is a proper subgroup. As $xy^{-1} \in H$ and $x \notin H$, H has index 3 as required.

§ 3. Proof of the Theorem

Throughout this section let G be a counter-example of minimum order to the theorem stated in section 1. Let x be an element of G such that $x^3 = 1$ and $C(x) = \langle x \rangle$. It is easily seen that $\langle x \rangle$ is a S_3 subgroup of G. We will eventually derive a contradiction from the assumed existence of G. This will be done in a series of Lemmas.

LEMMA 1. *G is a non-cyclic simple group.*

Proof. Suppose this is not the case and let H be a minimal normal subgroup of G. Suppose that 3 divides $|H|$. Then the Sylow theorems imply that $G = N(\langle x \rangle)H$. Thus $[G:H] = 2$ and $N(\langle x \rangle) \cap H = \langle x \rangle$. Hence by Burnside's transfer theorem H contains a normal 3-complement H_0. Thus $H_0 \triangleleft G$ and the minimality of H implies that $H_0 = 1$. Consequently G is isomorphic to $\mathfrak{S}_3$ contrary to assumption.

Assume now that 3 divides $[G:H]$. Then $\langle x \rangle H$ is a Frobenius group. Thus H is nilpotent ([2], page 91). It is easily seen that G/H satisfies the hypotheses of the theorem stated in section 1. Thus by induction G/H satisfies condition (I), (II), or (III). Therefore G contains a normal subgroup N such that G/N is isomorphic to $\mathfrak{A}_3$, $\mathfrak{S}_3$, $\mathfrak{A}_5$ or $PSL(2, 7)$. In any case $\langle x \rangle N$ is a Frobenius group and N is nilpotent ([2], page 91). If G/N is isomorphic to $\mathfrak{A}_3$ or $\mathfrak{S}_3$ nothing remains to be proved.

Let p be a prime dividing $|N|$. We will show that $p=2$ if G/N is isomorphic to $\mathfrak{A}_5$ while $|N|=1$ if G/N is isomorphic to $PSL(2, 7)$. By induction it may be assumed that N is an elementary abelian p-group. Suppose that A is a subgroup of G such that $A \cap N=1$, $A \neq A'$, $[A : A']=3$ and p does not divide $|A|$. Then A is a Frobenius group acting on N. Since x has no fixed points on N we get that A' acts trivially on N. Thus $N \subset C(N)$. Since $C(N) \lhd G$ and $\mathfrak{A}_5$, $PSL(2,7)$ are simple, this yields that $C(N)=G$. Thus $N \subseteq C(x)$ or $N=1$. As both $\mathfrak{A}_5$ and $PSL(2,7)$ contain a subgroup A which is isomorphic to $\mathfrak{A}_4$ this implies that $p=2$. As $PSL(2,7)$ contains a nonabelian subgroup of order 21 we get that $N=1$ in this case. The proof is complete in all cases.

LEMMA 2. *G contains only one conjugate class of elements of order three, and $|G|$ is even.*

Proof. Lemma 1 and Burnside's transfer theorem imply that $N(\langle x\rangle) \neq \langle x\rangle$. Thus $|N(\langle x\rangle)|=6$. The result is immediate.

LEMMA 3. *There exist exactly two non-principal irreducible characters θ, χ of G which do not vanish on x. They can be chosen so that $\theta(x)=1$, $\chi(x)=-1$ and $1+\theta(y)-\chi(y)=0$ for y not conjugate to x.*

Proof. Let λ be a nonprincipal irreducible character of $\langle x\rangle$. Let α be the generalized character of $N(\langle x\rangle)$ induced by $1_{\langle x\rangle}-\lambda$. Then it is easily seen that $\|\alpha^*\|^2=3$ and

$$(1) \qquad \begin{aligned} \alpha^*(x) &= \alpha(x)=3 \\ \alpha^*(y) &= 0 \qquad \text{for } y \text{ not conjugate to } x. \end{aligned}$$

Consequently $\alpha^*=1_G+\theta-\chi$, where χ, θ are distinct nonprincipal irreducible characters of G. Furthermore $1+\theta(y)-\chi(y)=0$ for y non conjugate to x. Furthermore by (1)

$$1=(\theta, \alpha^*)=\frac{1}{6}3\{\theta(x^{-1})+\theta(x)\}=\theta(x)$$

Thus by (1) $\chi(x)=-1$. Consequently

$$|C(x)|=3=1+|\theta(x)|^2+|\chi(x)|^2.$$

Hence the orthogonality relations imply that every irreducible character of G, distinct from 1_G, χ and θ vanishes on x. The proof is complete.

The next lemma is due to R. Brauer and M. Suzuki. We are indebted to them for informing us of the result.

LEMMA 4. *G contains exactly one class of involutions.*

Proof. Any two involutions which normalize a subgroup of G of order 3 are conjugate. Suppose that G contains two classes of involutions then there is one class containing involutions such that uv is not conjugate to x for any u, v in that class. Let C be the group algebra sum of this class of involutions and let K be the group algebra sum of the elements of order 3. Thus the coefficient of K in C^2 is zero. Hence by a well-known formula ([2], page 316)

$$\frac{|G|}{|C(u)|^2}\left[\sum\frac{\zeta_i(u)^2\overline{\zeta_i(x)}}{\zeta_i(1)}\right]=0,$$

where ζ_i ranges over all the irreducible characters of G. In view of Lemma 3 this implies that

$$1+\frac{\theta(u)^2}{\theta(1)}-\frac{\{\theta(u)+1\}^2}{\theta(1)+1}=0.$$

Therefore

$$\theta(1)^2+\theta(1)+\theta(1)\theta(u)^2+\theta(u)^2-\theta(1)\theta(u)^2-2\,\theta(1)\theta(u)-\theta(1)=0,$$

or equivalently

$$\theta(1)^2-2\,\theta(1)\theta(u)+\theta(u)^2=0.$$

Thus $\{\theta(1)-\theta(u)\}^2=0$ and $\theta(1)=\theta(u)$. This implies that u lies in a proper normal subgroup of G contrary to Lemma 1. The proof is complete.

Throughout the rest of this paper the following notation will be used.

K is the group algebra sum of all elements of order 3 in G.

C is the group algebra sum of all involutions in G.

u is a fixed involution in G.

$M_1, \ldots, M_{s+m}$ is a complete set of representatives of the conjugate classes of maximal solvable subgroups of G whose order is divisible by 3. By induction each M_i contains a normal nilpotent subgroup N_i. The notation is chosen so that

$$M_i/N_i \text{ is isomorphic to } \mathfrak{A}_3 \qquad \text{for } 1\leq i\leq k$$

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$$M_i/N_i \text{ is isomorphic to } \mathfrak{S}_3 \qquad \text{for } k<i\leq s+m$$

where $|N_i|$ is odd for $k+1\leq i\leq s$ and $|N_i|$ is even for $s+1\leq i\leq s+m$.

Let $N_i=H_i\times T_i$, where $|H_i|$ is odd and T_i is a 2-group. Define

$$h_i=|H_i|,\ t_i=|T_i| \qquad \text{for } 1\leq i\leq s+m.$$

LEMMA 5. *H_i is a Hall subgroup of G for $1\leq i\leq s+m$ and $(h_i, h_j)=1$ for $1\leq i<j\leq s+m$. N_i is a Hall subgroup of G for $1\leq i\leq k$ and $(|N_i|, |N_j|)=1$ for $1\leq i<j\leq k$.*

Proof. Let P be a S_p subgroup of N_i for some prime p. Lemma 1 and the maximality of M_i imply by induction that $N(P)=M_i$ if $p>2$ or if $1\leq i\leq k$. Thus in these cases P is a S_p subgroup of G. Hence H_i, N_i are Hall subgroups for $1\leq i\leq s+m$, $1\leq i\leq k$ respectively. If one of the other statements of the Lemma is false it may be assumed by taking conjugates that for some S_p subgroup P of G, $P\subseteq H_i\cap H_j$, $i\neq j$, or $P\subseteq N_i\cap N_j$ and $1\leq i<j\leq k$. Hence in either case $\langle M_i, M_j\rangle\subseteq N(P)$. By the first part of the lemma this implies that $M_i=M_j$ contrary to the definition of the groups M_i.

Lemma 5 yields that

$$g=|G|=3\cdot 2^n g_0\prod_{i=1}^{s+m}h_i, \qquad (g_0, 6)=1 \tag{2}$$

Furthermore $t_i\neq 1$ for at most one value of i with $1\leq i\leq k$. Choose the notation so that

$$\begin{aligned} &t_1=1 \text{ or } t_1=2^n \\ &t_i=1 \text{ for } 2\leq i\leq k \qquad (3)\\ &t_i\neq 1 \text{ for } s+1\leq i\leq s+m. \end{aligned}$$

$$h_{s+1}\geq h_i \qquad \text{for } s+1\leq i\leq s+m. \tag{4}$$

LEMMA 6.

$$\frac{g}{9}<\frac{g}{9}\left\{1+\frac{1}{\theta(1)}-\frac{1}{\theta(1)+1}\right\}\leq 1+2\sum_{i=1}^{k}(h_it_i-1)+\sum_{i=k+1}^{s+m}(h_it_i-1) \tag{5}$$

Proof. The first inequality is trivial. By Lemma 3 the second term in (5) is the multiplicity of K in K^2. Thus the second term in (5) is the number of ordered pairs (y, z) with $yz=x$ and y, z of order 3. Since $\langle y, z\rangle$ is a homomorphic image of (3, 3, 3) it is solvable by Theorem 1. Thus for every such pair, $\langle y, z\rangle$ is contained in a conjugate of some M_i, $1\leq i\leq s+m$.

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Suppose that $x \in M_i \cap w^{-1}M_i w$ for some $w \in G$. Then $wxw^{-1} \in M_i$. There exists $w_1 \in M_i$ such that $w_1\langle x\rangle w_1^{-1} = w\langle x\rangle w^{-1}$. Hence it may be assumed that $w \in N(\langle x\rangle)$. This implies that x is contained in exactly one conjugate of M_i for $k+1 \leq i \leq s+m$ and x is contained in exactly two conjugates of M_i for $1 \leq i \leq k$. The number of ordered pairs (y, z) with $yz = x$, y, z of order 3 and $y, z \in M_i$ is easily seen to be $h_i t_i$. If the pair (x^2, x^2) is counted just once the second inequality in (5) follows.

LEMMA 7. *Let a be the multiplicity of C in K^2. Then*

$$a \geq \sum_{i=s+1}^{s+m} \frac{|C(u)|}{2h_i t_i} h_i t_i.$$

Proof. Let (y, z) be an ordered pair of elements of order 3 such that $yz = u$. Then $\langle y, z\rangle$ is isomorphic to $(3, 3, 2) = \mathfrak{A}_4$.

Suppose that $\langle y, z\rangle$ is contained in two distinct subgroups which are respectively conjugate to M_i and M_j with $s+1 \leq i < j \leq s+m$. By changing notation it may be assumed that $\langle y, z\rangle \subseteq M_i \cap M_j$, where $M_i \cap M_j$ is maximal among all such intersections. Let $D = N_i \cap N_j$, then $N(\langle y\rangle) \subseteq N(D)$. Since $[\langle y, z\rangle : \langle y, z\rangle'] = 3$ it follows that $\langle y, z\rangle' \subseteq D$. Define

$$L_i = N(D) \cap N_i, \qquad L_j = N(D) \cap N_j.$$

Then $\langle L_i, L_j\rangle \subseteq N(D)$. Thus by Lemma 1 $\langle L_i, L_j\rangle \neq G$. Furthermore

$$N(\langle y\rangle) \subseteq N(L_i) \cap N(L_j) \subseteq N(\langle L_i, L_j\rangle).$$

Let M be a maximal solvable subgroup of G which contains $N(\langle y\rangle)\langle L_i, L_j\rangle$ and let N be the maximal normal nilpotent subgroup of M. By induction M/N is isomorphic to $\mathfrak{S}_3$. Since $N(\langle y\rangle) \cap L_i = \langle 1\rangle$ this implies that $L_i \subseteq N$. Since $N_i \subsetneqq N_j$ we have that $D \neq N_i$. Thus $D \neq L_i$ as N_i is nilpotent. Therefore $M_i \cap M_j \subset M_i \cap M$. A similar argument shows that $M_i \cap M_j \subset M_j \cap M$. As M cannot be conjugate to both M_i and M_j one of these inclusions contradicts the maximal nature of $M_i \cap M_j$. Thus $\langle y, z\rangle$ is not contained in two subgroups which are conjugate M_i, M_j respectively with $s+1 \leq i < j \leq s+m$.

If $\langle y, z\rangle \subseteq M_i \cap w^{-1}M_i w$ for $w \in G$ then $N(\langle y\rangle) \subseteq M_i \cap w^{-1}M_i w$. This implies that $M_i = w^{-1}M_i w$. Let u lie in exactly m_i subgroups conjugate to N_i. Since M_i contains at least $h_i t_i$ ordered pairs (y, z) with $y^3 = z^3 = 1$, $yz = u$, this implies that

$$a \geq \sum_{i-s+1}^{s+m} h_i t_i m_i.$$

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Clearly $m_i \geq [C(u) : C(u) \cap M_i] \geq \frac{|C(u)|}{2h_i t_i}$. The lemma follows.

LEMMA 8.

$$\sum_{i=s+1}^{s+m} \frac{|C(u)|}{2h_i t_i} h_i t_i \leq \frac{g}{3}.$$

Proof. Let a be the multiplicity of C in K^2. Then by Lemma 3

$$a = \frac{g}{9}\left\{1 + \frac{\theta(u)}{\theta(1)} + \frac{\chi(u)}{\chi(1)}\right\} \leq \frac{g}{3}.$$

The result now follows from Lemma 7.

LEMMA 9. $|C(u)| = 2^n h$ *with* $h \neq 1$.

Proof. By Lemma 4, u is in the center of a S_2-subgroup of G. Suppose that $h = 1$. Then ([4], p. 870, [5], Theorem 3) G is isomorphic to $PSL(2, 9)$, $PSL(3, 4)$ or $PSL(2, q)$ for q a prime or a power of 2. Since 9 does not divide g the first two possibilities cannot occur. If q is odd $PSL(2, q)$ contains cyclic subgroups of order $\frac{q+1}{2}$ and $\frac{q-1}{2}$. Thus one of p, $\frac{p-1}{2}$, $\frac{p+1}{2}$ equals 3. Hence $p = 3, 5, 7$. Since $PSL(2, 3)$, $PSL(2, 5)$ are respectively isomorphic to $\mathfrak{A}_4$, $\mathfrak{A}_5$ these possibilities cannot occur. If q is a power of 2, then $q \pm 1 = 3$ and so $q = 2$ or 4. As $PS(2, 2)$ is not simple and $PSL(2, 4)$ is isomorphic to $\mathfrak{A}_5$ we get that $h \neq 1$.

The proof of the main Theorem is now divided into three cases.

Case I. $h = h_{s+1}$, $t_1 \neq 2^n$

Case II. $h = h_{s+1}$, $t_1 = 2^n$

Case III. $h \neq h_{s+1}$.

In cases I and II $h_i = 1$ for $i > s+1$. In case II $h_1 = 1$. Thus in cases I and II Lemmas 6, 7 and 8 and equation (2) yield that

$$\frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{3} \leq 2\left\{h_1 t_1 + \sum_{i=2}^{k} h_i\right\} + h t_{s+1} + \frac{1}{h}\left\{2^n g_0 \prod_{i=1}^{s+m} h_i\right\}.$$

Since $(h, 6) = 1$ and $h \neq 1$ by Lemma 9 we get that $h \geq 5$. Thus in cases I or II we get

$$\frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{15} \leq \frac{1}{2}\left(\frac{1}{3} - \frac{1}{h}\right) 2^n g_0 \prod_{i=1}^{s+m} h_i \leq h_1 t_1 + \sum_{i=2}^{k} h_i + h t_{s+1}. \tag{6}$$

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Hence in Case I we get

$$\frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{15} \leq \frac{1}{2}\left(\frac{1}{3} - \frac{1}{h}\right) 2^n g_0 \prod_{i=1}^{s+m} h_i \leq \sum_{i=1}^{k} h_i + h 2^n. \tag{7}$$

Since $t_{s+1} \leq 2^{n-1}$ we get in case II that

$$h_1 t_1 + h t_{s+1} \leq 2^n + 2^{n-1} h < 2^n h.$$

Thus in case II

$$\frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{15} \leq \frac{1}{2}\left(\frac{1}{3} - \frac{1}{h}\right) 2^n g_0 \prod_{i=1}^{s+m} h_i \leq \sum_{i=2}^{k} h_i + h 2^n. \tag{8}$$

In case III let h_0 be the minimum value of h/h_i for $s+1 \leq i \leq s+m$. Hence $h_0 \geq 5$. Thus

$$\frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{3} \leq 2\left\{h_1 t_1 + \sum_{i=2}^{k} h_i\right\} + \frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{h_0}$$

or in case III

$$\frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{15} \leq \frac{1}{2}\left(\frac{1}{3} - \frac{1}{h_0}\right) 2^n g_0 \prod_{i=1}^{s+m} h_i \leq h_1 t_1 + \sum_{i=2}^{k} h_i. \tag{9}$$

For convenience the following notation is now introduced.

Case I $q = k+1$, $z = h$

$\{x_1, \ldots, x_q\}$ is the set $\{h_1, \ldots, h_k, h 2^n\}$

in ascending order, and

$$y = \frac{1}{h} g_0 \prod_{i=k+1}^{s} h_i.$$

Case II $q = k$, $z = h$

$\{x_1, \ldots, x_q\}$ is the set $\{h_2, \ldots, h_k, h 2^n\}$

in ascending order, and

$$y = \frac{1}{h} g_0 \prod_{i=k+1}^{s} h_i.$$

Case III $q = k$, $z = h_0$

$\{x_1, \ldots, x_q\}$ is the set $\{h_1 t_1, h_2, \ldots, h_k\}$

in ascending order, and

$$y = g_0 2^n \prod_{i=k+1}^{s+m} h_i \qquad \text{if } t_1 = 1$$

$$= g_0 \prod_{i=k+1}^{s+m} h_i \qquad \text{if } t_1 = 2^n.$$

In all cases we get that $x_1, \ldots, x_q, y, z$ are integers such that

(10) $$g = 3y \prod_{i=1}^{q} x_i$$

(11) $$(x_i, x_j) = 1 \qquad \text{for } 1 \le i < j \le q$$

(12) $$(3, y) = (x_i, y) = 1 \qquad \text{for } 1 \le i \le q$$

If $x_i \not\equiv 1 \pmod 3$ then $x_i = h2^n$ in cases I or II. Thus $x_i > 4h \ge 20$. Therefore

(13) $$x_i \equiv 1 \pmod 3 \text{ or } x_i > 20 \text{ for } 1 \le i \le q.$$

The inequalities (7), (8) and (9) become

(14) $$\frac{y \prod_{i=1}^{q} x_i}{15} \le \frac{1}{2}\left(\frac{1}{3} - \frac{1}{z}\right) y \prod_{i=1}^{q} x_i \le \sum_{i=1}^{q} x_i$$

LEMMA 10. $q \le 2$. *If* $q = 2$ *then* $y = 1$.

Proof. If $x_1 > 4$ then by (13) $x_1 \ge 7$. Hence (14) yields that

$$7^{q-1} x_q \le 15 \sum_{i=1}^{q} x_i \le 15\, q x_q.$$

Thus $7^{q-1} \le 15q$ and so $q \le 2$ in this case. If $x_1 \le 4$ then $x_1 = 4$ and

$$4^{q-1} x_q \le 15 \sum_{i=1}^{q} x_i \le 15\, q x_q$$

Thus $4^{q-1} \le 15q$ and $q \le 3$. Hence $q = 3$ and by (14)

$$4x_2 x_3 < 15(x_1 + x_2 + x_3) < 45 x_3.$$

Hence $x_2 < 12$. Thus by (11) and (12) $x_2 = 7$ and so $28 x_3 < 15(4 + 7 + x_3)$ or $13 x_3 < 165$. Hence $x_3 < 13$ contrary to $7 < x_3$, (11) and (12). Thus $q \le 2$.

Suppose that $q = 2$. Then (14) yields that $y x_1 x_2 \le 15(x_1 + x_2)$. If $y \ge 5$ this implies that $x_1 x_2 \le 3(x_1 + x_2) < 6x_2$. Thus $x_1 = 4$ and $4x_2 \le 12 + 3x_2$. Hence $x_2 = 7$. Therefore $28y \le 15(11) = 165$ and $y < 6$. Therefore $y = 5$ and by (10) $g = 3 \cdot 4 \cdot 5 \cdot 7 = 420$. This is impossible since there is no simple group of order

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420. Thus $y<5$. If $y \neq 1$, then $y=2$ or $y=4$. If $y=2$, then $x_1 x_2$ is odd and by (10) 4 does not divide g contrary to the simplicity of G. Thus $y=4$. Hence $x_1 x_2$ is odd and so $x_1 \geq 7$. If $x_1>7$ then $x_1 \geq 13$ and $52 x_2 \leq 4 x_1 x_2 \leq 15(x_1+x_2)<30 x_2$ which is not the case. If $x_1=7$ then $28 x_2 \leq 15(7+x_2)$ or $13 x_2<15.7$. Hence $x_2<13$ which is not the case. The lemma is proved in all cases.

Lemma 11. *In case I or case II*

$$\frac{11}{75} y \prod_{i=1}^{q} x_i \leq \sum_{i=1}^{q} x_i.$$

Proof. H_{s+1} admits $\mathfrak{S}_3$ as a group of automorphisms, thus H_{s+1} is not cyclic. Hence $z=h=|H_{s+1}| \geq 25$ and the result follows from (14).

Lemma 12. $q=2$, $y=1$.

Proof. Suppose that $q=1$. Assume first that we have case I or II. Then Lemma 11 implies that $y<7$. Furthermore $|C(u)|=x_1$ and $[G: C(u)]=3y$. Thus $y \neq 1$ and so by (12) $y=5$. Hence in case I (6) becomes

$$\frac{11}{75} \cdot 5 \cdot 2^n h \leq h t_{s+1} \leq h\, 2^{n-1},$$

or $\frac{22}{15} \leq 1$ which is not the case. In case II $|N(H_1)|=3 x_1$ and so $[G: N(H_1)]=5$, thus G is isomorphic to a subgroup of $\mathfrak{S}_5$. Hence G is isomorphic to $\mathfrak{A}_5$ contrary to assumption.

Assume now that $q=1$ and we are in case III. Then (14) implies that $y \leq 15$. Since G is simple $4 \mid g$. Thus by (10) and (12) either y is odd or $4 \mid y$. Hence $y=4, 8, 5, 7, 11$ or 13 and $g=3x_1 y$. If x_1 is even then $x_1 \,\Big|\, |C(u)|$ and $x_1 \neq |C(u)|$. Since in this case $y=5, 7, 11$ or 13, it is a prime. Hence $|C(u)| = x_1 y$ and $[G: C(u)]=3$ which is impossible. If x_1 is odd then $x_1 \equiv 1 \pmod 3$, $y=4$ or 8 and $[G: N(H_1)]=y$. Thus $y=8$ and G is isomorphic to subgroup of $\mathfrak{S}_8$. As H_1 is nilpotent the Sylow theorems imply that the only prime dividing x_1 is 7. As 49 does not divide 8! this implies that $x_1=7$. Hence $g=3 \cdot 7 \cdot 8$ and G is isomorphic to $PSL(2, 7)$ contrary to assumption.

Hence $q=2$ and by Lemma 10 $y=1$.

The proof of the main Theorem will now be completed.

By Lemma 12 $g=3 x_1 x_2$. In case I or II H_{s+1} is not cyclic. Thus $z=h= |H_{s+1}| \geq 25$. By (14) we get that

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$$\frac{11}{75} x_1 x_2 \le x_1 + x_2 < 2 x_2.$$

Hence $x_1 < 14$. Thus x_1 is odd and $x_1 \equiv 1 \pmod 3$. This implies that $x_1 = 7$ or $x_1 = 13$. If $x_1 = 13$ then $\frac{11 \cdot 13}{75} x_2 \le 13 + x_2$ or $25 \le h \le x_2 \le \frac{75}{68} \cdot 13$ which is not the case. Suppose that $x_1 = 7$. In case I (7) implies that

$$2^n h < \frac{11}{75} 7 \cdot 2^n h \le 7 + 2^{n-1} h.$$

Hence $25 < 2^{n-1} h \le 7$ which is not the case. In case II (6) implies that $2^n h < \frac{11}{75} 7 \cdot 2^n h \le 2^n + 7 + h t_{s+1} \le 2^n + 7 + 2^{n-1} h$. Hence $25 \le 2^{n-1} h \le 2^n + 7$. So that $2^n > 7$ and $2^{n-1} \cdot 25 \le 2^{n-1} h < 2^{n+1}$ which is not the case.

Assume now that we have Case III. Then $x_1 \equiv x_2 \equiv 1 \pmod 3$, and by (14) $\frac{x_1 x_2}{15} \le x_1 + x_2 < 2 x_2$. Hence $x_1 < 30$. If x_1 is even then $x_1 \equiv 0 \pmod 4$. If x_1 is a prime then $|C(u)| = x_1 x_2$ and $[G : C(u)] = 3$ which is not the case. Thus $x_1 = 4, 16, 25, 28$. $[G : N(H_2)] = x_1$, thus $x_1 \ne 4$. If $x_1 = 28$ then (14) yields that $\frac{28}{15} x_2 \le 28 + x_2$ or $x_2 \le \frac{28 \cdot 15}{13} < 33$. Hence $x_2 = 31$ is a prime. Thus $|C(u)| = x_1 x_2$ and $[G : C(u)] = 3$ which is not the case. If $x_1 = 25$ then $\frac{25}{15} x_2 \le 25 + x_2$ or $x_2 \le \frac{3}{2} 25 < 38$. Since $x_2 \equiv 0 \pmod 4$, $x_2 = 28$. The Sylow theorems now imply that some divisor d of 25 satisfies $d \equiv 1 \pmod 7$ which is not the case. Assume finally that $x_1 = 16$. If P is a Sylow subgroup of H_2 for some prime p then $|N(P)| = 3 x_2$ as $N(H_2)$ is a maximal solvable subgroup of G. Thus $16 \equiv 1 \pmod p$. Hence $p = 5$. Since $x_2 \equiv 1 \pmod 3$, $x_2 = 5^{2a}$ for some integer a. By (14) $\frac{16}{15} x_2 \le 16 + x_2$ or $x_2 \le 240$. Thus $x_2 = 25$ and $g = 3 \cdot 16 \cdot 25 = 1200$. There is no simple group of order 1200.

This final contradiction establishes the main theorem of the paper.

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