ON PONTRYAGIN DUALITY

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Introduction. The main aim of this article is to discuss the relationship between Pontryagin duality and pro-objects. The basic idea arises from K. H. Hofmann's articles [**7**] and [**8**] where it is shown that the elementary abelian (Lie) groups are "dense" in the category of locally compact hausdorff abelian groups.

We commence with a good symmetric monoidal closed category $\mathscr{V}$ and a full sub-$\mathscr{V}$-category $\mathscr{A} \subset \mathscr{V}$ of "elementary" objects. We then build pro-$\mathscr{A}$-objects as suitable projective limits of these elementary objects. This is done with a view to extending Pontryagin duality to pro-$\mathscr{A}$-objects once it holds in $\mathscr{A}$ with respect to some basic dualising object which we call Ω. The actual pro-$\mathscr{A}$-objects constructed are relative to a subcategory $\mathscr{E}$ of $\mathscr{V}_0$ which, in practice, is usually taken to be some good class of epimorphisms in $\mathscr{V}_0$. This is done in Sections 2 and 3.

In Section 4 we discuss to what extent projective limits of pro-$\mathscr{A}$-objects are again pro-$\mathscr{A}$-objects. This at least explains one of Kaplan's results [**10**]; namely, that the product of locally compact hausdorff abelian groups satisfies Pontryagin duality. Kaplan's second result [**11**] remains to be fitted into this context.

In the examples of Section 5 we apply the results of the preceding sections to prove that Pontryagin duality holds for any abelian group object in the category of compactly generated spaces which is a suitable projective limit of its elementary Lie quotients. We also reproduce the duality of Hofman, Mislove and Stralka [**9**] between semilattices and compact zero-dimensional semilattices.

For basic notation and terminology we refer the reader to Day and Kelly [**3**], Eilenberg and Kelly [**5**] and Mac Lane [**12**].

1. Preliminaries. Let $\mathscr{V} = (\mathscr{V}_0, V, \otimes, I, [-,-], \ldots)$ be a complete and cocomplete symmetric monodial closed category in the sense of Eilenberg and Kelly [**5**]. This means that we have at our disposal the calculus of $\mathscr{V}$-ends discussed in Day and Kelly [**3**] and in Dubuc [**4**].

Let $\mathscr{E}ns$ denote "the" category of small sets and set maps. We denote the X-fold power, respectively copower, of $X \in \mathscr{E}ns$ with $C \in \mathscr{V}$ by $\{X, C\}$, respectively $X . C$.

We now assume that $V: \mathscr{V}_0 \to \mathscr{E}ns$ is *faithful*. The effect of this assumption is the following.

LEMMA 1.1. *Suppose $\mathscr{A}$ is a small $\mathscr{V}$-category and $S: \mathscr{A}^{op} \otimes \mathscr{A} \to \mathscr{V}$ is a $\mathscr{V}$-functor. Let S' denote the composite.*

$$V_* \mathscr{A}^{op} \times V_* \mathscr{A} \xrightarrow{\tilde{V}_*} V_*(\mathscr{A}^{op} \otimes \mathscr{A}) \xrightarrow{V_* S} V_* \mathscr{V} = \mathscr{V}_0.$$

Then

$$\int_{A \in V_* \mathscr{A}} S'(AA) \cong \int_{A \in \mathscr{A}} S(AA).$$

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Proof. Following the notation of Eilenberg and Kelly [5] we write $\mathscr{A}_0 = V_*\mathscr{A}$. Because $V:\mathscr{V}_0 \to \mathscr{E}ns$ is faithful a family $\alpha_A : C \to S(AA)$ is $\mathscr{V}$-natural in $A \in \mathscr{A}$ if and only if it is $\mathscr{E}ns$-natural in $A \in \mathscr{A}_0$ since

$$\begin{array}{ccc} \mathscr{A}(AB) & \xrightarrow{S(A-)} & [S(AA), S(AB)] \\ {\scriptstyle S(-B)}\downarrow & & \downarrow{\scriptstyle [\alpha_A, 1]} \\ [S(BB), S(AB)] & \xrightarrow[{[\alpha_B, 1]}]{} & [C, S(AB)] \end{array}$$

commutes if and only if

$$\begin{array}{ccc} \mathscr{A}_0(AB) & \xrightarrow{VS(A-)} & \mathscr{V}_0(S(AA), S(AB)) \\ {\scriptstyle VS(-B)}\downarrow & & \downarrow{\scriptstyle \mathscr{V}_0(\alpha_A, 1)} \\ \mathscr{V}_0(S(BB), S(AB)) & \xrightarrow[{\mathscr{V}_0(\alpha_B, 1)}]{} & \mathscr{V}_0(C, S(AB)) \end{array}$$

commutes. Thus the equaliser of the canonical pair

$$\prod_{A \in \mathscr{A}} S(AA) \rightrightarrows \prod_{A,B \in \mathscr{A}} [\mathscr{A}(A,B), S(AB)]$$

which is, by definition, $\int_{A \in \mathscr{A}} S(AA)$ coincides with the equaliser of the canonical pair

$$\prod_{A \in \mathscr{A}_0} S(AA) \rightrightarrows \prod_{A,B \in \mathscr{A}_0} \{\mathscr{A}_0(A,B), S(AB)\}$$

which is, by definition, $\int_{A \in \mathscr{A}_0} S'(AA)$.

Henceforth we shall denote $S'(AB)$ simply by $S(AB)$.

The assumption that $V:\mathscr{V}_0 \to \mathscr{E}ns$ is faithful allows us, in effect, to "mix" $\mathscr{V}$-ends with ordinary $\mathscr{E}ns$-ends.

2. Pro-$\mathscr{A}$-objects. Let $\mathscr{A} \subset \mathscr{V}$ be a full sub-$\mathscr{V}$-category of $\mathscr{V}$. Let $\mathscr{C}$ be a subcategory of $\mathscr{V}_0$ and let $\mathscr{K} = \mathscr{C} \cap \mathscr{A}_0$.

DEFINITION 2.1. (i) A *pro-$\mathscr{A}$-object* in $\mathscr{V}$ relative to $\mathscr{C}$ is an object $C \in \mathscr{V}$ such that $C \cong \int_{A \in \mathscr{K}} \{\mathscr{C}(C, A), A\}$.

(ii) A *strong pro-$\mathscr{A}$-object* in $\mathscr{V}$ relative to $\mathscr{C}$ is a pro-$\mathscr{A}$-object $C \in \mathscr{V}$ such that $\int^{A \in \mathscr{K}} \mathscr{C}(C, A) . [A, B] \to [C, B]$ is an epimorphism for all $B \in \mathscr{A}$.

The $\mathscr{V}$-category of pro-$\mathscr{A}$-objects is denoted by $\mathscr{P}\mathscr{A}(\mathscr{C})$ while the $\mathscr{V}$-category of strong pro-$\mathscr{A}$-objects is denoted by $\mathscr{S}\mathscr{P}\mathscr{A}(\mathscr{C})$.

LEMMA 2.2. $\mathscr{A} \subset \mathscr{S}\mathscr{P}\mathscr{A}(\mathscr{C})$.

Proof. If $A' \in \mathscr{A}$ then $A' \cong \int_{A \in \mathscr{K}} \{\mathscr{K}(A', A), A\}$ by the representation theorem applied to $A \in \mathscr{K}$, $\cong \int_{A \in \mathscr{K}} \{\mathscr{C}(A', A), A\}$. Similarly $\int^{A \in \mathscr{K}} \mathscr{C}(A', A).[A, B] = \int^{A \in \mathscr{K}} \mathscr{K}(A', A) . [A, B] \cong [A', B]$ by the representation theorem applied to $A \in \mathscr{K}$.

THEOREM 2.3. *The inclusion $\mathscr{A} \subset \mathscr{S}\mathscr{P}\mathscr{A}(\mathscr{C})$ is $\mathscr{V}$-codense ($= \mathscr{V}$-coadequate).*

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Proof. For $\mathscr{V}$-codensity we require that $C \cong \int_{A\in\mathscr{A}} [[C, A], A]$ for all $C \in \mathscr{S}\mathscr{P}\mathscr{A}(\mathscr{C})$. But, by Definition 2.1(ii), there is a monomorphism

$$[[C, B], B] \to \left[\int^{A\in\mathscr{K}} \mathscr{C}(C, A) \,.\, [A, B], B\right]$$

for each $B \in \mathscr{A}$. This gives us a monomorphism

$$\int_{B\in\mathscr{A}_0} [[C, B], B] \to \int_{B\in\mathscr{A}_0} \left[\int^{A\in\mathscr{K}} \mathscr{C}(C, A) \,.\, [A, B], B\right]$$

since limits of monomorphisms are monomorphisms. Moreover, the codomain of this monomorphism becomes:

$$\int_{B\in\mathscr{A}_0} \left[\int^{A\in\mathscr{K}} \mathscr{C}(C, A) \,.\, [A, B], B\right] \cong \int_{B\in\mathscr{A}_0} \int_{A\in\mathscr{K}} \{\mathscr{C}(C, A), [[A, B], B]\}$$

$$\cong \int_{A\in\mathscr{K}} \int_{B\in\mathscr{A}_0} \{\mathscr{C}(C, A), [[A, B], B]\}$$

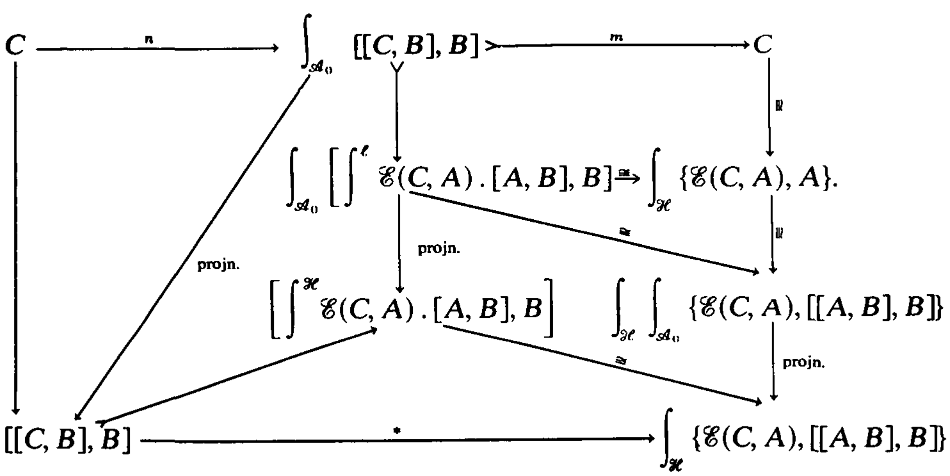
on interchanging ends,

$$\cong \int_{A\in\mathscr{K}} \left\{\mathscr{C}(C, A), \int_{B\in\mathscr{A}_0} [[A, B], B]\right\}$$

where $\int_{B\in\mathscr{A}_0} [[A, B], B] \cong \int_{B\in\mathscr{A}} [[A, B], B]$ (by Lemma 1.1) $\cong A$ by the $\mathscr{V}$-representation theorem applied to $B \in \mathscr{A}$. Thus we obtain a monomorphism

$$m : \int_{B\in\mathscr{A}_0} [[C, B], B] \to \int_{A\in\mathscr{K}} \{\mathscr{C}(C, A), A\} \cong C$$

by Definition 2.1(i). However, this monomorphism is left inverse to the canonical transformation η from C to $\int_{B\in\mathscr{A}_0} [[C, B], B]$ by consideration of the following diagram:



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where $*$ commutes for obvious reasons (project both legs at $A \in \mathscr{K}$ and $f \in \mathscr{C}(C, A)$). Thus $C \cong \int_{B \in \mathscr{A}_0} [[C, B], B] \cong \int_{B \in \mathscr{A}} [[C, B], B]$ by Lemma 1.1, as required.

3. Duality. Given $\mathscr{A} \subset \mathscr{V}$ we can form the Pontryagin closure of $\mathscr{A}$:

$$\bar{\mathscr{A}} = \left\{ C \in \mathscr{V};\, C \cong \int_{A \in \mathscr{A}} [[C, A], A] \text{ in } \mathscr{V} \right\}.$$

PROPOSITION 3.1. *$\mathscr{A} \subset \bar{\mathscr{A}}$ and $\bar{\mathscr{A}} = \bar{\bar{\mathscr{A}}}$.*

Proof. If $B \in \mathscr{A}$ then $B \cong \int_{A \in \mathscr{A}} [[B, A], A]$ by the $\mathscr{V}$-representation theorem applied to $A \in \mathscr{A}$. Thus $\mathscr{A} \subset \bar{\mathscr{A}}$ so $\bar{\mathscr{A}} \subset \bar{\bar{\mathscr{A}}}$. But $C \in \bar{\bar{\mathscr{A}}}$ implies $C \cong \int_{A' \in \bar{\mathscr{A}}} [[C, A'], A']$ where $A' \in \bar{\mathscr{A}}$ implies $A' \cong \int_{A \in \mathscr{A}} [[A', A], A]$. So

$$\begin{aligned} C &\cong \int_{A' \in \bar{\mathscr{A}}} \left[[C, A'], \int_{A \in \mathscr{A}} [[A', A], A] \right] \\ &\cong \int_{A \in \mathscr{A}} \left[\int^{A' \in \bar{\mathscr{A}}} [C, A'] \otimes [A', A], A \right] \\ &\cong \int_{A \in \mathscr{A}} [[C, A], A] \end{aligned}$$

by the $\mathscr{V}$-representation theorem applied to $A' \in \bar{\mathscr{A}}$.

COROLLARY 3.2. *The inclusion $\mathscr{A} \subset \bar{\mathscr{A}}$ is $\mathscr{V}$-codense and if $\mathscr{C} \subset \mathscr{V}$ with $\mathscr{A} \subset \mathscr{C}$ being $\mathscr{V}$-codense then $\mathscr{C} \subset \bar{\mathscr{A}}$.*

Given $\mathscr{A} \subset \mathscr{V}$ and $\Omega \in \mathscr{A}$ such that $[[A, \Omega], \Omega] \cong A$ we have:

PROPOSITION 3.3. *Pontryagin duality with respect to $\Omega \in \mathscr{A}$ holds in $\bar{\mathscr{A}}$.*

Proof. If $C \in \bar{\mathscr{A}}$ then

$$\begin{aligned} C &\cong \int_{A \in \mathscr{A}} [[C, A], A] \\ &\cong \int_{A \in \mathscr{A}} [[C, A], [[A, \Omega], \Omega]] \\ &\cong \left[\int^{A \in \mathscr{A}} [C, A] \otimes [A, \Omega], \Omega \right] \\ &\cong [[C, \Omega], \Omega] \end{aligned}$$

by the $\mathscr{V}$-representation theorem applied to $A \in \mathscr{A}$.

COROLLARY 3.4. *Pontryagin duality with respect to Ω holds in $\mathscr{S}\mathscr{P}\mathscr{A}(\mathscr{C})$.*

Proof. By Theorem 2.3.

We now examine the case where $\int^{A \in \mathscr{K}} \mathscr{C}(C, A) \,.\, [A, B] \to [C, B]$ is an *isomorphism* for

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all $C \in \mathcal{V}$ and $B \in \mathcal{A}$. When this is so $\mathcal{PA}(\mathcal{E})$ and $\mathcal{SPA}(\mathcal{E})$ coincide and we can define an endofunctor $P: \mathcal{V} \to \mathcal{V}$ by any one of the formulas

$$PC = \int_{A \in \mathcal{K}} \{\mathcal{E}(C, A), A\} \cong \int_{A \in \mathcal{A}} [[C, A], A] \cong [[C, \Omega], \Omega].$$

We recall from Day [**1**] that a category $\mathcal{C}$ is *$\mathcal{M}$-complete* if $\mathcal{M}$ is a subcategory of monomorphisms in $\mathcal{C}$ such that $\mathcal{C}$ has the following inverse limits and $\mathcal{M}$ contains each monomorphism so formed:

(*a*) equalisers of pairs of morphisms,

(*b*) pullbacks of $\mathcal{M}$-monomorphisms,

(*c*) the intersection of any family of $\mathcal{M}$-monomorphisms with a common codomain.

A functor $T: \mathcal{C} \to \mathcal{B}$ is *$\mathcal{M}$-continuous* if it preserves these inverse limits in $\mathcal{C}$.

PROPOSITION 3.5. *If $\mathcal{V}$ is $\mathcal{M}$-complete for some system $\mathcal{M}$ of monomorphisms then $\mathcal{PA}(\mathcal{E}) \subset \mathcal{V}$ is reflective if $[-, \Omega]: \mathcal{V}^{op} \to \mathcal{V}$ preserves suitable colimits.*

Proof. Basically we require that $[-, \Omega]^{op}: \mathcal{V} \to \mathcal{V}^{op}$ be $\mathcal{M}$-continuous and that $[-, \Omega]: \mathcal{V}^{op} \to \mathcal{V}$ preserve linear colimits. The effect of the first requirement is that $\mathcal{PA}(\mathcal{E})$ is $\mathcal{M}$-complete and the inclusion $\mathcal{PA}(\mathcal{E}) \subset \mathcal{V}$ is $\mathcal{M}$-continuous. Thus we can apply Day [**1**, Theorem 2.2] provided $P: \mathcal{V} \to \mathcal{V}$ is a suitable boundary functor. But the canonical morphism $\eta_C: C \to PC$ gives us:

$$C \xrightarrow{\eta_C} PC \xrightarrow{\eta_{PC}} P^2C \longrightarrow \ldots \longrightarrow P^nC \longrightarrow \ldots$$

where $P^{n+1}C = [[P^nC, \Omega], \Omega]$. If $[-, \Omega]$ preserves linear colimits then $P^\omega C =$ colim $P^{n+1}C = \text{colim}[[P^nC, \Omega], \Omega] = [\lim[P^nC, \Omega], \Omega] = [[\text{colim}\, P^nC, \Omega], \Omega] = [[P^\omega C, \Omega], \Omega]$. Thus $P^\omega C$ lies in $\mathcal{PA}(\mathcal{E})$ for all $C \in \mathcal{V}$. This implies (by Day [**1**, Theorem 2.2]) that $\mathcal{PA}(\mathcal{E}) \subset \mathcal{V}$ is reflective and the reflection of $C \in \mathcal{V}$ is the intersection in $\mathcal{V}$ of all the $\mathcal{PA}(\mathcal{E})$-$\mathcal{M}$-subobjects of $P^\omega C$ through which the resultant canonical transformation $\beta_C: C \to P^\omega C$ factors.

We recall that $\Omega \in \mathcal{V}$ is said to be a *strong $\mathcal{V}$-cogenerator* for $\mathcal{V}$ if $[-, \Omega]: \mathcal{V}^{op} \to \mathcal{V}$ reflects isomorphisms.

COROLLARY 3.6. *If Ω is a strong $\mathcal{V}$-cogenerator for $\mathcal{V}$ then $\mathcal{PA}(\mathcal{E}) = \mathcal{V}$ if and only if $[-, \Omega]: \mathcal{V}^{op} \to \mathcal{V}$ preserves colimits.*

Proof. If $[-, \Omega]$ preserves colimits then $\mathcal{PA}(\mathcal{E}) \subset \mathcal{V}$ is $\mathcal{V}$-reflective hence is closed under $\mathcal{V}$-limits. Thus, if Ω is a strong $\mathcal{V}$-cogenerator then every object of $\mathcal{V}$ is a $\mathcal{V}$-limit of copies of Ω. Thus $\mathcal{PA}(\mathcal{E}) = \mathcal{V}$. Conversely, if $\mathcal{PA}(\mathcal{E}) = \mathcal{V}$ then Pontryagin duality holds in $\mathcal{V}$ with respect to Ω. Thus $\text{colim}[A_\lambda, \Omega] \cong [\lim A_\lambda, \Omega]$ because $[\text{colim}[A_\lambda, \Omega], \Omega] \cong \lim[[A_\lambda, \Omega], \Omega] \cong \lim A_\lambda \cong [[\lim A_\lambda, \Omega], \Omega]$ where Ω is a strong $\mathcal{V}$-cogenerator so that $[-, \Omega]$ reflects isomorphisms.

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4. Strong $\mathscr{A}$-limits. A *strong $\mathscr{A}$-limit* relative to $\mathscr{E}$ is a limit $\lim C_\lambda$ in $\mathscr{V}_0$ such that the canonical morphisms

$$\sum \mathscr{E}(C_\lambda, A) \to \mathscr{E}(\lim C_\lambda, A), \qquad \sum [C_\lambda, A] \to [\lim C_\lambda, A]$$

are *epimorphisms* for all $A \in \mathscr{A}$.

PROPOSITION 4.1. *A strong $\mathscr{A}$-limit of strong pro-$\mathscr{A}$-objects is a strong pro-$\mathscr{A}$-object.*

Proof. The morphism $\sum \mathscr{E}(C_\lambda, A) \to \mathscr{E}(\lim C_\lambda, A)$ is a surjection if and only if the canonical morphism $\operatorname{colim} \mathscr{E}(C_\lambda, A) \to \mathscr{E}(\lim C_\lambda, A)$ is an epimorphism. This gives a monomorphism

$$\begin{aligned} \int_{A \in \mathscr{K}} \{\mathscr{E}(\lim C_\lambda, A), A\} \to \int_{A \in \mathscr{K}} &\{\operatorname{colim} \mathscr{E}(C_\lambda, A), A\} \\ &\cong \lim \int_{A \in \mathscr{K}} \{\mathscr{E}(C_\lambda, A), A\} \\ &\cong \lim C_\lambda. \end{aligned}$$

Moreover, this monomorphism is left inverse to the canonical morphism from $\lim C_\lambda$ to $\int_{A \in \mathscr{K}} \{\mathscr{E}(\lim C_\lambda, A), A\}$ hence it is an isomorphism. The fact that $\lim C_\lambda$ is a strong pro-$\mathscr{A}$-object now follows from consideration of the following diagram:

$$\begin{array}{ccc} \int^{A \in \mathscr{K}} \sum \mathscr{E}(C_\lambda, A) \,.\, [A, B] & \longrightarrow\!\!\!\!\rightarrow & \sum [C_\lambda, B] \\ \big\downarrow & & \big\downarrow \\ \int^{A \in \mathscr{K}} \mathscr{E}(\lim C_\lambda, A) \,.\, [A, B] & \dashrightarrow & [\lim C_\lambda, B]. \end{array}$$

Here the dotted arrow is an epimorphism because the diagonal is an epimorphism.

5. Examples.

EXAMPLE 5.1. Let $\mathscr{V}$ be the symmetric monoidal closed category $\mathscr{C}\mathscr{A}\mathscr{b}$ of abelian group objects in the category $\mathscr{C}$ of all convergence spaces (i.e. limit spaces). Let $\mathscr{A} = \{\mathbf{R}^m \oplus (\mathbf{R}/\mathbf{Z})^n \oplus G;\ m, n \in \mathbf{N}$ and G discrete$\}$. Let $\Omega = \mathbf{R}/\mathbf{Z}$ and let $\mathscr{E}$ be the category of identification maps.

PROPOSITION 5.1.1. *Each locally compact hausdorff abelian group is a strong pro-$\mathscr{A}$-object.*

Proof. Each locally compact hausdorff abelian group C is a pro-$\mathscr{A}$-object by the Lie-group approximation theorem: see Hofmann [7]. Secondly, each continuous homomorphism $f: C \to B$, $B \in \mathcal{A}$, factors as

$$C \xrightarrow{e} C/\ker f \xrightarrow{m} B$$

where $C/\ker f$ is a locally compact hausdorff group, hence is a Lie group (see Hochschild [**6**, Chapter VIII]), so $C/\ker f \in \mathcal{A}$. This implies that $\int^{A \in \mathcal{K}} \mathcal{E}(C, A) . [A, B] \to [C, B]$ is a surjection, as required.

COROLLARY 5.1.2. *Pontryagin duality in $\mathcal{CAb}$ holds for locally compact hausdorff abelian groups.*

A *strong projective limit* in $\mathcal{T}$, the category of topological spaces and continuous maps, is a limit $\lim_{\lambda \in \Lambda} C_\lambda$ over a cofiltered index category Λ such that each projection $p_\lambda : \lim C_\lambda \to C_\lambda$ is an identification map. For example, a product $\prod C_\lambda$ may be regarded as a strong limit cofiltered by the set of finite subsets of Λ.

LEMMA 5.1.3. *Given a strong projective limit in $\mathcal{TAb}$, with projections $p_\lambda : \lim C_\lambda \to C_\lambda$, the collection $\{\ker p_\lambda ; \lambda \in \Lambda\}$ is a filter base on $\lim C_\lambda$ which converges to zero.*

Proof. Since Λ is cofiltered the collection $\{p_\lambda^{-1}(V);\ V \text{ open in } C_\lambda\}$ is a *base* for the topology on $\lim C_\lambda$ in $\mathcal{TAb}$. Thus $\{\ker p_\lambda\} \to 0$.

PROPOSITION 5.1.4. *A strong projective limit $\lim C_\lambda$ in $\mathcal{TAb}$ is a strong $\mathcal{A}$-limit in $\mathcal{CAb}$.*

Proof. For each $A \in \mathcal{A}$, the canonical morphisms $\sum \mathcal{E}(C_\lambda, A) \to \mathcal{E}(\lim C_\lambda, A)$ and $\sum [C_\lambda, A] \to [\lim C_\lambda, A]$ are epimorphisms by Lemma 5.1.3 and the fact that each $A \in \mathcal{A}$ has no small subgroups.

COROLLARY 5.1.5. *A product of locally compact hausdorff groups satisfies Pontryagin duality in $\mathcal{CAb}$.*

EXAMPLE 5.2. *Let* $\mathcal{V} = \mathcal{HAb}_2$ be the category of hausdorff abelian group objects in the category $\mathcal{K}$ of k-spaces. With $\mathcal{A}$ and Ω as in Example 5.1 let $\mathcal{E}$ consist of *all* epimorphisms in $\mathcal{HAb}_2$.

PROPOSITION 5.2.1. *Pontryagin duality holds for pro-$\mathcal{A}$-objects*

Proof. Each pro-$\mathcal{A}$-object is now strong because any morphism $f: C \to B$ factors as

$$C \xrightarrow{e} \overline{C/\ker f} \xrightarrow{m} B$$

where e is an epimorphism and m is a *closed* subspace. Thus $\overline{C/\ker f} \in \mathcal{A}$.

It is actually possible to show that each locally compact hausdorff abelian group is a pro-$\mathcal{A}$-object for this $\mathcal{E}$ on $\mathcal{HAb}_2$; this we leave to the reader.

EXAMPLE 5.3. Let K be a discrete field and let $\mathcal{V}$ be the category of K-vector spaces in $\mathcal{K}$. Let $\mathcal{A} \subset \mathcal{V}$ be the full subcategory determined by $\{K^n ; n \in \mathbf{N}\}$. Then Pontryagin

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duality holds in $\mathscr{A}$ with respect to $\Omega = K$. Let $\mathscr{E}$ be the category of strong epimorphisms in $\mathscr{V}$. Then each map $f: C \to K^n$ factors as

$$C \xrightarrow{e} C/\ker f \xrightarrow{m} K^n$$

where e is a strong epimorphism and $C/\ker f$ is of the form K^p for some $p \in \mathbf{N}$. Thus $\int^{A \in \mathscr{K}} \mathscr{E}(C, A) . [A, B] \to [C, B]$ is an epimorphism for *all* $C \in \mathscr{V}$ and $B \in \mathscr{A}$.

Proposition 5.3.1. *Pontryagin duality with respect to* $\Omega = K$ *holds for pro-*$\mathscr{A}$*-objects.*

Example 5.4. Let K be a topological field in $\mathscr{K}$ and let $\mathscr{V}$ be the category of K-vector spaces in $\mathscr{K}$. Let $\mathscr{A}$ consist of 0 and K and let $\Omega = K$. Let $\mathscr{E}$ be the category of epimorphisms in $\mathscr{V}$. Now each map $f: C \to B$, $B \in \mathscr{A}$, factors

$$C \xrightarrow{e} C/\ker f \xrightarrow{m} B$$

where e is an epimorphism and $C/\ker f$ is either 0 or K.

Proposition 5.4.1. *Pontryagin duality with respect to* $\Omega = K$ *holds for pro-*$\mathscr{A}$*-objects.*

Example 5.5. Let $\mathscr{V}$ be the symmetric monoidal closed category of semilattices in $\mathscr{K}$ (see Hofmann, Mislove and Stralka [**9**]). Let $\mathscr{A}$ be the finite discrete semilattices in $\mathscr{V}$ and let $\mathscr{E}$ be the category of strong epimorphisms. Also let $\Omega = 2 \in \mathscr{A}$.

Once again every pro-$\mathscr{A}$-object is strong because any morphism $f: C \to B$ in $\mathscr{V}$ factors as

$$C \xrightarrow{e} C/\ker f \xrightarrow{m} B$$

where $C/\ker f$ is finite since B is finite.

For any compact zero-dimensional semilattice C in $\mathscr{K}$ we have

$$C \cong \int_{A \in \mathscr{K}} \{\mathscr{E}(C, A), A\}$$

because this is true in the category of topological semilattices (see Numakura [**13**] and Hofmann, Mislove and Stralka [**9**]). Thus, if $\mathscr{Z}$ denotes the category of compact zero-dimensional semilattices we have:

Proposition 5.5.1. *Pontryagin duality with respect to* $\Omega = 2$ *holds for each object of* $\mathscr{Z}$.

Proof. Pontryagin duality with respect to $\Omega = 2$ holds in $\mathscr{A}$ by [**9**, Chapter I, Lemma 3.8].

In his example $\mathscr{Z}$ has an explicit dual category, namely the category $\mathscr{S}$ of semilattices and semilattice morphisms (see [**9**, Chapter I]). This is so because we have $(\varepsilon, \eta): F \dashv R \cdot \mathscr{Z}^{op} \to \mathscr{S}$ given by $R = [-, \Omega]$ and $F = [-, \Omega]^{op}$. Because Pontryagin duality holds in $\mathscr{Z}$

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we have $\varepsilon : FR \cong 1 : \mathscr{X} \to \mathscr{X}$. To prove $\eta : 1 \to RF : \mathscr{S} \to \mathscr{S}$ is an isomorphism note that $F : \mathscr{S} \to \mathscr{X}^{op}$ reflects isomorphisms because $\Omega = 2$ is a (strong) cogenerator in $\mathscr{S}$ (see [**9**, Chapter I, Proposition 1.4]). Thus it suffices to prove that $F\eta : F \to FRF$ is an isomorphism. But this follows from the triangle identity

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FRF \\ & {\scriptstyle 1_F}\searrow & \downarrow{\scriptstyle \varepsilon F} \\ & & F \end{array}$$

In [**9**, Chapter I] it is shown that $\mathscr{S}$ is symmetric monoidal closed. This puts us in the situation of Proposition 3.5 and Corollary 3.6 because $\int^{A \in \mathscr{X}} \mathscr{E}(C, A) \times \mathscr{S}(A, B) \to \mathscr{S}(C, B)$ is easily seen to be an *isomorphism* for all $B \in \mathscr{A}$ (as before) and $C \in \mathscr{S}$; $\mathscr{E}$ denotes the category of (strong) epimorphisms. Noting again that 2 is a strong cogenerator of $\mathscr{S}$ we have that $\mathscr{P}\mathscr{A}(\mathscr{E}) = \mathscr{S}$. Thus Pontryagin duality for $\mathscr{S}$ could be shown directly by proving that $[-, 2] : \mathscr{S}^{op} \to \mathscr{S}$ preserves colimits; as it is, this is a consequence of duality in $\mathscr{S}$ as derived from the duality in $\mathscr{X}$.

Example 5.6. Let $\mathscr{V} = R\text{-}\mathscr{M}od$ be the category of R-modules over a principal ideal domain R. Let $\mathscr{A}$ be determined by the free R-modules of finite rank, and let $\Omega = R$. Then Pontryagin duality holds in $\mathscr{A}$ with respect to $\Omega = R$. If $\mathscr{E}$ is the category of (strong) epimorphisms in $\mathscr{V} = R\text{-}\mathscr{M}od$ then $\int^{A \in \mathscr{X}} \mathscr{E}(C, A) . [A, B] \to [C, B]$ is an epimorphism for *all* $C \in R\text{-}\mathscr{M}od$ and $B \in \mathscr{A}$. A pro-$\mathscr{A}$-object (= a strong pro-$\mathscr{A}$-object) is called a pro-free R-module.

Proposition 5.6.1. *Pontryagin duality with respect to $\Omega = R$ holds for pro-free R-modules.*

Example 5.7. It is clear that the calculations in Sections 1, 2, 3 and 4 can be carried out with $\mathscr{E}ns$ replaced by an arbitrary base category $\mathscr{W}$ which is symmetric monoidal closed and complete and cocomplete. As an example, let $\mathscr{W}$ be $\mathscr{A}b$, the category of abelian groups. Now let R be a commutative topological ring in $\mathscr{K}$. Let $\mathscr{V}$ be R-modules in $\mathscr{K}$ and let $\mathscr{A}$ comprise R alone as a full subcategory of $\mathscr{V}$. Let $\mathscr{E}$ be all "maps" in $\mathscr{V}_0$ (now a $\mathscr{W}$-category). Then every pro-$\mathscr{A}$-object is strong and R^n is a pro-$\mathscr{A}$-object for all $n \in \mathbf{N}$ because

$$R^n \cong \int_{A \in \mathscr{K}} \{\mathscr{V}_0(R^n, A), A\}$$

since $\mathscr{V}_0(R^n, A) \cong \bigoplus_n \mathscr{V}_0(R, A)$. Here of course $\{X, A\}$ denotes $\mathscr{A}b$-cotensoring of $X \in \mathscr{A}b$ with $\mathscr{A} \in \mathscr{V}$.

Example 5.8. It is worth noting to what extent Example 5.5 can be generalised. Let $\mathscr{V}$ be the category of algebras in $\mathscr{K}$ for some commutative algebraic $\mathscr{K}$-theory and let $\mathscr{A}$ be the category of finite discrete algebras in $\mathscr{V}$. Then, by Theorem 2.3, $\mathscr{A}$ is $\mathscr{V}$-codense in

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the full sub-$\mathscr{V}$-category $\mathscr{PA}(\mathscr{E})$ of pro-finite algebras in $\mathscr{V}$ ($\mathscr{E}$ is the category of strong epimorphisms and all pro-$\mathscr{A}$-objects relative to this $\mathscr{E}$ are strong). Thus $C \cong \int_n [[C, n], n]$ for C pro-finite. This may be regarded as a form of Pontryagin duality in which there is generally no basic dualising object Ω in $\mathscr{A}$. The actual duality is between $\mathscr{PA}(\mathscr{E})$ and a full sub-$\mathscr{V}$-category of the $\mathscr{V}$-functor category $[\mathscr{A}, \mathscr{V}]$.

Examples are easily obtained. For instance let $\mathscr{V}$ be the category of algebras for the theory of commutative semigroups or the theory of distributive lattices. Then, by Numakura [**13**], this form of Pontryagin duality holds for the compact zero-dimensional objects of $\mathscr{V}$. For further examples see Hofmann [**8**].

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