DECOMPOSITION OF THE n-DIMENSIONAL LATTICE-GRAPH INTO HAMILTONIAN LINES

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1. Statement of the Problem

A *graph* G consists, for the purposes of this paper, of two disjoint sets $V(G)$, $E(G)$, whose elements are called *vertices* and *edges* respectively of G, together with a relationship whereby with each edge is associated an unordered pair of distinct vertices (called its *end-vertices*) which the edge is said to *join*, and whereby no two vertices are joined by more than one edge. An edge λ and vertex ξ are *incident* if ξ is an end-vertex of λ. A *monomorphism* [*isomorphism*] of a graph G into [onto] a graph H is a one-to-one function ϕ from $V(G) \cup E(G)$ into [onto] $V(H) \cup E(H)$ such that $\phi(V(G)) \subset V(H)$, $\phi(E(G)) \subset E(H)$ and an edge and vertex of G are incident in G if and only if their images under ϕ are incident in H. G and H are *isomorphic* (in symbols, $G \cong H$) if there exists an isomorphism of G onto H. A *subgraph* of G is a graph H such that $V(H) \subset V(G)$, $E(H) \subset E(G)$ and an edge and vertex of H are incident in H if and only if they are incident in G; if $V(H) = V(G)$, H is a *spanning subgraph*. A collection of graphs are *edge-disjoint* if no two of them have an edge in common. A *decomposition* of G is a set of edge-disjoint subgraphs of G which between them include all the edges and vertices of G. L^n is a graph whose vertices are the lattice points of n-dimensional Euclidean space, two vertices A and B being joined by an edge if and only if AB is of unit length (and therefore necessarily parallel to one of the co-ordinate axes). An *endless Hamiltonian line* of a graph G is a spanning subgraph of G which is isomorphic to L^1. The object of this paper is to prove that L^n is decomposable into n endless Hamiltonian lines, a result previously established (1) for the case where n is a power of 2.

2. Preliminary Lemmas

Definitions. The set whose elements are $a_1, a_2, \ldots, a_n$ will be denoted by $\{a_1, a_2, \ldots, a_n\}$. If A, B are sets, $A \ominus B$ will denote the set of those elements of A which do not belong to B. The number of elements of A will be denoted by ord A. The set of all real numbers, the set of all integers, the set of all non-negative integers and the set of all positive integers will be denoted by R, I, J and P respectively. We shall suppose given an infinite sequence $\boldsymbol{e}^1, \boldsymbol{e}^2, \ldots$ of vectors forming a basis of an infinite-dimensional real vector space U. Let $x \in R$, $\boldsymbol{u} \in U$ and Q, T be subsets of R, U respectively. Then $\boldsymbol{u} + T$ [$Q\boldsymbol{u}$] will denote the set of all vectors of the form $\boldsymbol{u}+\boldsymbol{t}$ [$q\boldsymbol{u}$], where $\boldsymbol{t} \in T$ [$q \in Q$]; and xQ [$Q+x$, $Q-x$] will denote the set of all real numbers of the form xq [$q+x$, $q-x$], where $q \in Q$. We shall write $I+\frac{1}{2} = \hat{I}$, $J+\frac{1}{2} = \hat{J}$, $\{1, 2, \ldots, n\} = P_n$. A set of n consecutive elements of $\hat{I}$ (where n is a positive integer) will be called a *string* of *length n*—e.g. $\{2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2}\}$ is a string of length 3. If $\boldsymbol{z} \in U$, z_i will denote its 'ith component', i.e. the coefficient of $\boldsymbol{e}^i$ in the unique relation $\boldsymbol{z} = z_1\boldsymbol{e}^1+z_2\boldsymbol{e}^2+\ldots$ Furthermore, ${}_i\boldsymbol{z}$ will denote the vector $\boldsymbol{z}-z_i\boldsymbol{e}^i$. V^n will denote the set of all vectors of the form $\lambda_1\boldsymbol{e}^1+\lambda_2\boldsymbol{e}^2+\ldots+\lambda_n\boldsymbol{e}^n$, where the λ_i are *integers*. It will be convenient to re-define the graph L^n as follows: $V(L^n) = V^n$, and two elements $\boldsymbol{u}$, $\boldsymbol{v}$ of V^n are joined by an edge of L^n if and only if $\boldsymbol{u}-\boldsymbol{v} = \pm\boldsymbol{e}^i$ for some i, in which case the edge joining them will be denoted by the vector $\frac{1}{2}(\boldsymbol{u}+\boldsymbol{v})$. This definition is essentially similar to that of § 1; but we have arranged for convenience that (i) our "n-dimensional space" is contained in our "$(n+1)$-dimensional space", and (ii) each edge of L^n is referred to by what may be thought of as the position vector of its mid-point. I define a *one-ended* [*endless*] *Hamiltonian function* for a graph G to be a one-to-one function f: $\frac{1}{2}J$ [$\frac{1}{2}I$]$\rightarrow V(G)\cup E(G)$ such that $f(J)$ [$f(I)$] $= V(G)$ and, for every $n \in J$ [I], $f(n+\frac{1}{2})$ is an edge joining $f(n)$ to $f(n+1)$ in G. (If f is an endless Hamiltonian function for G, the elements of $f(\frac{1}{2}I)$ clearly form an endless Hamiltonian line of G.) If f is a one-ended [endless] Hamiltonian function for G and T is a subset of $E(G)$, $\Delta_f(T)$ will denote the number of elements of T which do not belong to $f(\hat{J})$ [$f(\hat{I})$].

Lemma 1. *Let N be a positive integer. For any subset A of $\hat{I}$, let $\mathscr{S}_A$ denote the set of all strings of length $\leqq N$ which are disjoint from A. Call a Hamiltonian function f for L^n "admissible" if, for every $\boldsymbol{u} \in V^n$ and $i \in P_n$, there is a finite subset A of $\hat{I}$ such that $\Delta_f(\boldsymbol{u}+S\boldsymbol{e}^i) \leqq 3^{n-2}$ for every $S \in \mathscr{S}_A$. Then, if $n \geqq 2$, there exist both a one-ended and an endless admissible Hamiltonian function for L^n.*

The proof will use a technique taken from (**2**).

Proof. The result is diagrammatically obvious if $n = 2$; cf. figs. 1 and 2, which are drawn for the illustrative case $n = 2$, $N = 4$. Assume, therefore, that the result is true for $2 \leqq n \leqq k-1$, where $k \geqq 3$. Then we can select admissible endless Hamiltonian functions g, h for L^{k-1}, L^2 respectively. Let ϕ be the monomorphism of L^2 into L^k defined by

$$\phi(\boldsymbol{z}) = g(z_1)+z_2\boldsymbol{e}^k \quad (\boldsymbol{z} \in V^2 \cup E(L^2)).$$

Then ϕh is clearly an endless Hamiltonian function for L^k; we will prove it to be admissible.

Let $\boldsymbol{u} \in V^k$, and let ${}_k\boldsymbol{u} = \boldsymbol{v}$, $\phi^{-1}(\boldsymbol{u}) = \boldsymbol{w}$. Since h is admissible, there is a finite subset A of $\hat{I}$ such that, for all $S \in \mathscr{S}_A$,

$$\Delta_{\phi h}(\boldsymbol{u}+S\boldsymbol{e}^k) = \Delta_h(\boldsymbol{w}+S\boldsymbol{e}^2) \leqq 1 < 3^{k-2}. \quad \ldots\ldots\ldots\ldots(1)$$

Moreover, if $i \in P_{k-1}$, the admissibility of g and h implies that there are finite

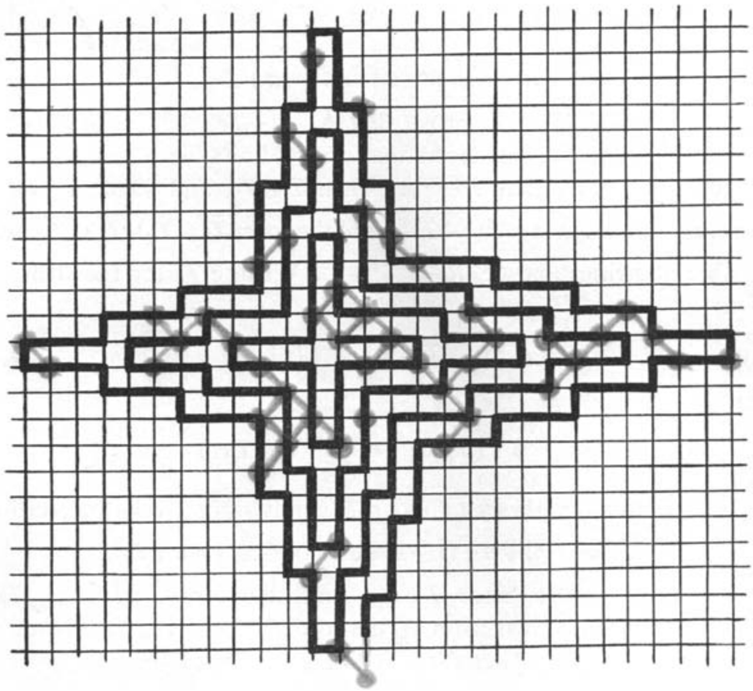

FIG. 1.

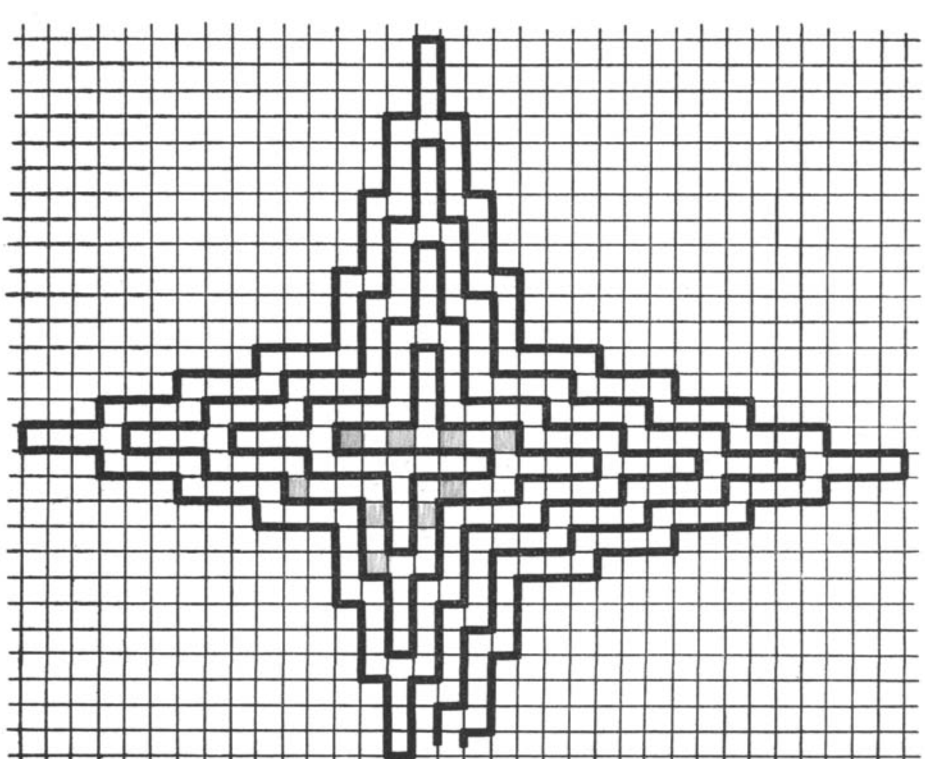

FIG. 2.

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subsets B, C of $\hat{I}$ such that $\Delta_g(v+Se^i) \leqq 3^{k-3}$, $\Delta_h(w+S'e^1) \leqq 1$ for all $S \in \mathscr{S}_B$, $S' \in \mathscr{S}_C$. Let

$$\phi(w+Ce^1) \cap (u+\hat{I}e^i) = u+De^i, \qquad \text{(2)}$$

and let $F = B \cup D$. We will prove that $\Delta_{\phi h}(u+Te^i) \leqq 3^{k-2}$ for every $T \in \mathscr{S}_F$; this result, together with (1), shows that ϕh is admissible.

Suppose, therefore, that $T \in \mathscr{S}_F$. Then $T \in \mathscr{S}_B$ and so $\Delta_g(v+Te^i) \leqq 3^{k-3}$, which clearly implies that $g^{-1}(v+Te^i)$ is of the form $\tilde{T}_1 \cup \tilde{T}_2 \cup \ldots \cup \tilde{T}_R$, where the $\tilde{T}_r$ are disjoint strings and $0 \leqq R \leqq 3^{k-3}+1$. Writing T_r for the string $\tilde{T}_r - w_1$, this gives

$$g^{-1}(v+Te^i) = \bigcup_{r=1}^{R} (T_r + w_1), \qquad \text{(3)}$$

whence

$$\phi^{-1}(u+Te^i) = \bigcup_{r=1}^{R} (w+T_r e^1). \qquad \text{(4)}$$

By (2) and (4), the hypotheses $t \in C$, $t \in T_r$ imply respectively the conclusions

$$\phi(w+te^1) \notin u+(\hat{I} \ominus D)e^i, \quad \phi(w+te^1) \in u+Te^i,$$

which are incompatible since $T \in \mathscr{S}_F$ and therefore $T \subset \hat{I} \ominus D$. Therefore $T_r \cap C = \varnothing$. It is also clear from (3) that ord $T_r \leqq$ ord $T \leqq N$. Hence $T_r \in \mathscr{S}_C$, and so $\Delta_h(w+T_r e^1) \leqq 1$. Therefore, by (4),

$$\Delta_{\phi h}((u+Te^i) \cap \phi(w+\hat{I}e^1)) = \Delta_h(\phi^{-1}(u+Te^i)) \leqq R \leqq 3^{k-3}+1. \qquad \text{(5)}$$

Moreover,

$$\text{ord } [(u+Te^i) \ominus \phi(w+\hat{I}e^1)] = \Delta_g(v+Te^i) \leqq 3^{k-3}. \qquad \text{(6)}$$

By (5) and (6),

$$\Delta_{\phi h}(u+Te^i) \leqq (3^{k-3}+1)+3^{k-3} \leqq 3^{k-2}.$$

Hence ϕh is admissible.

A similar argument shows that the composition of ϕ with any one-ended admissible Hamiltonian function for L^2 is a one-ended admissible Hamiltonian function for L^k. So Lemma 1 is now proved by induction on n.

Lemma 2. *If $n \geqq 2$, there exists a one-ended Hamiltonian function f for L^n such that $f(0) = \mathbf{0}$ and, for every $u \in V^n$, $i \in P_n$, the set $\{x \in \hat{I} \mid u+xe^i \in f(2P-\tfrac{1}{2})\}$ is unbounded above and below.*

Proof. Taking $N = 2.3^{n-2}+2$, let f' be an admissible one-ended Hamiltonian function for L^n in the sense of Lemma 1. Then, for any $u \in V^n$, $I \in P_n$, there is a finite subset A of $\hat{I}$ such that $\Delta_{f'}(u+Se^i) \leqq 3^{n-2}$ for every $S \in \mathscr{S}_A$, which implies that every string of length $2.3^{n-2}+2$ and disjoint from A includes two consecutive elements θ, $\theta+1$ such that $u+\theta e^i$ and $u+(\theta+1)e^i$ belong to $f'(\hat{J})$. These must clearly be images under f' of successive elements of $\hat{J}$; hence one of them belongs to $f'(2P-\tfrac{1}{2})$. Thus the set $\{x \in \hat{I} \mid u+xe^i \in f'(2P-\tfrac{1}{2})\}$ is unbounded above and below. Writing $f(\alpha) = f'(\alpha)-f'(0)$ for every $\alpha \in \tfrac{1}{2}J$, we clearly obtain an f which meets our requirements.

Definitions. Z will denote the set of all ordered pairs (t, y) such that $t \in I$, $y \in \hat{J}$, $t \equiv y+\frac{1}{2}(\text{mod } 2)$ and *either* $t = 0$ *or* $|t|>y$. An edge $\boldsymbol{u}$ of L^n will be called an *i-edge* if $u_i \in \hat{I}$ (i.e. if $\boldsymbol{u}$ joins vertices $\boldsymbol{v}$, $\boldsymbol{w}$ such that $\boldsymbol{v}-\boldsymbol{w} = \pm\boldsymbol{e}^i$). K, M will denote the subgraphs of L^2 defined by

$$V(M) = V(K) = \{\boldsymbol{u} \in V^2 \mid u_2 \geqq 0\}, \quad E(K) = \{\boldsymbol{u} \in E(L^2) \mid u_2 \geqq 0\},$$
$$E(M) = \{\boldsymbol{u} \in E(L^2) \mid u_2 \in J\}.$$

(Thus all edges of M are 1-edges.) An *i-couple* of L^n is a pair $\{\boldsymbol{u}, \boldsymbol{v}\}$ of i-edges of L^n such that $\boldsymbol{u}-\boldsymbol{v} = \pm\boldsymbol{e}^j$ for some $j \neq i$. A *couple* of L^n is a pair of edges of L^n which is an i-couple for some value of i. If $x, y \in \hat{I}$, $\delta(x, y)$ is defined to be the 1-couple of L^2 consisting of the edges $x\boldsymbol{e}^1+(y\pm\frac{1}{2})\boldsymbol{e}^2$. If c is an i-couple of L^n consisting of the edges $\boldsymbol{u}\pm\frac{1}{2}\boldsymbol{e}^j$ (where the vector $\boldsymbol{u}$ has necessarily just two non-integral components), the *conjugate* couple c' is defined to consist of the edges $\boldsymbol{u}\pm\frac{1}{2}\boldsymbol{e}^i$; geometrically speaking, c and c' are the two pairs of opposite sides of a unit square. Let C be a set of couples of L^n, S be the union of these couples (i.e. a subset of $E(L^n)$), S' be the union of their conjugates and H be a subgraph of L^n such that $S \subset E(H)$. Then $H*C$ will denote the subgraph of L^n defined by

$$V(H*C) = V(H), \quad E(H*C) = (E(H) \ominus S) \cup S'.$$

Lemma 3. *Let $x: Z \rightarrow \hat{I}$ be a function such that the inequalities*

$$x(-2\alpha+1, 2\alpha-\tfrac{3}{2}) < x(0, 2\alpha-\tfrac{1}{2}) < x(2\alpha-1, 2\alpha-\tfrac{3}{2})$$

hold for every positive integer α and the inequalities

$$x(\beta, \gamma-\tfrac{1}{2}) < x(\beta+1, \gamma+\tfrac{1}{2}) < x(\beta+2, \gamma-\tfrac{1}{2})$$
$$x(-\beta-2, \gamma-\tfrac{1}{2}) < x(-\beta-1, \gamma+\tfrac{1}{2}) < x(-\beta, \gamma-\tfrac{1}{2})$$

hold for every pair of positive integers β, γ such that $\beta-\gamma \in 2J$. Let C be the set of all couples of the form $\delta(x(t, y), y)$, where $(t, y) \in Z$. Then $M*C \cong L^1$.

A detailed formal proof would be tedious; but it is thought that a sufficient indication is given by fig. 3, which is drawn for the illustrative case in which x is defined by $x(t, y) = 3t+\frac{1}{2}$.

3. The Main Result

We shall now prove that L^n is decomposable into n endless Hamiltonian lines. Since this result is trivial for $n = 1$ and easily established diagrammatically [(**1**), fig. 1] for $n = 2$, we shall henceforward assume that $n \geqq 3$. Let f be a one-ended Hamiltonian function for L^{n-1} such that $f(0) = \boldsymbol{0}$ and, for every $\boldsymbol{u} \in V^{n-1}$, $i \in P_{n-1}$, the set $\{x \in \hat{I} \mid \boldsymbol{u}+x\boldsymbol{e}^i \in f(2P-\frac{1}{2})\}$ is unbounded above and below. (Such an f exists by Lemma 2.) Let ϕ^i be the monomorphism of K into L^n defined by

$$\phi^i(\boldsymbol{z}) = \sum_{j=1}^{i-1} f_j(z_2)\boldsymbol{e}^j + z_1\boldsymbol{e}^i + \sum_{j=i+1}^{n} f_{j-1}(z_2)\boldsymbol{e}^j$$

for every $\boldsymbol{z} \in V(K) \cup E(K)$, where $f_j(\theta)$ is the jth component of $f(\theta)$. We shall write

$\pi(i, x, y)$ for $\phi^i(\delta(x, y))$, where $x \in \hat{I}$, $y \in \hat{J}$. (Thus $\pi(i, x, y)$ is an i-couple of L^n.) For a fixed $i \in P_n$ and $y \in \hat{J}$, $\pi(i, \hat{I}, y)$ will denote the set of all couples of the form $\pi(i, x, y)$, $x \in \hat{I}$. A couple c of L^n will be called *admissible* if $c' = \pi(i, x, y)$ for some $i \in P_n$, $x \in \hat{I}$, $y \in \hat{J}$, and *good* if $c' = \pi(i, x, y)$ for some $i \in P_n$, $x \in \hat{I}$, $y \in 2P-\frac{1}{2}$. If $c = \pi(i, x, y)$, we define $Y(c)$ to be y.

Lemma 4. *For every $i \in P_n$, $y \in \hat{J}$, the set of those $x \in \hat{I}$ for which $\pi(i, x, y)$ is good is unbounded above and below.*

Proof. Let $i \in P_n$ and $y \in \hat{J}$. Then for any $x \in \hat{I}$, clearly

$$\pi(i, x, y)' = \phi^i(\delta(x, y)'),$$

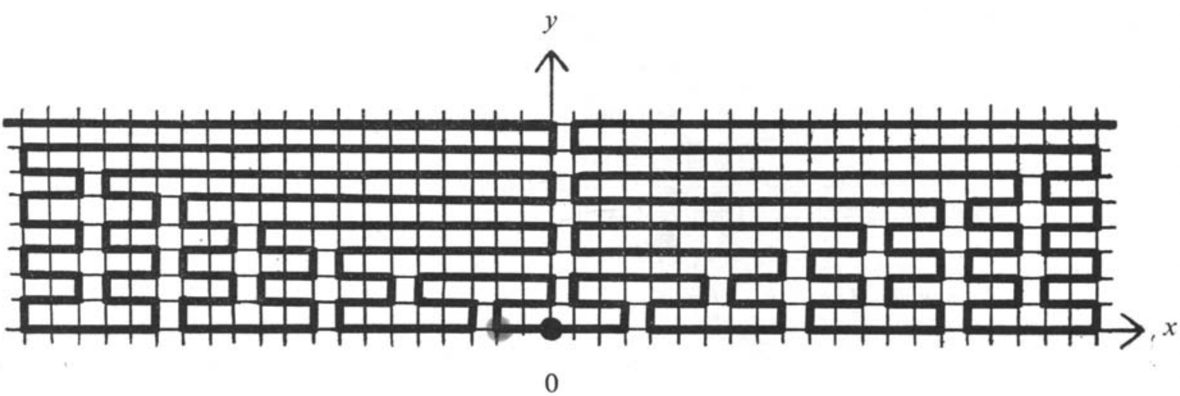

FIG. 3.

which consists of the edges

$$\phi^i((x\pm\tfrac{1}{2})e^1+ye^2) = \boldsymbol{u}+(x\pm\tfrac{1}{2})e^i, \quad\text{.........................(7)}$$

where $\boldsymbol{u} = \phi^i(ye^2)$. Let $\boldsymbol{u}$ be a j-edge; then clearly $j \neq i$. If $\alpha \in \hat{J}$, $\pi(j, u_j, \alpha)$ consists of the edges

$$\phi^j(u_je^1+(\alpha\pm\tfrac{1}{2})e^2) = \phi^j((\alpha\pm\tfrac{1}{2})e^2)+u_je^j,$$

and so coincides with (7) if the pairs of vectors $\phi^j((\alpha\pm\frac{1}{2})e^2)$, ${}_j\boldsymbol{u}+(x\pm\frac{1}{2})e^i$ coincide. These pairs of vectors are the pairs of end-vertices of the edges $\phi^j(\alpha e^2)$, ${}_j\boldsymbol{u}+xe^i$ respectively, and so coincide if ${}_j\boldsymbol{u}+xe^i = \phi^j(\alpha e^2)$, i.e. if $\boldsymbol{v}+xe^k = f(\alpha)$, where

$$\boldsymbol{v} = u_1e^1+u_2e^2+\ldots+u_{j-1}e^{j-1}+u_{j+1}e^j+u_{j+2}e^{j+1}+\ldots+u_ne^{n-1}$$

and $k = i$ or $i-1$ according as $i<j$ or $i>j$ respectively. So $\pi(i, x, y)' = \pi(j, u_j, \alpha)$ if $\boldsymbol{v}+xe^k = f(\alpha)$, and hence $\pi(i, x, y)$ is good if $\boldsymbol{v}+xe^k \in f(2P-\frac{1}{2})$. But, by our choice of f, this last property holds for a set of x-values unbounded above and below. Thus the lemma is proved.

Lemma 5. *If $\pi(i, x, y)' = \pi(j, \bar{x}, \bar{y})$, then $|\bar{x}| \leq y$.*

Proof. The hypothesis clearly implies that $i \neq j$. Let $k = j$ or $j-1$ according

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as $j<i$ or $j>i$ respectively. Clearly $\pi(i, x, y)' = \phi^i(\delta(x, y)')$, which consists of the edges $\phi^i((x\pm\frac{1}{2})e^1+ye^2)$; the jth component of each of these vectors is, by the definition of ϕ^i, equal to $f_k(y)$. But $\pi(j, \bar{x}, \bar{y})$ consists of the edges $\phi^j(\bar{x}e^1+(\bar{y}\pm\frac{1}{2})e^2)$, and the jth component of each of these vectors is, by the definition of ϕ^j, equal to $\bar{x}$. Hence $\bar{x} = f_k(y)$. But, since $f(0) = \mathbf{0}$, it is clear that $|f_k(y)|\leqq y$; hence $|\bar{x}|\leqq y$.

Definition. If, for each $m \in P$, S_m denotes the finite sequence $a_{m1}, a_{m2}, \ldots, a_{m\psi(m)}$, then $S_1\ S_2\ S_3 \ldots$ will denote the infinite sequence

$$a_{11}, a_{12}, \ldots, a_{1\psi(1)}, a_{21}, a_{22}, \ldots a_{2\psi(2)}, a_{31}, a_{32}, \ldots, a_{3\psi(3)}, \ldots.$$

For any *positive* integer α, there are only finitely many elements (t, y) of Z for which $|t| = \alpha$; hence these elements of Z can be arranged in a finite sequence s_α. Let s denote the sequence

$$\bar{s}_1 s_1 s_2 \bar{s}_2 s_3 s_4 \bar{s}_3 s_5 s_6 \bar{s}_4 s_7 s_8 \ldots,$$

where $\bar{s}_\alpha$ is the sequence whose only term is $(0, 2\alpha-\frac{1}{2})$; thus s is an arrangement of the elements of Z in an infinite sequence. Let (t_m, y_m) be the mth term of s. Let σ be the infinite sequence $\sigma_1\sigma_2\sigma_3\ldots$, where σ_m denotes the finite sequence

$$(1, t_m, y_m), (2, t_m, y_m), \ldots, (n, t_m, y_m).$$

Thus σ is a sequence of ordered triples; let (i_r, τ_r, η_r) be its rth term.

Lemma 6. *If* $q<r$, $i_q = i_r$ *and* $\tau_r = 0$, *then* $\eta_r \geqq \eta_q+2$.

Proof. The above hypotheses imply that (τ_q, η_q) is an earlier term of s than $(0, \eta_r)$, which, by the definitions of Z and s, clearly implies that $\eta_r \geqq \eta_q+2$.

We shall now select in succession admissible couples $c_1, c_2, c_3, \ldots$ of L^n. For each r, c_r will be $\pi(i_r, \xi_r, \eta_r)$ for some $\xi_r \in \hat{I}$; so the selection of c_r will be determined by that of ξ_r and vice-versa. First, we take c_1 to be a good element of $\pi(i_1, \hat{I}, \eta_1)$; this is possible by Lemma 4. Suppose we have selected the admissible couples $c_1, c_2, \ldots, c_{r-1}$ (and the associated numbers $\xi_1, \xi_2, \ldots, \xi_{r-1}$), where $r\geqq 2$. Let

$$A_r = \max_{q=1}^{r-1} \max(|\xi_q|, \eta_q, Y(c_q')).$$

($Y(c_q')$ is defined since c_q is admissible.) We now choose ξ_r (or c_r) in accordance with the following instructions; the possibility of the choice in Cases (i)-(iii) follows from Lemma 4.

(i) If $\tau_r>0$, choose ξ_r so that $c_r = \pi(i_r, \xi_r, \eta_r)$ is good and $\xi_r > A_r$.

(ii) If $\tau_r<0$, choose ξ_r so that c_r is good and $\xi_r < -A_r$.

(iii) If $\tau_r = 0$ and none of $c_1', c_2', \ldots, c_{r-1}'$ belongs to $\pi(i_r, \hat{I}, \eta_r)$, choose ξ_r so that c_r is good and $|\xi_r| > A_r$.

(iv) If $\tau_r = 0$ and at least one c_q' $(q<r)$ belongs to $\pi(i_r, \hat{I}, \eta_r)$, let c_r be one such c_q'.

Then c_r is certainly admissible since it is good in Cases (i)-(iii) and conjugate to some $\pi(i_q, \xi_q, \eta_q)$ in Case (iv).

We now prove that, if $q<r$, c_q and c_r are disjoint (i.e. have no edge in common). Since this is obvious if $i_q \neq i_r$ from the fact that c_q is an i_q-couple and c_r an i_r-couple, we shall assume that $i_q = i_r$. Then $c_q = \pi(i_r, \xi_q, \eta_q)$ and $c_r = \pi(i_r, \xi_r, \eta_r)$ are disjoint if $\delta(\xi_q, \eta_q)$, $\delta(\xi_r, \eta_r)$ are disjoint, which is the case if either $\xi_q \neq \xi_r$ or $\eta_r \geqq \eta_q + 2$. But, in Cases (i)-(iii), $|\xi_r| > A_r \geqq |\xi_q|$ while, in Case (iv), $\eta_r \geqq \eta_q + 2$ by Lemma 6.

We next prove that, in Case (iv), there can in fact have been only one possible choice for c_r. For suppose, if possible, that $\tau_r = 0$, $p<q<r$ and c'_p, c'_q both belong to $\pi(i_r, \hat{I}, \eta_r)$. By Lemma 6, there is no $\rho<r$ such that $i_\rho = i_r$ and $\eta_\rho = \eta_r$; hence $c_\rho \notin \pi(i_r, \hat{I}, \eta_r)$ for any $\rho<r$. Therefore there is no $\rho<r$ (and hence *a fortiori* no $\rho<q$) such that $c_\rho = c'_q$; so c_q cannot have been chosen according to the rule for Case (iv). We therefore have, by Lemma 5,

$$Y(c'_q) \geqq |\xi_q| > A_q \geqq Y(c'_p),$$

contrary to the hypothesis that $Y(c'_p) = Y(c'_q) = \eta_r$; this contradiction proves that the choice of c_r must have been uniquely determined.

We now show that, for each $r \in P$, c'_r is a term of the sequence (c_m). If c_r was chosen by Rule (iv), this is immediate. In all other cases, c_r is good and so $c'_r \in \pi(j, \hat{I}, y)$ for some $j \in P_n$, $y \in 2P - \frac{1}{2}$. By the definition of σ, there is a unique p such that $(i_p, \tau_p, \eta_p) = (j, 0, y)$. Since the conjugate of an i_r-couple cannot be an i_r-couple, $i_r \neq j = i_p$ and hence $p \neq r$. Moreover, $p<r$ would imply that

$$y = \eta_p \leqq A_r < |\xi_r| \leqq Y(c'_r)$$

by Lemma 5, contrary to the assumption that $Y(c'_r) = y$. Therefore $p>r$. Since $\tau_p = 0$, $p>r$ and $c'_r \in \pi(j, \hat{I}, y) = \pi(i_p, \hat{I}, \eta_p)$, c_p must be chosen by Rule (iv), and, since we have just shown that Rule (iv) in fact gives only one choice, it follows that $c_p = c'_r$; hence c'_r is a term of our sequence (c_m).

If $(t, y) \in Z$, $i \in P_n$, there is a unique r such that $(i, t, y) = (i_r, \tau_r, \eta_r)$; define $x_i(t, y)$ to be ξ_r for this value of r.

Lemma 7. *If (t, y), $(\bar{t}, \bar{y}) \in Z$ and $(\bar{t}, \bar{y})$ is a later term of s than (t, y), then*

(a) $x_i(\bar{t}, \bar{y}) > x_i(t, y)$ if $\bar{t} > 0$,
(b) $x_i(\bar{t}, \bar{y}) < x_i(t, y)$ if $\bar{t} < 0$.

Proof. The hypotheses imply that $(i, t, y) = (i_q, \tau_q, \eta_q)$, $(i, \bar{t}, \bar{y}) = (i_r, \tau_r, \eta_r)$ for some q, r such that $q<r$. Moreover, if $\bar{t}>0$, i.e. $\tau_r>0$, then ξ_r is chosen in accordance with Rule (i) and so $\xi_r > A_r \geqq \xi_q$, which, since $\xi_r = x_i(\bar{t}, \bar{y})$, $\xi_q = x_i(t, y)$, establishes (*a*). Similarly, if $\bar{t} = \tau_r < 0$, ξ_r is chosen in accordance with Rule (ii) and so $\xi_r < -A_r \leqq \xi_q$, which establishes (*b*).

Let H_i be the spanning subgraph of L^n whose edges are precisely the i-edges of L^n. Let S_i be the set of all i-couples in the sequence (c_m). Since we have seen that the terms of this sequence are disjoint and that the conjugate of each term of the sequence is also a term of the sequence, it follows that the subgraphs $H_i * S_i$ constitute a decomposition of L^n. Since, moreover, these are spanning subgraphs of L^n, it suffices to prove that they are all isomorphic to L^1. But H_i, S_i are the images under ϕ^i of M, C_i respectively, where C_i is the set of all couples of the form $\delta(x_i(t, y), y)$, $(t, y) \in Z$. Therefore $H_i * S_i$ is the image of $M * C_i$, and so it suffices to prove that $M * C_i \cong L^1$. To do this, it suffices, by Lemma 3, to show that the hypotheses of that lemma are satisfied by the function $x_i: Z \rightarrow \hat{I}$. But this follows at once from Lemma 7.

REFERENCES

(1) G. Ringel, Über drei kombinatorische Probleme am n-dimensionalen Würfel und Würfelgitter. *Abh. Math. Sem. Univ. Hamburg* **20** (1955), 10-19.

(2) E. Vázsonyi, Über Gitterpunkte des mehrdimensionalen Raumes. *Acta Litt. Sci. Szeged, Sect. Sci. Math.* **9** (1939), 163-173.

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