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EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS OF THE CAUCHY PROBLEM FOR PARABOLIC DELAY-DIFFERENTIAL EQUATIONS

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In this paper, a class of systems governed by second order linear parabolic partial delay-differential equations in "divergence form" with Cauchy conditions is considered. Existence and uniqueness of a weak solution is proved and its *a priori* estimate is established.

1. Introduction

In the absence of time delayed argument, the existence and uniqueness of solutions for systems governed by parabolic partial differential equations with Cauchy conditions have been studied in [1] to [7] and others.

In this paper, we consider questions on the existence and uniqueness of weak solutions of a class of systems governed by the following parabolic partial delay-differential equations with Cauchy conditions

$$(1.1)\quad \begin{cases} L\phi(x,\ t) = \sum_{k=0}^{N} \left\{ \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(F_{kj}(x,\ t-h_k)\right) + f_k(x,\ t-h_k) \right\} , \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,\ t) \in R^n \times (0,\ T) , \\ \phi(x,\ t) = \Phi(x,\ t) ,\quad (x,\ t) \in R^n \times [-h_N,\ 0] , \end{cases}$$

where $h_1, h_2, \ldots, h_N$ and T are constants so that

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$$0 = h_0 < h_1 < \ldots < h_N < T < \infty , \quad N \text{ is finite,}$$

and the operator L is defined by

$$(1.2) \quad L\psi(x, t) \triangleq \frac{\partial\psi(x,t)}{\partial t} - \sum_{k=0}^{N} \left\{ \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[\sum_{i=1}^{n} a_{kij}(x, t-h_k) \cdot \frac{\partial\psi(x,t-h_k)}{\partial x_i} + a_{kj}(x, t-h_k) \cdot \psi(x, t-h_k) \right] + \sum_{j=1}^{n} b_{kj}(x, t-h_k) \cdot \frac{\partial\psi(x,t-h_k)}{\partial x_j} + c_k(x, t-h_k) \cdot \psi(x, t-h_k) \right\} .$$

Weak solutions of system (1.1) are defined in the sense of Ladyženskaja, Solonnikov, Ural'ceva [7, p. 171]. The result on the existence and uniqueness of a weak solution is presented in Theorem 4.1 of §4.

2. Notations

Let R^s denote the s-dimensional Euclidean space. For any $z \in R^s$, let $|z| = \left(\sum_{i=1}^{s} |z_i|^2 \right)^{\frac{1}{2}}$. "a.e." means almost everywhere with respect to Lebesgue measure. $\bar{B}$ denotes the closure of the set B.

$L^2(R^n)$ is the Banach space consisting of all measurable functions $z : R^n \to R^1$ that are second power integrable on R^n. Its norm is defined by

$$\|z\|_{2,R^n} \triangleq \left(\int_{R^n} |z(x)|^2 dx \right)^{\frac{1}{2}} .$$

$L^{q,r}(R^n \times I)$ $(1 \le q, r \le \infty)$, is the Banach space of all measurable functions $z : R^n \times I \to R^1$ with finite norm $\|z\|_{q,r,R^n \times I}$, where

$$\|z\|_{q,r,R^n\times I} \triangleq \left\{ \int_I \left(\int_{R^n} |z(x, t)|^q dx \right)^{r/q} dt \right\}^{1/r} \quad \text{for } 1 \le q , \; r < \infty ,$$

$$\|z\|_{q,\infty,R^n\times I} \triangleq \operatorname*{ess\,sup}_{t\in I} \|z(\cdot, t)\|_{q,R^n} \quad \text{for } 1 \le q < \infty , \; r = \infty ,$$

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$$\|z\|_{\infty,r,R^n\times I} \triangleq \left\{\int_I \left(\|z(\cdot,\ t)\|_{\infty,R^n}\right)^r dt\right\}^{1/r} \quad \text{for} \quad q = \infty\ , \quad 1 \le r < \infty\ ,$$

and

$$\|z\|_{\infty,\infty,R^n\times I} \triangleq \operatorname*{ess\ sup}_{(x,t)\in R^n\times I} |z(x,\ t)| \quad \text{for} \quad q = r = \infty\ .$$

$W^{2,r}(R^n \times I)$ $(r \ge 1)$, is the Banach space of all functions z from $L^{2,r}(R^n \times I)$ having a generalized derivative z_x and a finite norm $|||z|||_r$, where

$$|||z|||_r \triangleq \left\{\int_I \left(\|z(\cdot,\ t)\|^r_{2,R^n}+\|z_x(\cdot,\ t)\|^r_{2,R^n}\right)dt\right\}^{1/r} \quad \text{for} \quad 1 \le r < \infty\ ,$$

and

$$|||z|||_\infty \triangleq \operatorname*{ess\ sup}_{t\in I} \left(\|z(\cdot,\ t)\|_{2,R^n}+\|z_x(\cdot,\ t)\|_{2,R^n}\right) \quad \text{for} \quad r = \infty\ ,$$

while $\|z_x(\cdot,\ t)\|_{2,R^n} \triangleq \left\{\int_{R^n} \sum_{i=1}^n |z_{x_i}(x,\ t)|^2 dx\right\}^{\frac{1}{2}}$ and $\|z(\cdot,\ t)\|_{2,R^n}$ is as defined before.

$W_2^{1,0}(R^n \times I)$ is the Hilbert space with scalar product

$$(z,\ y)_{W_2^{1,0}(R^n\times I)} \triangleq \iint_{R^n\times I} \left\{z\cdot y + \sum_{i=1}^n \frac{\partial z}{\partial x_i} \cdot \frac{\partial y}{\partial x_i}\right\}dxdt$$

and $W_2^{1,1}(R^n \times I)$ is the Hilbert space with scalar product

$$(x,\ y)_{W_2^{1,1}(R^n\times I)} \triangleq \iint_{R^n\times I} \left\{z\cdot y + \sum_{i=1}^n \frac{\partial z}{\partial x_i} \cdot \frac{\partial y}{\partial x_i} + \frac{\partial z}{\partial t} \cdot \frac{\partial y}{\partial t}\right\}dxdt\ .$$

$V_2(R^n \times I)$ is the Banach space consisting of all functions z from $W_2^{1,0}(R^n \times I)$ having a finite norm

$$|z|_{R^n\times I} \triangleq \|z\|_{2,\infty,R^n\times I} + \|z_x\|_{2,2,R^n\times I}\ ,$$

where

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$$\|z_x\|_{2,2,R^n\times I} \triangleq \left\{\iint_{R^n\times I} \sum_{i=1}^{n} \left|\frac{\partial z(x,t)}{\partial x_i}\right|^2 dxdt\right\}^{\frac{1}{2}} .$$

$V_2^{1,0}(R^n \times I)$ is the Banach space consisting of all functions $z \in V_2(R^n \times I)$ that are continuous in t in the norm of $L^2(R^n)$, with norm

$$|z|_{R^n\times I} \triangleq \max_{t\in\bar{I}} \|z(\cdot,\ t)\|_{2,R^n} + \|z_x\|_{2,2,R^n\times I} .$$

The continuity in t of a function z in the norm $L^2(R^n)$ means that

$$\|z(\cdot,\ t+\Delta t)-z(\cdot,\ t)\|_{2,R^n} \to 0 \quad \text{as} \quad \Delta t \to 0 .$$

The space $V_2^{1,0}(R^n \times I)$ is obtained by completing the set $W_2^{1,1}(R^n \times I)$ in the norm of $V_2(R^n \times I)$.

$V_2^{1,\frac{1}{2}}(R^n \times I)$ is the Banach space of all functions $z \in V_2^{1,0}(R^n \times I)$ for which

$$\int_0^{T-h} \int_{R^n} \frac{1}{h} \left(z(x,\ t+h)-z(x,\ t)\right)^2 dxdt \to 0 \quad \text{as} \quad h \to 0 .$$

$$\psi_t \triangleq \frac{\partial\psi}{\partial t} , \quad \psi_{x_i} \triangleq \frac{\partial\psi}{\partial x_i} , \quad (\ )_{x_j} \triangleq \frac{\partial}{\partial x_j} (\ ) .$$

3. Definitions and basic assumptions

Let h_k $(k = 0, 1, \ldots, N)$, and T be fixed constants so that $0 = h_0 < h_1 < \ldots < h_N < T < \infty$, N is finite. Let $Q = R^n \times (0, T)$, $Q_0 = R^n \times [-h_N, 0]$ and $Q_1 = R^n \times [-h_N, T]$.

For brevity, we introduce the following notations

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$$(3.1)\quad \langle L\Psi,\ Z\rangle_Q \triangleq \int_Q\int \Bigg[-\psi(x,\ t)\cdot Z_t(x,\ t) + \sum_{k=0}^{N}\Bigg\{\sum_{j=1}^{n}\Bigg(\sum_{i=1}^{n} a_{kij}(x,\ t-h_k)$$

$$\cdot\Psi_{x_i}(x,\ t-h_k)+a_{kj}(x,\ t-h_k)\cdot\Psi(x,\ t-h_k)\Bigg)\cdot Z_{x_j}(x,\ t) - \sum_{j=1}^{n} b_{kj}(x,\ t-h_k)$$

$$\cdot\Psi_{x_j}(x,\ t-h_k)\cdot Z(x,\ t)-c_k(x,\ t-h_k)\cdot\Psi(x,\ t-h_k)\cdot Z(x,\ t)\Bigg\}\Bigg]dxdt\ ,$$

for any functions $\Psi \in W^{2,2}(Q_1)$ and $Z \in W_2^{1,1}(Q)$, where L is as defined in (1.2).

$$(3.2)\quad \langle F,\ Z\rangle_Q \triangleq \int_Q\int \Bigg[\sum_{k=0}^{N}\Bigg\{\sum_{j=1}^{n} F_{kj}(x,\ t-h_k)\cdot Z_{x_j}(x,\ t)$$

$$-f_k(x,\ t-h_k)\cdot Z(x,\ t)\Bigg\}\Bigg]dxdt$$

for any function $Z \in W_2^{1,1}(Q)$, where F is defined by

$$(3.3)\qquad F(x,\ t) = \sum_{k=0}^{N}\Bigg\{\sum_{j=1}^{n}\big(F_{kj}(x,\ t-h_k)\big)_{x_j}+f_k(x,\ t-h_k)\Bigg\}\ .$$

Corresponding to system (1.1) we need

DEFINITION 3.1. A function $\phi : Q_1 \to R^1$ is said to be a *weak solution* from $V_2^{1,\frac{1}{2}}(Q)$ in the sense of Ladyženskaja, Solonnikov Ural'ceva [7, p. 171] if

(i) $\phi|_Q \in V_2^{1,\frac{1}{2}}(Q)$,

(ii) $\phi(x,\ t) = \Phi(x,\ t)$ on Q_0 , and

(iii) $\langle L\phi+F,\ \eta\rangle_Q = \int_{R^n} \Phi(x,\ 0)\cdot\eta(x,\ 0)dx$ for any $\eta \in W_2^{1,1}(Q)$ that is equal to zero at $t = T$, where $\phi|_Q$ denotes the restriction of ϕ on Q .

The following assumptions will be referred to as *assumptions* (A):

(i) for each $k \in \{0,\ 1,\ \ldots,\ N\}$ and $i,\ j \in \{1,\ \ldots,\ n\}$,

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the functions a_{kij}, a_{kj}, b_{kj}, c_k, F_{kj} and f_k are measurable on $R^n \times [-h_k, T-h_k]$ with values in R^1 ;

(ii) there exist constants $\nu, \mu > 0$ such that

$$\nu|\xi|^2 \le \sum_{i,j=1}^{n} a_{0ij}(x, t)\cdot\xi_i\cdot\xi_j \le \mu|\xi|^2$$

a.e. in $R^n \times [0, T]$ for all $\xi \in R^n$;

(iii) there exist constants $\mu_1, \mu_2 > 0$ such that

$$\left\| \sum_{j=1}^{n} a_{0j}^2, \sum_{j=1}^{n} b_{0j}^2, c_0 \right\|_{q,r,Q} \le \mu_1 ,$$

in which q and r are arbitrary numbers satisfying the conditions

$$(3.4) \qquad \begin{cases} \frac{1}{r} + \frac{n}{2q} = 1 , \\ q \in \left(\frac{n}{2}, \infty\right) , \quad r \in [1, \infty) \quad \text{for} \quad n \ge 2 , \\ q \in [1, \infty] , \quad r \in [1, 2] \quad \text{for} \quad n = 1 ; \end{cases}$$

and $|a_{kij}, a_{kj}, b_{kj}, c_k| \le \mu_2$ $(i, j = 1, \ldots, n)$, a.e. on $R^n \times [-h_k, T-h_k]$ for each $k = 1, \ldots, n$;

(iv) for each $k \in \{0, 1, \ldots, N\}$,

$$F_{kj} \in L^{2,2}\left(R^n \times (-h_k, T-h_k)\right) \quad (j = 1, \ldots, n) ,$$

and $f_k \in L^{2,s}\left(R^n \times (-h_k, T-h_k)\right)$ where $s \in [1, 2]$; and

(v) $\Phi \in W^{2,2}(Q_0)$ and $\Phi(\cdot, 0) \in L^2(R^n)$.

4. Existence of weak solutions

In this section we shall show the existence and uniqueness of a weak solution of system (1.1). Further, an *a priori* estimate of the weak solution will be also established.

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THEOREM 4.1. *Consider system* (1.1). *Let the assumptions* (A) *be satisfied. Then system* (1.1) *admits a unique weak solution* ϕ *from* $V_2^{1,\frac{1}{2}}(Q)$. *Further,* ϕ *satisfies the following* a priori *estimate*

$$(4.1)\quad |\phi|_Q \le M\Bigg\{\|\Phi(\cdot,\ 0)\|_{2,R^n}+\|\Phi\|_{2,2,R^n\times(-h_N,0)}+\|\Phi_x\|_{2,2,R^n\times(-h_N,0)} + \sum_{k=0}^{N}\left[\sum_{j=1}^{n}\|F_{kj}\|_{2,2,R^n\times(-h_k,T-h_k)}+\|f_k\|_{2,s,R^n\times(-h_k,T-h_k)}\right]\Bigg\} ,$$

where $|\cdot|_Q$ *is the norm in* $V_2^{1,\frac{1}{2}}(Q)$ *and the positive constant* M *depends only on* ν, μ, μ_1, μ_2, n, N, q, s, h_1 *and* T .

Proof. Let K be an integer such that $Kh_1 < T \le (K+1)h_1$. Let us consider system (1.1) on $R^n \times [(l-1)h_1,\ lh_1)$ successively in the order of $l = 1, 2, \ldots, K$ and on $R^n \times [Kh_1,\ T)$. Then it is clear that system (1.1) reduces to systems without time delayed argument given by

$$(4.2)\quad \begin{cases} L_0\phi(x,\ t) = \sum_{j=1}^{n}\left(F_j^l(x,\ t)\right)_{x_j} + f^l(x,\ t) \\ \qquad\qquad\qquad\qquad \text{on } Q^l \triangleq R^n \times ((l-1)h_1,\ lh_1) , \\ \phi(x,\ (l-1)h_1) = \phi^{l-1}(x,\ (l-1)h_1) ,\quad x \in R^n , \end{cases}$$

for $l = 1, 2, \ldots, K$, and

$$(4.3)\quad \begin{cases} L_0\phi(x,\ t) = \sum_{j=1}^{n}\left(F_j^{K+1}(x,\ t)\right)_{x_j} + f^{K+1}(x,\ t) \\ \qquad\qquad\qquad\qquad \text{on } Q^{K+1} \triangleq R^n \times (Kh_1,\ T) , \\ \phi(x,\ Kh_1) = \phi^K(x,\ Kh_1) ,\quad x \in R^n , \end{cases}$$

where

(i) L_0 is defined by

(4.4) $L_0\psi(x, t)$

$$\triangleq \psi_t(x, t) - \sum_{j=1}^{n}\left\{\sum_{i=1}^{n} a_{0ij}(x, t)\cdot\psi_{x_i}(x, t)+a_{0j}(x, t)\cdot\psi(x, t)\right\}_{x_j} - \sum_{j=1}^{n} b_{0j}(x, t)\cdot\psi_{x_j}(x, t) - c_0(x, t)\cdot\psi(x, t) ;$$

(ii) for each $l = 1, 2, \ldots, K+1$,

(4.5) $$F_j^l(x, t) = \sum_{k=1}^{N}\left\{\sum_{i=1}^{n} a_{kij}(x, t-h_k)\cdot\tilde{\phi}_{x_i}^{l-1}(x, t-h_k)+a_{kj}(x, t-h_k)\cdot\tilde{\phi}^{l-1}(x, t-h_k)+F_{kj}(x, t-h_k)\right\} + F_{0j}(x, t) ,$$

(4.6) $$f^l(x, t) = \sum_{k=1}^{N}\left\{\sum_{j=1}^{n} b_{kj}(x, t-h_k)\cdot\tilde{\phi}_{x_j}^{l-1}(x, t-h_k) + c_k(x, t-h_k)\cdot\tilde{\phi}^{l-1}(x, t-h_k)+f_k(x, t-h_k)\right\} + f_0(x, t) ;$$

(iii) ϕ^l $(l = 1, \ldots, K)$, are weak solutions from $V_2^{1,\frac{1}{2}}(Q^l)$ of system (4.2) on $R^n \times [(l-1)h_1, lh_1)$ $(l = 1, \ldots, K)$, respectively;

(iv) $\phi^0 = \tilde{\phi}^0 = \Phi$; and

(v) for each $l = 1, \ldots, K$,

$$\tilde{\phi}^l(x, t) = \begin{cases} \Phi(x, t) , & (x, t) \in Q_0 , \\ \phi^\iota(x, t) , & (x, t) \in R^n \times [(\iota-1)h_1, \iota h_1) , \quad \iota = 1, 2, \ldots, l . \end{cases}$$

Note that it can be easily verified that

(4.7) $$\left\{\int_Q \int \sum_{i=1}^{n} \Gamma_i^2(x, t)dxdt\right\}^{\frac{1}{2}} \le n^{\frac{1}{2}} \sum_{i=1}^{n} \|\Gamma_i\|_{2,2,Q} .$$

By virtue of the definitions of $\tilde{\phi}^l$ $(l = 0, 1, \ldots, K)$, and the assumptions A (iii), A (iv) and A (v), it can be easily shown by using inequality (4.7), Minkowski's inequality and Cauchy's inequality that, for

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each $l = 1, 2, \ldots, K, K+1$,

$$(4.8)\quad \left(\int_{Q^l}\int \sum_{j=1}^{n} \left\{F_j^l(x, t)\right\}^2 dxdt\right)^{\frac{1}{2}}$$

$$\le n^{\frac{1}{2}} \sum_{j=1}^{n} \left[\|F_{0j}(\cdot, \cdot)\|_{2,2,Q^l} + \sum_{k=1}^{N} \left\{\left\|\sum_{i=1}^{n} a_{kij}(\cdot, \cdot-h_k)\cdot\tilde{\phi}_{x_i}^{l-1}(\cdot, \cdot-h_k)\right\|_{2,2,Q^l} + \left\|a_{kj}(\cdot, \cdot-h_k)\cdot\tilde{\phi}^{l-1}(\cdot, \cdot-h_k)\right\|_{2,2,Q^l} + \|F_{kj}(\cdot, \cdot-h_k)\|_{2,2,Q}\right\}\right]$$

$$\le n^{\frac{1}{2}} \sum_{j=1}^{n} \left[\|F_{0j}(\cdot, \cdot)\|_{2,2,Q^l} + \sum_{k=1}^{N} \left\{n^{\frac{1}{2}}\mu_2\left\|\tilde{\phi}_x^{l-1}(\cdot, \cdot-h_k)\right\|_{2,2,Q^l} + \mu_2\left\|\tilde{\phi}^{l-1}(\cdot, \cdot-h_k)\right\|_{2,2,Q^l} + \|F_{kj}(\cdot, \cdot-h_k)\|_{2,2,Q^l}\right\}\right]$$

$$\triangleq n^{\frac{1}{2}}\cdot\left\{\sum_{k=0}^{N} \sum_{j=1}^{n} \|F_{kj}(\cdot, \cdot-h_k)\|_{2,2,Q^l} + \sum_{k=1}^{N} \left[n^{3/2}\mu_2\left\|\tilde{\phi}_x^{l-1}(\cdot, \cdot-h_k)\right\|_{2,2,Q^l} + n\mu_2\left\|\tilde{\phi}^{l-1}(\cdot, \cdot-h_k)\right\|_{2,2,Q^l}\right]\right\} .$$

Next, by using the definitions of $\tilde{\phi}^l$ $(l = 0, 1, \ldots, K)$, and the assumptions A (iii), A (iv) and A (v), we can deduce from Minkowski's inequality and Hölder's inequality that, for each $l = 1, \ldots, K, K+1$,

$$(4.9) \quad \|f^l\|_{2,s,Q^l}$$

$$\le \Big\{\|f_0\|_{2,s,Q^l} + \sum_{k=1}^{N} \Big\{\|f_k(\cdot, \cdot-h_k)\|_{2,s,Q^l} + \sum_{j=1}^{n} \Big\|b_{kj}(\cdot, \cdot-h_k)\cdot\tilde{\phi}_{x_j}^{l-1}(\cdot, \cdot-h_k)\Big\|_{2,s,Q^l} + \Big\|c_k(\cdot, \cdot-h_k)\cdot\tilde{\phi}^{l-1}(\cdot, \cdot-h_k)\Big\|_{2,s,Q^l}\Big\}\Big\}$$

$$\le \Big\{\sum_{k=0}^{N} \|f_k(\cdot, \cdot-h_k)\|_{2,s,Q^l} + \sum_{k=1}^{N} \Big\{n\mu_2 h_1^{(2-s)/2s}\Big\|\tilde{\phi}_x^{l-1}(\cdot, \cdot-h_k)\Big\|_{2,2,Q^l} + \mu_2 h_1^{(2-s)/2s}\Big\|\tilde{\phi}^{l-1}(\cdot, \cdot-h_k)\Big\|_{2,2,Q^l}\Big\}\Big\} .$$

Further, since $\phi^0 = \Phi$ and since ϕ^l $(l = 1, \ldots, K)$, are weak solutions from $V_2^{1,\frac{1}{2}}(Q^l)$ of system (4.2) on Q^l $(l = 1, \ldots, K)$, respectively, it follows readily that $\phi^0(\cdot, 0) \in L^2(R^n)$ and, for $l = 1, \ldots, K$,

$$(4.10) \qquad \|\phi^l(\cdot, lh)\|_{2,R^n} \le |\phi^l|_{Q^l} .$$

Thus, by applications of Theorem 5.2 of [7, p. 171] to system (4.2) $(l = 1, \ldots, K)$, and system (4.3) successively, we obtain that, for each $l = 1, \ldots, K$, system (4.2) admits a unique weak solution ϕ^l from $V_2^{1,\frac{1}{2}}(Q^l)$ and system (4.3) also admits a unique weak solution ϕ^{K+1} from $V_2^{1,\frac{1}{2}}(Q^{K+1})$. Since the constant in the estimate (2.2) of Lemma 2.1 of [7, p. 139] does not depend on Ω , we examine easily that the proof of Lemma 2.1 remains valid when Ω is replaced by R^n . Thus Theorem 2.1 of [7, p. 143] remains valid when Ω is replaced by R^n . Therefore, by virtue of this modified version of Theorem 2.1 of [7, p. 143], ϕ^l satisfies the estimate

$$(4.11)\quad |\phi^l|_{Q^l} \le M_l\left\{\left\|\phi^{l-1}(\cdot,\ (l-1)h_1)\right\|_{2,R^n} + \left(\iint_{Q^l} \sum_{j=1}^{n} \left(F_j^l(x,\ t)\right)^2 dxdt\right)^{\frac{1}{2}} + \|f^l\|_{2,s,Q^l}\right\},$$

where the constant $M_l > 0$ depends only on n, ν, μ, μ_1, and q from the assumptions A (ii) - A (iii).

Let ϕ be defined on Q_1 by

$$(4.12)\quad \phi(x,\ t) = \begin{cases} \Phi(x,\ t)\ , & (x,\ t) \in Q_0\ , \\ \phi^l(x,\ t)\ , & (x,\ t) \in R^n \times [(l-1)h_1,\ lh_1)\ , \quad l = 1,\ \ldots,\ K\ , \\ \phi^{K+1}(x,\ t)\ , & (x,\ t) \in R^n \times [Kh_1,\ T)\ . \end{cases}$$

We shall show that ϕ is a unique weak solution from $V_2^{1,\frac{1}{2}}(Q)$ of system (1.1). Clearly, ϕ satisfies the conditions (i) and (ii) of Definition 3.1. Let $\eta \in W_2^{1,1}(Q)$ be arbitrary and equal to zero at $t = T$. Let η^l $(l = 1,\ \ldots,\ K+1)$, denote, respectively, the restrictions of η on $R^n \times [(l-1)h_1,\ lh_1]$ $(l = 1,\ \ldots,\ K)$ and on $R^n \times [Kh_1,\ T)$. Since ϕ^l is the weak solution from $V_2^{1,\frac{1}{2}}(Q^l)$ of system (4.2) on Q^l $(l = 1,\ \ldots,\ K)$, and ϕ^{K+1} is the weak solution from $V_2^{1,\frac{1}{2}}(Q^{K+1})$ of system (4.3) on Q^{K+1}, it follows that

$$(4.13)\quad \int_{R^n} \phi^l(x,\ lh_1)\cdot\eta^l(x,\ lh_1)dx + \left\langle L_0\phi^l + F^l,\ \eta^l\right\rangle_{Q^l} = \int_{R^n} \phi^{l-1}(x,\ (l-1)h_1)\cdot\eta^l(x,\ (l-1)h_1)dx$$

for $l = 1,\ \ldots,\ K$ and

$$(4.14)\quad \left\langle L_0\phi^{K+1} + F^{K+1},\ \eta^{K+1}\right\rangle_{Q^{K+1}} = \int_{R^n} \phi^K(x,\ Kh_1)\cdot\eta^{K+1}(x,\ Kh_1)dx\ ,$$

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where F^l $(l = 1, \ldots, K+1)$ is defined by

$$F^l(x, t) = \sum_{j=1}^{n} \left\{F_j^l(x, t)\right\}_{x_j} + f^l(x, t) \tag{4.15}$$

while F_j^l and f^l are as defined in (4.5) and (4.6), respectively.

By virtue of the definitions of η^l and ϕ^0, (4.4), (4.12), (4.13), (4.14) and (4.15), we obtain that

$$\langle L\phi+F, \eta\rangle_Q = \int_{R^n} \Phi(x, 0)\cdot\eta(x, 0)dx .$$

Thus ϕ is a weak solution from $V_2^{1,\frac{1}{2}}(Q)$ of system (1.1). Uniqueness of ϕ follows from uniqueness of ϕ^l $(l = 1, \ldots, K+1)$.

Next we shall show that ϕ satisfies estimate (4.1). Substituting (4.8) and (4.9) into (4.11), we obtain

$$\begin{aligned}
(4.16)\quad |\phi^l|_{Q^l} &\le M_l n^{\frac{1}{2}}\Bigg\{\left\|\phi^{l-1}(\cdot, (l-1)h_1)\right\|_{2,R^n} + \sum_{k=1}^{N}\Bigg\{\left\{n^{3/2}\mu_2+n\mu_2h_1^{(2-s)/2s}\right\} \\
&\quad \cdot\left\|\tilde{\phi}_x^{l-1}(\cdot, \cdot-h_k)\right\|_{2,2,Q^l}+\left\{n\mu_2+\mu_2h_1^{(2-s)/2s}\right\}\cdot\left\|\tilde{\phi}^{l-1}(\cdot, \cdot-h_k)\right\|_{2,2,Q^l}\Bigg\} \\
&\quad + \sum_{k=0}^{N}\left\{\sum_{j=1}^{n}\|F_{kj}(\cdot, \cdot-h_k)\|_{2,2,Q^l}+\|f_k(\cdot, \cdot-h_k)\|_{2,s,Q^l}\right\}\Bigg\} \\
&\le M_0\Bigg\{\left\|\phi^{l-1}(\cdot, (l-1)h_1)\right\|_{2,R^n} + \sum_{k=1}^{N}\left\{\left\|\tilde{\phi}_x^{l-1}(\cdot, \cdot-h_k)\right\|_{2,2,Q^l}\right. \\
&\quad \left.+\left\|\tilde{\phi}^{l-1}(\cdot, \cdot-h_k)\right\|_{2,2,Q^l}\right\} \\
&\quad + \sum_{k=0}^{N}\left\{\sum_{j=1}^{n}\|F_{kj}(\cdot, \cdot-h_k)\|_{2,2,Q^l}+\|f_k(\cdot, \cdot-h_k)\|_{2,s,Q^l}\right\}\Bigg\} ,
\end{aligned}$$

where the constant M_0 is defined by

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$$(4.17)\qquad M_0 = \max_{l\in\{1,\ldots,K+1\}} M_l n^{\frac{1}{2}}\left[\max\left\{1,\ n\mu_2\left(n^{\frac{1}{2}}+h_1^{(2-s)/2s}+1\right)\right\}\right] .$$

Note that, for each $l = 1, \ldots, K$,

$$(4.18)\qquad \|\phi^l\|_{2,2,Q^l} = \left(\int_{(l-1)h_1}^{lh_1}\int_{R^n} |\phi^l(x,\ t)|^2 dxdt\right)^{\frac{1}{2}}$$

$$\le \left(\int_{(l-1)h_1}^{lh_1}\left\{\max_{t\in[(l-1)h_1,lh_1]}\left(\int_{R^n} |\phi^l(x,\ t)|^2 dx\right)^{\frac{1}{2}}\right\}^2 dt\right)^{\frac{1}{2}}$$

$$\triangleq h_1^{\frac{1}{2}}\cdot \max_{t\in[(l-1)h_1,lh_1]} \|\phi^l(\cdot,\ t)\|_{2,R^n} .$$

Similarly

$$(4.19)\qquad \|\phi^{K+1}\|_{2,2,Q^{K+1}} \le h_1^{\frac{1}{2}}\cdot \max_{t\in[Kh_1,T]} \|\phi^{K+1}(\cdot,\ t)\|_{2,R^n} .$$

Further, it can be easily deduced from the definitions of $\tilde{\phi}^l$ and estimate (4.7) that

$$(4.20)\qquad \left\|\tilde{\phi}^{l-1}(\cdot,\ \cdot-h_k)\right\|_{2,2,Q^l}$$

$$\le \|\tilde{\phi}^{l-1}\|_{2,2,R^n\times(-h_N,(l-1)h_1)}$$

$$\triangleq \left(\int_{-h_N}^{0}\int_{R^n} |\Phi(x,\ t)|^2 dxdt + \sum_{\iota=1}^{l-1}\int_{(\iota-1)h_1}^{\iota h_1}\int_{R^n} |\phi^\iota(x,\ t)|^2 dxdt\right)^{\frac{1}{2}}$$

$$\le l^{\frac{1}{2}}\left(\|\Phi\|_{2,2,R^n\times(-h_N,0)} + \sum_{\iota=1}^{l-1} \|\phi^\iota\|_{2,2,Q^\iota}\right) ,$$

for all $k = 1, \ldots, N$ and $l = 2, \ldots, K+1$. Similarly as above, we have

$$(4.21)\qquad \left\|\tilde{\phi}_x^{l-1}(\cdot,\ \cdot-h_k)\right\|_{2,2,Q^l} \le l^{\frac{1}{2}}\left(\|\Phi_x\|_{2,2,R^n\times(-h_N,0)} + \sum_{\iota=1}^{l-1}\left\|\phi_x^\iota\right\|_{2,2,Q^\iota}\right)$$

for all $k = 1, \ldots, N$ and $l = 2, \ldots, K+1$.

Let

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$$(4.22)\quad C \triangleq \Big\{\|\Phi(\cdot,\ 0)\|_{2,R^n}+\|\Phi\|_{2,2,R^n\times(-h_N,0)}+\|\Phi_x\|_{2,2,R^n\times(-h_N,0)} + \sum_{k=0}^{N}\Big(\sum_{j=1}^{n}\|F_{kj}\|_{2,2,R^n\times(-h_k,T-h_k)}+\|f_k\|_{2,s,R^n\times(-h_k,T-h_k)}\Big)\Big\}$$

Then, by letting $l = 1$ in estimate (4.16), it follows from the fact that $\phi^0 = \tilde{\phi}^0 = \Phi$, and inequalities (4.20) and (4.21) that

$$(4.23)\qquad |\phi^1|_{Q^1} \le M_0NC \triangleq dC ,$$

where M_0 and C are as defined in (4.17) and (4.18), respectively.

Now, by letting $l = 2$ in estimate (4.16), we deduce from (4.10), (4.20), (4.21), (4.18) and (4.23) that

$$\begin{aligned}(4.24)\quad |\phi^2|_{Q^2} \le d\Big\{dC &+2^{\frac{1}{2}}\Big(\|\Phi_x\|_{2,2,R^n\times(-h_N,0)}+\Big\|\phi_x^1\Big\|_{2,2,Q^1}\Big) \\ &+2^{\frac{1}{2}}\Big(\|\Phi\|_{2,2,R^n\times(-h_N,0)}+h_1^{\frac{1}{2}}\max_{t\in[0,h_1]}\|\phi^1(\cdot,\ t)\|_{2,R^n}\Big) \\ &+\sum_{k=0}^{N}\Big(\sum_{j=1}^{n}\|F_{kj}(\cdot,\ \cdot-h_k)\|_{2,2,Q^1}+\|f_k(\cdot,\ \cdot-h_k)\|_{2,s,Q^1}\Big)\Big\} \\ \le{}& 2^{\frac{1}{2}}d\big(dC+C+h^{\frac{1}{2}}dC\big) \\ \le{}& 2^{\frac{1}{2}}d(1+hd)C ,\end{aligned}$$

where $h = 1 + h_1^{\frac{1}{2}}$ and C is as defined in (4.22).

By the same token, we can show successively in the order of $l = 3, 4, \ldots, K+1$ that

$$(4.25)\qquad |\phi^l|_{Q^l} \le (l!)^{\frac{1}{2}}dC(1+hd)^{l-1} ,$$

where C and the constants d and h are as defined before.

On the other hand, we deduce from inequality (4.7) that

$$(4.26)\quad |\phi|_Q = \max_{t\in[0,T]} \|\phi(\cdot,\ t)\|_{2,R^n} + \|\phi_x\|_{2,2,Q}$$

$$\leq (K+1)^{\frac{1}{2}} \cdot \left\{ \sum_{l=1}^{K} \left(\max_{t\in[(l-1)h_1, lh_1]} \|\phi^l(\cdot,\ t)\|_{2,R^n} + \left\|\phi_x^l\right\|_{2,2,Q^l} \right) \right.$$

$$\left. + \max_{t\in[Kh_1, T]} \|\phi^{K+1}(\cdot,\ t)\|_{2,R^n} + \left\|\phi_x^{K+1}\right\|_{2,2,Q^{K+1}} \right\}$$

$$\triangleq (K+1)^{\frac{1}{2}} \sum_{l=1}^{K+1} |\phi^l|_{Q^l} .$$

Thus by substituting inequalities (4.23), (4.24) and (4.25) into the right hand side of (4.26) we obtain estimate (4.1) with

$$M \triangleq (K+1)^{\frac{1}{2}}\{d + 2^{\frac{1}{2}}d(1+hd) + (3!)^{\frac{1}{2}}d(1+hd)^2 + \ldots + \left((K+1)!\right)^{\frac{1}{2}}d(1+hd)^K\} .$$

This completes the proof.

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