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Global Well-Posedness and Convergence Results for the 3D-Regularized Boussinesq System

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Abstract. Analytical study of the regularization of the Boussinesq system is performed in frequency space using Fourier theory. Existence and uniqueness of weak solutions with minimum regularity requirement are proved. Convergence results of the unique weak solution of the regularized Boussinesq system to a weak Leray–Hopf solution of the Boussinesq system are established as the regularizing parameter α vanishes. The proofs are done in the frequency space and use energy methods, the Arzelà–Ascoli compactness theorem and a Friedrichs-like approximation scheme.

1 Introduction

Let us consider the following three-dimensional incompressible Boussinesq system denoted (Bq) and given by

$$\partial_t \theta - \kappa\Delta\theta + (u\cdot\nabla)\theta = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3$$

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u = -\nabla p + \theta e_3 \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3$$

$$\operatorname{div} u = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3$$

$$(u,\theta)|_{t=0} = (u^0,\theta^0) \quad \text{in } \mathbb{T}^3,$$

where $\nu > 0$ is the viscosity of the fluid and $\kappa > 0$ its thermal conductivity. The unknown vector field u and the unknown scalars p and θ denote respectively the velocity, the pressure and the temperature of the fluid at the point $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^3$. The data u^0 and θ^0 are the given initial velocity and temperature, where u^0 is divergence free. If u_0 and θ_0 are quite regular, the divergence-free condition determines the pressure p, and

$$p = -\Delta^{-1}\Big(\sum_{i,j=1}^{3}\partial_i\partial_j(u^i u^j) - \partial_3\theta\Big). \tag{1.1}$$

As for physical interpretation, the Boussinesq system is used as a toy model for geophysical fluids whenever rotation and stratification play important roles. The scalar θ

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may represent temperature variation in a gravity field and the vector θe_3 is the buoyancy force. It is well known that actually neither the available theory nor the analytical advances are sufficient to prove the global well-posedness of three-dimensional Navier–Stokes-like equations, namely the Boussinesq system, which is a coupling between the fluid velocity and temperature. To overcome this serious difficulty and to make practical progress, researchers interested in such fields incorporate many numerical regularizations of three-dimensional fluid mechanic equations. The aim was usually to obtain models that produce smooth and regular numerical or analytical solutions which agree with experimental studies in practical situations. In this framework, many models were proposed, including the hyperviscosity [13], the nonlinear viscosity [10], [11], [21], and the alpha-models [6], [8], [9], [12], [7]. The latter, denoted also by α-models, produce solutions that are in excellent agrement with empirical data. They also can be implemented in a relatively simple way in numerical computation of three-dimensional fluid equations. Numerical simulation of fluid equations in their original formulation introduced by Euler, Navier, and Stokes are known to be prohibitive, disputable, and very sensitive to initial data [8], [7].

In this paper, filtering only the linear part of the fluid equation, we obtain the so-called simplified Bardina model, which is one particular case of the family of α-models. It will be denoted by (Bq_α) and reads

$$\partial_t \theta - \kappa\Delta\theta + (u\cdot\nabla)\theta = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3 \tag{1.2}$$

$$\partial_t v - \nu\Delta v + (u\cdot\nabla)u = -\nabla p + \theta e_3 \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3 \tag{1.3}$$

$$v = u - \alpha^2\Delta u \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3 \tag{1.4}$$

$$\operatorname{div} u = \operatorname{div} v = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3 \tag{1.5}$$

$$(u,\theta)|_{t=0} = (u^0,\theta^0) \quad \text{in } \mathbb{T}^3, \tag{1.6}$$

with periodic boundary conditions and hence periodic solutions. For a detailed introduction to α-models, especially the Bardina one, readers can see [6] and the extensive bibliography therein. In such references, existence and uniqueness results were proved using the Galerkin approximation scheme. To put the α-models in their historical framework and see some of their numerical applications, we can see the paper [8]. Convergence results are derived for α-geostrophic equations in [9] and certain α-MHD in [12] where Aubin–Lions compactness methods are used in the framework of the phase space.

Throughout this paper, and for both existence and convergence results, we use frequency space computations, since they are well suited for unlimited domains such as $\mathbb{T}^3$ and $\mathbb{R}^3$. Such domains describe well the physical situations where the Boussinesq system is properly used, namely atmospheric and oceanographic turbulence, where rotation and stratification play an important role. We believe that our proofs are new, simpler and shorter than those done for other α-systems in phase space. In the case of geophysical magnetohydrodynamic systems, we have already used frequency space analysis to deal with existence, uniqueness and convergence results as a small parameter (Rossby number) vanishes [18], [1], [2], [19], [20]. Using the Fourier transform as a principal tool, we also gave an asymptotic study and stability results for both two-dimensional Leray weak solutions [4] and three-dimensional Fujita–Kato strong solutions [3] to the periodic Navier stokes equation in critical spaces as time goes to infinity. Mainly, we gave an alternative simpler and shorter proof for the vanishing limit of the 2D weak solution of the Navier–Stokes equation using only the frequency structure of the solution without any additional assumptions [4]. Here, we are dealing with the periodic case. The case of the whole space can be treated using the same method with some minor modifications.

Our first result is a theorem that addresses the problem of existence of the weak solution to the system (Bq_α) within the minimal regularity of the initial data.

Theorem 1.1 *Let $\theta_0 \in L^2(\mathbb{T}^3)$ and let $u_0 \in \dot{H}^1(\mathbb{T}^3)$ be a divergence-free vector field. Then for $T = T(u_0, \theta_0) > 0$, there exists a weak solution (u, θ) of system (Bq_α) such that*

$$u \in C\big([0,T], \dot{H}^1(\mathbb{T}^3)\big) \cap L^2\big([0,T], \dot{H}^2(\mathbb{T}^3)\big)$$

and

$$\theta \in C\big([0,T], L^2(\mathbb{T}^3)\big) \cap L^2\big([0,T], \dot{H}^1(\mathbb{T}^3)\big).$$

Moreover, this solution satisfies the following energy estimates

$$\begin{aligned} (1.7)\quad & \|u(t)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u(t)\|^2_{L^2(\mathbb{T}^3)} + \|\theta(t)\|^2_{L^2(\mathbb{T}^3)} \\ & \qquad + 2\int_0^t \nu\big(\|\nabla u(t)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u(t)\|^2_{L^2(\mathbb{T}^3)}\big) + \kappa\|\theta(\tau)\|^2_{L^2(\mathbb{T}^3)}\, d\tau \\ & \le \|u^0\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u^0\|^2_{L^2(\mathbb{T}^3)} + \|\theta^0\|^2_{L^2(\mathbb{T}^3)} + 2\rho_\alpha(T) \end{aligned}$$

where ρ_α is a positive increasing function of time t, defined by

$$\rho_\alpha(t) = \frac{1}{2}(e^{2t} - 1)(\|u^0\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u^0\|^2_{L^2(\mathbb{T}^3)}) + \left(\left(t - \frac{1}{2}\right)e^{2t} + t + \frac{1}{2}\right)\|\theta^0\|^2_{L^2(\mathbb{T}^3)}.$$

The proof is done in the frequency space and uses the so-called Friedrichs method, an approximating method that is well suited for the frequency calculation. First, we approximate (Bq_α) by a system for which we can apply the classical theory of ordinary differential equations to construct an approximate solution. Next, we obtain uniform estimates that are independent of the approximating parameter n regarding this approximate solution. To do so, we use conservation laws and product lemmas. While trying to close the energy estimates, the buoyancy force presents some difficulties that we overcome by a Gronwall-type technique. After that, we run a compactness method based on the Arzelà–Ascoli theorem. Cantor's diagonal extraction process allows constructing a convergent subsequence in suitable spaces. Properties of Friedrichs' operator are widely used in the proof.

Our second result deals with one of the main objectives behind the regularization, that is, uniqueness of weak solution in the three-dimensional case to the regularized system. Such uniqueness is not available for original Boussinesq system. More than uniqueness, we get continuous dependence of the weak solution on the initial data. Mainly, we have the following theorem.

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Theorem 1.2 *The weak solution dealt with in Theorem* 1.1 *is continuously dependent on the initial data on any bounded interval* $[0, T]$. *In particular, that solution is unique.*

To prove this theorem, we consider the system satisfied by the difference of two solutions. The energy method is applied. The nonlinear terms are tracked one by one using results of the existence part, Young product inequalities, and suitable Sobolev product laws. Finally, a Gronwall-type differential inequality is derived. This allows us to infer continuous dependence of the solutions on initial data in any bounded interval $[0, T]$, and in particular, the solution is unique.

Our third result discusses the weak and strong convergence of the unique weak solution of the regularized system (Bq_α) to a Leray–Hopf weak solution of the original system (Bq) as the regularizing parameter α goes to zero. The latter solution is known to exist [5]. Actually, in addition to uniqueness, these convergence results make this regularization very useful for numerical study of the original fluid equation. Those convergence results are summarized in the following theorem.

Theorem 1.3 *Let* $T > 0$, $u_0 \in \dot{H}^1(\mathbb{T}^3)$ *a divergence-free vector field,* $\theta_0 \in L^2(\mathbb{T}^3)$ *and* u_α, θ_α, *the solutions of system* (Bq_α) *and* $v_\alpha = u_\alpha - \alpha^2\Delta u_\alpha$. *Then there are subsequences* u_{α_k}, v_{α_k}, θ_{α_k}, *and a scalar function* $\breve{\theta}$ *and a divergence-free vector field* $\breve{u}$ *both of them belonging to* $L^\infty\big([0,T], L^2(\mathbb{T}^3)\big) \cap L^2\big([0,T], \dot{H}^1(\mathbb{T}^3)\big)$, *such that as* $\alpha_k \to 0^+$ *we have*

- u_{α_k} *converges to* $\breve{u}$ *and* θ_{α_k} *converges to* $\breve{\theta}$ *weakly in* $L^2\big([0,T], \dot{H}^1(\mathbb{T}^3)\big)$ *and strongly in* $L^2\big([0,T], L^2(\mathbb{T}^3)\big)$.
- v_{α_k} *converges to* $\breve{u}$ *weakly in* $L^2\big([0,T], L^2(\mathbb{T}^3)\big)$ *and strongly in* $L^2\big([0,T], \dot{H}^{-1}(\mathbb{T}^3)\big)$.
- u_{α_k} *converges to* $\breve{u}$ *and* θ_{α_k} *converges to* $\breve{\theta}$ *weakly in* $L^2(\mathbb{T}^3)$ *and uniformly over* $[0,T]$. *Furthermore,* $(\breve{u}, \breve{\theta})$ *is a Leray–Hopf weak solution of the Boussinesq system* (Bq) *and satisfies for all* $t \in [0,T]$ *the energy inequality*

$$
\|\breve{u}(t)\|^2_{L^2(\mathbb{T}^3)} + \|\breve{\theta}(t)\|^2_{L^2(\mathbb{T}^3)} + 2\int_0^t \nu\|\nabla\breve{u}(t)\|^2_{L^2(\mathbb{T}^3)} + \kappa\|\nabla\breve{\theta}(\tau)\|^2_{L^2(\mathbb{T}^3)}\,d\tau \leq \|u^0\|^2_{L^2(\mathbb{T}^3)} + \|\theta^0\|^2_{L^2(\mathbb{T}^3)} + 2\rho_0(t). \tag{1.8}
$$

The idea of the proof is to extract subsequences that converge to the solution of (Bq) as $\alpha \to 0^+$. First, we derive a uniform bound independent of the parameter α. This gives the weak convergence. Then, following the lines of the existence proof, we establish strong convergence of such subsequences in suitable spaces. This strong convergence allows to take the limit in the quadratic terms, and hence a weak convergence of the unique weak solution of (Bq_α) to a weak solution of (Bq) is proved and the associated energy estimate is derived. Using such energy estimates, we further ameliorate the strong convergence results derived earlier.

The remainder of our paper is organized as follows. For the sake of completeness, and to be read independently, we start by recalling some preliminary background and stating useful definitions. Section 3 is devoted to the proof of the existence result. In Section 4, we prove continuous dependence of the weak solution on the initial data and in particular uniqueness. Section 5 is devoted to proving several convergence results.

2 Notations and Preliminary Results

This section collects some known results that will be useful in the sequel.

- For $\mathbb{T}^3 = [0, 2\pi[^3$, the Fourier transformation of a suitable distribution f is normalized as

$$\mathcal{F}(f)(k) = \widehat{f}(k) = \int_{\mathbb{T}^3} \exp(-ix.k) f(x)\, dx, \quad k = (k_1, k_2, k_3) \in \mathbb{Z}^3$$

and the inverse Fourier formula is

$$\mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \sum_{\mathbb{Z}^3} \exp(ik.x) \widehat{f}(k), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

- For $s \in \mathbb{R}$, $\dot{H}^s(\mathbb{T}^3)$ denotes the usual homogeneous Sobolev space on $\mathbb{T}^3$ and $(\,\cdot\,,\,\cdot\,)_{\dot{H}^s(\mathbb{T}^3)}$ the associated inner product;

$$(f, g)_{\dot{H}^s(\mathbb{T}^3)} = \sum_{\mathbb{Z}^3} |k|^{2s} \mathcal{F}(f)(k) \overline{\mathcal{F}(g)(k)}.$$

- For any Banach space $(B, \|\cdot\|)$, any real number $1 \leq p \leq \infty$ and any time $T > 0$, we will denote by $L^p_T(B)$ the space of all measurable time-dependent functions defined on $[0, T]$ with value in B such that $\big(t \to \|f(t)\|\big)$ belongs to $L^p([0, T])$. We denote by $\mathcal{C}^{1/2}_{\text{loc}}(\mathbb{R}_+, B)$ the space of functions u that belong to $L^\infty(\mathbb{R}_+, B)$ and

$$\sup_{t \neq t'} \frac{\|u(t) - u(t')\|_B}{|t - t'|^{1/2}} < +\infty.$$

- If $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$ are two smooth vector fields and θ is a smooth scalar function, we set

$$f \otimes g := (g_1 f, g_2 f, g_3 f),$$

and

$$\operatorname{div}(f \otimes g) := \big(\operatorname{div}(g_1 f), \operatorname{div}(g_2 f), \operatorname{div}(g_3 f)\big)\,.$$

Moreover, if $\operatorname{div} f = 0$ we have

$$\operatorname{div}(f \otimes g) = (f \cdot \nabla) g,$$
$$\operatorname{div}(\theta f) = f \cdot \nabla \theta,$$
$$(f \cdot \nabla g, g)_{L^2(\mathbb{T}^3)} = 0$$

and

$$(f \cdot \nabla \theta, \theta)_{L^2(\mathbb{T}^3)} = 0.$$

The two last equalities can be proved by symmetry.

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- For any subset X of a set E, the symbol $\mathbf{1}_X$ denotes the characteristic function of X defined by

$$\mathbf{1}_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{elsewhere.} \end{cases}$$

Finally, we recall the version of the Arzelà–Ascoli theorem that we are using in this paper.

Theorem 2.1 *Let X be a compact metric space, let Y be a metric space and $H \subset C(X,Y)$ the set of continuous functions from X to Y. Then the following statements are equivalent.*

(i) *The closure of H is compact in $C(X,Y)$.*

(ii) *H is equicontinuous and $\forall x \in X$, and the set $H(x) = \{f(x), f \in H\}$ is relatively compact in Y.*

3 Existence Result

For a strictly positive integer n, we introduce the Friedrichs operator J_n, defined for any suitable distribution f by

$$J_n f := \mathcal{F}^{-1}\big(\mathbf{1}_{\{|k|<n\}}\mathcal{F}(f)\big).$$

One of the advantages of the so-called Friedrichs approximating method is that a partial differential equation will be approximated by an ordinary differential equation. In fact, derivatives in phase space became multiplications in frequency space. At this step, we should note that the frequency variable k_i, $1 \leq i \leq 3$, can take any value in $\mathbb{Z}$. However, when truncation applies, the expression $\mathbf{1}_{\{|k|<n\}}\mathcal{F}(f)$ is no longer a symbol of derivative and then $J_n u$ is no more a derivative but simply a distribution with a spectrum supported in the ball $B(0,n)$. Mainly, for the case of (Bq) we approximate the space $L^2(\mathbb{T}^3)$ by the sequence of increasing subspaces

$$L^2_n = \Big\{ f \in \big(L^2(\mathbb{T}^3)\big)^3, \operatorname{supp}\mathcal{F}(f) \subset B(0,n)\Big\},$$

and should then approximate the distribution u by a sequence u_n that belongs to L^2_n. Setting $u_n = J_n u$, we obtain the nonlinear ordinary differential equations denoted $(Bq_\alpha)_n$ and given by

$$\partial_t\theta_n - \kappa\Delta\theta_n + J_n \operatorname{div}(\theta_n u_n) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3, \tag{3.1}$$

$$\begin{aligned}\partial_t v_n - \nu\Delta v_n + J_n \operatorname{div}(u_n \otimes u_n) - \theta_n e_3 \\ = J_n \nabla\Delta^{-1}\Big(\sum_{i,j=1}^{3} \partial_i\partial_j(u_n^i u_n^j) - \partial_3\theta_n\Big) \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3,\end{aligned} \tag{3.2}$$

$$v_n = u_n - \alpha^2\Delta u_n \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3, \tag{3.3}$$

$$\operatorname{div} u_n = \operatorname{div} v_n = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{T}^3, \tag{3.4}$$

$$(u_n,\theta_n)|_{t=0} = (u_n^0,\theta_n^0) := (J_n u^0, J_n\theta^0) \quad \text{in } \mathbb{T}^3. \tag{3.5}$$

Before going further in our proof, we give some comments on how to apply the Friedrichs operator to the convective term. First, the distribution u needs to be truncated to belong to L^2_n and then replaced everywhere by $J_n u$. Second, since even when u_n belongs to L^2_n it is not clear that the nonlinear terms such as $u_n \cdot \nabla u_n$ belong also to L^2_n, we need to reapply the Friedrichs operator in front. As for the continuous problem, we obtain the expression of the approximate pressure as a function of the velocity and the temperature by applying the divergence operator to equations (3.2)–(3.5) under the hypothesis $\operatorname{div}(J_n u) = 0$. Let us return to our approximating scheme; system $(Bq_\alpha)_n$ is an ODE and can be rewritten in the abstract form

$$\frac{d}{dt}(v_n, \theta_n) = G_n(v_n, \theta_n),$$

where the expression of G_n is given by $(Bq_\alpha)_n$. By classical computation, we can verify that G_n is a continuous function from L^2_n into itself; the key idea is that we are using distributions with bounded spectrum. Hence, the Cauchy–Lipshitz theorem (known also as the Picard–Lindelöf theorem) asserts that system $(Bq_\alpha)_n$ has a unique maximal solution (u_n, θ_n) in the space $\mathcal{C}^1\big([0, T_n^*), L^2_n\big)$, where T_n^* is the maximal existence time. Taking the inner product in $L^2(\mathbb{T}^3)$ of equation (3.1) by θ_n and equation (3.2) by u_n, we obtain, under the incompressibility condition of u_n, for $t \in [0, T_n^*)$,

$$\frac{1}{2}\frac{d}{dt}\|\theta_n\|^2_{L^2(\mathbb{T}^3)} + \kappa\|\nabla\theta_n\|^2_{L^2(\mathbb{T}^3)} = 0,$$

$$\frac{1}{2}\frac{d}{dt}(\|u_n\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n\|^2_{L^2(\mathbb{T}^3)}) + \nu(\|\nabla u_n\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u_n\|^2_{L^2(\mathbb{T}^3)}) = \langle \theta_n e_3, u_n\rangle.$$

Integrating over time, it follows that for all $t \in [0, T_n^*)$,

$$\|\theta_n(t)\|^2_{L^2(\mathbb{T}^3)} + 2\kappa\int_0^t \|\nabla\theta_n\|^2_{L^2(\mathbb{T}^3)}\,d\tau = \|\theta_n^0\|^2_{L^2(\mathbb{T}^3)} \tag{3.6}$$

and

$$\begin{aligned}\|u_n(t)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n\|^2_{L^2(\mathbb{T}^3)} + 2\nu\int_0^t (\|\nabla u_n\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u_n\|^2_{L^2(\mathbb{T}^3)})\,d\tau \\ = \|u_n^0\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n^0\|^2_{L^2(\mathbb{T}^3)} + 2\int_0^t\int_{\mathbb{T}^3}\theta_n e_3 u_n\,dx\,d\tau.\end{aligned} \tag{3.7}$$

Let us first prove that these estimates imply that the maximal solution of the ODE $(Bq_\alpha)_n$ is global. In fact, applying respectively the Cauchy–Schwartz inequality and Young product inequality, we obtain

$$\begin{aligned}\int_0^t\int_{\mathbb{T}^3}\theta_n e_3 u_n\,dx\,d\tau &\le \int_0^t \|\theta_n(\tau)\|_{L^2(\mathbb{T}^3)}\,\|u_n(\tau)\|_{L^2(\mathbb{T}^3)}\,d\tau \\ &\le \int_0^t \|\theta_n(\tau)\|^2_{L^2(\mathbb{T}^3)} + \|u_n(\tau)\|^2_{L^2(\mathbb{T}^3)}\,d\tau \\ &\le \int_0^t \|\theta_n(\tau)\|^2_{L^2(\mathbb{T}^3)} + \big(\|u_n(\tau)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u(\tau)\|^2\big)\,d\tau.\end{aligned}$$

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By the energy inequality (3.6), we have

$$\|\theta_n(t)\|^2_{L^2(\mathbb{T}^3)} \le \|\theta_n^0\|^2_{L^2(\mathbb{T}^3)}.$$

Dropping the non-negative term from the left-hand side of (3.7), we obtain

$$\|u_n(t)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n(t)\|^2_{L^2(\mathbb{T}^3)} \le \|u_n^0\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n^0\|^2_{L^2(\mathbb{T}^3)} + 2t\|\theta_n^0\|^2_{L^2(\mathbb{T}^3)} + \int_0^t 2\big(\|u_n(\tau)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n(\tau)\|^2_{L^2(\mathbb{T}^3)}\big)\,d\tau.$$

Since the function

$$t \mapsto \|u_n^0\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n^0\|^2_{L^2(\mathbb{T}^3)} + 2t\|\theta_n^0\|^2_{L^2(\mathbb{T}^3)}$$

is non-decreasing, by a Gronwall-type inequality we infer that

$$\|u_n(t)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n(t)\|^2_{L^2(\mathbb{T}^3)} \le (\|u_n^0\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n^0\|^2_{L^2(\mathbb{T}^3)} + 2t\|\theta_n^0\|^2_{L^2(\mathbb{T}^3)})e^{2t}.$$

Hence,

$$\int_0^t \|u_n(\tau)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n(\tau)\|^2_{L^2(\mathbb{T}^3)}\,d\tau \le \frac{1}{2}(e^{2t}-1)(\|u_n^0\|^2_{L^2} + \alpha^2\|\nabla u_n^0\|^2_{L^2}) + \|\theta_n^0\|^2_{L^2(\mathbb{T}^3)}\Big(\Big(t-\frac{1}{2}\Big)e^{2t} + \frac{1}{2}\Big).$$

This implies that

$$\int_0^t\int_{\mathbb{T}^3} \theta_n e_3 u_n\,dx\,d\tau \le \rho_\alpha(t),$$

where for all non-negative time t, the function ρ_α is defined by

$$\rho_\alpha(t) = \frac{1}{2}(e^{2t}-1)(\|u_n^0\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n^0\|^2_{L^2(\mathbb{T}^3)}) + \Big(\Big(t-\frac{1}{2}\Big)e^{2t} + t + \frac{1}{2}\Big)\|\theta_n^0\|^2_{L^2(\mathbb{T}^3)}.$$

A classical computation asserts that for all positive time t, the function $\rho_\alpha(t)$ is an increasing one with minimum value equal to zero and reached as time vanishes. Hence, the energy estimate (3.7) reads:

$$\|u_n(t)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_n\|^2_{L^2(\mathbb{T}^3)} + 2\nu\int_0^t(\|\nabla u_n\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u_n\|^2_{L^2(\mathbb{T}^3)})\,d\tau \le \|u^0\|^2_{L^2(\mathbb{T}^3)} + 2\rho_\alpha(t). \tag{3.8}$$

Summing up estimate (3.8) together with estimate (3.6), we deduce that the maximal solution of the ODE is global. That is, $T_n^* = +\infty$. Otherwise, if T_n^* is finite, the right-hand side of estimation (3.8) will be so and u_n would have a life span strictly larger than T_n^*, which is in contradiction with maximality. Estimates above imply that θ_n is bounded in $L^\infty\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big) \cap L^2\big(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)\big)$ and u_n is bounded in

$$L^\infty\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big) \cap L^2\big(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)\big) \cap L^\infty\big(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)\big) \cap L^2\big(\mathbb{R}_+, \dot{H}^2(\mathbb{T}^3)\big).$$

In particular, the facts that the operator $I-\alpha^2\Delta$ is bounded from $H^2(\mathbb{T}^3)$ into $L^2(\mathbb{T}^3)$ and that u_n belongs to $L^\infty\big(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)\big) \cap L^2\big(\mathbb{R}_+, \dot{H}^2(\mathbb{T}^3)\big)$ imply that v_n is bounded in $L^\infty\big(\mathbb{R}_+, \dot{H}^{-1}(\mathbb{T}^3)\big) \cap L^2\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big)$. Using the approximating system, Sobolev embedding, and product laws, we infer that for any fixed time $T > 0$, $\frac{d}{dt}\theta_n$ is bounded in $L^2\big([0,T], \dot{H}^{-1}(\mathbb{T}^3)\big)$. At this point, we note that for a fixed positive time, the diffusion term $\Delta\theta_n$ belongs to $L^2\big(\dot{H}^{-1}([0,T], \mathbb{T}^3)\big)$ and the convective one satisfies

$$\begin{aligned}\|\operatorname{div}\theta_n u_n\|_{\dot{H}^{-1}} &\le \|\theta_n\|_{\dot{H}^1}\,\|u_n\|_{\dot{H}^{1/2}}\\ &\le \|\theta_n\|_{\dot{H}^1}\,\|u_n\|_{L^2}^{1/2}\,\|u_n\|_{\dot{H}^1}^{1/2},\end{aligned}$$

where we used respectively Sobolev product laws and interpolation inequality. Thus,

$$\begin{aligned}\int_0^T \|\operatorname{div}\theta_n u_n\|_{\dot{H}^{-1}}^2 &\le \int_0^T \|\theta_n\|_{\dot{H}^1}^2\,\|u_n\|_{\dot{H}^1}\,\|u_n\|_{L^2}\\ &\le \|\theta_n\|_{L^2_T(\dot{H}^1)}^2\,\|u_n\|_{L^\infty_T(\dot{H}^1)}\,\|u_n\|_{L^\infty_T(L^2)}.\end{aligned}$$

Here, we note that the maximal regularity associated to the temperature derivative is imposed by its diffusion process.

For the time derivative of the velocity, on one hand, v_n belongs to $L^2\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big)$, so Δv_n belongs to $L^2\big(\mathbb{R}_+, H^{-2}(\mathbb{T}^3)\big)$, the temperature θ_n belongs to $L^2\big(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)\big)$ and

$$\begin{aligned}\|\operatorname{div}(u_n\otimes u_n)\|_{\dot{H}^{-2}} &\le \|\operatorname{div}(u_n\otimes u_n)\|_{\dot{H}^{-3/2}}\\ &\le \|u_n\otimes u_n\|_{\dot{H}^{-1/2}}\\ &\le \|u_n\|_{L^2}\,\|u_n\|_{\dot{H}^1}.\end{aligned}$$

So it follows that

$$\int_0^T \|\operatorname{div}(u_n\otimes u_n)\|_{\dot{H}^{-2}}^2 \le \|u\|_{L^\infty_T(L^2)}^2\,\|u\|_{L^2_T(\dot{H}^1)}^2.$$

Hence, $\frac{d}{dt}v_n$ is bounded in $L^2\big([0,T], \dot{H}^{-2}(\mathbb{T}^3)\big)$.

On the other hand, the operator $(I-\alpha^2\Delta)^{-1}$ is bounded from $\dot{H}^{-2}(\mathbb{T}^3)$ into $L^2(\mathbb{T}^3)$, so

$$\begin{aligned}\Big\|\frac{d}{dt}u_n\Big\|_{L^2(\mathbb{T}^3)} &= \Big\|(I-\alpha^2\Delta)^{-1}\frac{d}{dt}v_n\Big\|_{L^2(\mathbb{T}^3)}\\ &\le c\Big\|\frac{d}{dt}v_n\Big\|_{\dot{H}^{-2}(\mathbb{T}^3)}.\end{aligned}$$

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Hence, for any fixed time $T > 0$,

$$\frac{d}{dt}u_n \text{ is bounded in } L^2\big([0,T], L^2(\mathbb{T}^3)\big).$$

To pass to the limit in the approximating system, we use a compactness method based on the Arzelà–Ascoli theorem. We first note that, for any times s and t,

$$\begin{aligned}\|u_n(t) - u_n(s)\|_{L^2} &\le \left|\int_s^t \|\partial_t u_n\|_{L^2}\, d\tau\right| \\ &\le |t-s|^{1/2}\, \|\partial_t u_n\|_{L^2(\mathbb{R}_+, L^2(\mathbb{T}^3))} \\ &\le M|t-s|^{1/2},\end{aligned}$$

where M is a positive real number. So the approximate velocity (u_n) is equicontinuous and belongs to the space $\mathcal{C}^{1/2}_{\mathrm{loc}}\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big)$. Likewise, the temperature (θ_n) is equicontinuous and it belongs to $\mathcal{C}^{1/2}_{\mathrm{loc}}\big(\mathbb{R}_+, \dot{H}^{-1}(\mathbb{T}^3)\big)$. Let $\mathbb{R}_+ = \bigcup_{p\in\mathbb{N}}[0, T_p]$, where T_p is an increasing sequence such that $\lim_{t\to+\infty} T_p = +\infty$. Using compactness of the Sobolev embedding of $\dot{H}^{s_1}(\mathbb{T}^3)$ into $\dot{H}^{s_2}(\mathbb{T}^3)$ whenever $s_1 > s_2$, we deduce that for any time $t \in [0, T_0]$, the set $E_1(t) = \{u_n(t), n \in \mathbb{N}\}$ is relatively compact in $\dot{H}^{-1}(\mathbb{T}^3)$ and the set $E_2(t) = \{\theta_n(t), n \in \mathbb{N}\}$ is relatively compact in $\dot{H}^{-2}(\mathbb{T}^3)$. By the Arzelà–Ascoli theorem, we infer that $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $C\big([0,T_0], \dot{H}^{-1}(\mathbb{T}^3)\big)$ and $(\theta_n)_{n\in\mathbb{N}}$ is relatively compact in $C\big([0,T_0], \dot{H}^{-2}(\mathbb{T}^3)\big)$. So there exists two extraction maps φ_0 and ψ_0 from $\mathbb{N}$ to $\mathbb{N}$ such that

$$u_{\varphi_0(n)} \to \tilde{u}_0 \quad \text{strongly in } C\big([0,T_0], \dot{H}^{-1}(\mathbb{T}^3)\big)$$

and

$$\theta_{\psi_0(n)} \to \tilde{\theta}_0 \text{ strongly in } C\big([0,T_0], \dot{H}^{-2}(\mathbb{T}^3)\big).$$

Recalling that sequences $(u_n)_{n\in\mathbb{N}}$ and $(\theta_n)_{n\in\mathbb{N}}$ are respectively bounded in the space $L^\infty\big(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)\big)$ and the space $L^\infty\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big)$, we use an interpolation argument to deduce that, for any real number $\varepsilon > 0$,

$$u_{\varphi_0(n)} \to \tilde{u}_0 \quad \text{strongly in } C\big([0,T_0], \dot{H}^{1-\varepsilon}(\mathbb{T}^3)\big)$$

and

$$\theta_{\psi_0(n)} \to \tilde{\theta}_0 \quad \text{strongly in } C\big([0,T_0], \dot{H}^{-\varepsilon}(\mathbb{T}^3)\big).$$

This is the initial step of an induction process that leads in its step number p to construction of a sequence of velocity fields $(\tilde{v}_p)_{p\in\mathbb{N}}$, temperature fields $(\tilde{\theta}_p)_{p\in\mathbb{N}}$ and a sequence of increasing maps from $\mathbb{N}$ to $\mathbb{N}$, $(\varphi_p)_{p\in\mathbb{N}}$ and $(\psi_p)_{p\in\mathbb{N}}$ such that for all $p \in \mathbb{N}$ and any real number $\varepsilon > 0$,

$$u_{\varphi_0\circ\varphi_1\circ\cdots\circ\varphi_p(n)} \to \tilde{u}_p \quad \text{strongly in } C\big([0,T_p], \dot{H}^{1-\varepsilon}(\mathbb{T}^3)\big)$$

and

$$\theta_{\psi_0\circ\psi_1\circ\cdots\circ\psi_p(n)} \to \tilde{\theta}_p \quad \text{strongly in } C\big([0,T_p], \dot{H}^{-\varepsilon}(\mathbb{T}^3)\big).$$

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Then, at this point in the proof, there exists a unique velocity field

$$u \in L^\infty\big(\mathbb{R}_+, \dot H^1(\mathbb{T}^3)\big) \cap C\big(\mathbb{R}_+, \dot H^{1-\varepsilon}(\mathbb{T}^3)\big)$$

and a unique temperature $\theta \in L^\infty\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big) \cap C\big(\mathbb{R}_+, \dot H^{\varepsilon}(\mathbb{T}^3)\big)$, defined for all $p \in \mathbb{N}$ by

$$(u,\theta)(x,t) = (\tilde u, \tilde\theta)(x,t), \quad \text{for } (t,x) \in [0,T_p] \times \mathbb{T}^3.$$

Using the Cantor diagonal extraction process [17], letting $\Phi(n) = \varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_n(n)$ and $\Psi(n) = \psi_0 \circ \psi_1 \circ \cdots \circ \psi_n(n)$, it follows that

$$\lim_{n\to\infty} u_{\Phi(n)} = \tilde u \ \text{ strongly in the space } C\big([0,T_p], \dot H^{1-\varepsilon}(\mathbb{T}^3)\big) \tag{3.9}$$

and

$$\lim_{n\to\infty} \theta_{\Psi(n)} = \tilde\theta \ \text{ strongly in the space } C\big([0,T_p], \dot H^{-\varepsilon}(\mathbb{T}^3)\big). \tag{3.10}$$

This is due to the fact that, for a given integer p, when n tends to infinity we are able to consider $n \geq p$. So it follows that $(v_{\Phi(n)})$ and $(\theta_{\Psi(n)})$ are subsequences of $(u_{\varphi_0 o \varphi_1 o \cdots o \varphi_p(n)})_n$ and $(\theta_{\psi_0 o \psi_1 o \cdots o \psi_p(n)})_n$ respectively. In particular, by taking a test function supported in $[0,T_p] \times \mathbb{T}^3$, we use the above limits to deduce that

$$\lim_{n\to\infty} v_{\Phi(n)} = \tilde v \quad \text{in } \mathcal{D}'(\mathbb{R}_+{}^* \times \mathbb{T}^3) \tag{3.11}$$

and

$$\lim_{n\to\infty} \theta_{\Psi(n)} = \tilde\theta \quad \text{in } \mathcal{D}'(\mathbb{R}_+{}^* \times \mathbb{T}^3). \tag{3.12}$$

It remains to take the limit in $\mathcal{D}'(\mathbb{R}_+{}^* \times \mathbb{T}^3)$ as n goes to $+\infty$ of the approximate system. The linear terms follow immediately from (3.11)–(3.12). As usual, the challenging term is the nonlinear one. At this step, we note that the strong convergence is compulsory for taking the limit. In the beginning, let us prove that

$$\lim_{n\to+\infty} u^i_{\Phi(n)} u^j_{\Phi(n)} = u^i u^j \quad \text{in } \mathcal{D}'(\mathbb{R}_+{}^* \times \mathbb{T}^3). \tag{3.13}$$

To do so, consider a distribution g supported in $]0,T[\, \times \mathbb{T}^3$. We have

$$g u^i_{\Phi(n)} u^j_{\Phi(n)} - g u^i u^j = (u^i_{\Phi(n)} - u^i) g u^j_{\Phi(n)} - (u^j_{\Phi(n)} - u^j) g u^i_{\Phi(n)}.$$

To estimate the first term of the right-hand side, we note that

$$\begin{aligned} &\|(u^i_{\Phi(n)} - u^i) g u^j_{\Phi(n)}\|_{L^2(\mathbb{R}_+, \dot H^{3/2-\varepsilon}(\mathbb{T}^3))} \\ &\qquad \leq C \|u^i_{\Phi(n)} - u^i\|_{L^\infty([0,T_p], \dot H^{1-\varepsilon}(\mathbb{T}^3))} \times \|g u^j_{\Phi(n)}\|_{L^2(\mathbb{R}_+, \dot H^2(\mathbb{T}^3))}. \end{aligned}$$

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The same holds for the $(u^j_{\Phi(n)} - u^j)gu^i_{\Phi(n)}$ and we deduce that

$$\lim_{n\to+\infty} gu^i_{\Phi(n)}u^j_{\Phi(n)} = gu^iu^j \quad \text{in } L^2\big(\mathbb{R}_+, \dot{H}^{3/2-\varepsilon}(\mathbb{T}^3)\big).$$

This holds for any integer p and any test function g. Thanks to a density argument, (3.13) is proved.

Now, let us prove that

$$\lim_{n\to+\infty} J_n(u^i_nu^j_n) = u^iu^j \quad \text{in } \mathcal{D}'(\mathbb{R}_+{}^* \times \mathbb{T}^3).$$

First, note that

$$J_n(u^i_nu^j_n) - u^iu^j = J_n(u^i_nu^j_n - u^iu^j) + (J_n - \mathrm{Id})u^iu^j.$$

The first term of the right-hand side is not problematic thanks to the application of the truncation operator J_n. However, the second one presents some difficulties since frequencies are not bounded; it will be dealt with as

$$\|(J_n - \mathrm{Id})u^iu^j\|_{L^2(\mathbb{R}_+,\dot{H}^{3/2-\varepsilon}(\mathbb{T}^3))} \leq n^{-\varepsilon}\|u\|_{L^\infty(\mathbb{R}_+,\dot{H}^1(\mathbb{T}^3))}\,\|u\|_{L^2(\mathbb{R}_+,\dot{H}^2(\mathbb{T}^3))}.$$

So we can take the limit in the velocity convective term. Likewise for the temperature advection, we have

$$g\theta_{\Psi(n)}u^i_{\Phi(n)} - g\theta u^i = (\theta_{\Psi(n)} - \theta)gu^i_{\Phi(n)} + (u^i_{\Phi(n)} - u^i)g\theta. \tag{3.14}$$

The following estimate holds:

$$\begin{aligned}\|(\theta_{\Psi(n)} - \theta)gu^i_{\Phi(n)}\|&_{L^2(\mathbb{R}_+,\dot{H}^{-1/2-\varepsilon}(\mathbb{T}^3))}\\ &\leq C\|\theta^i_{\Psi(n)} - \theta\|_{L^\infty([0,T_p],\dot{H}^{-\varepsilon}(\mathbb{T}^3))} \times \|gu^i_{\Phi(n)}\|_{L^2(\mathbb{R}_+,\dot{H}^1(\mathbb{T}^3))}.\end{aligned}$$

Since, as n goes to infinity, $\theta_{\Psi(n)}$ converges to θ in $L^\infty\big([0,T_p],\dot{H}^{-\varepsilon}(\mathbb{T}^3)\big)$ and since $u^i_{\Phi(n)}$ belongs to $L^2\big(\mathbb{R}_+,\dot{H}^1(\mathbb{T}^3)\big)$,

$$\lim_{n\to+\infty}\|(\theta_{\Psi(n)} - \theta)gu^i_{\Phi(n)}\|_{L^2(\mathbb{R}_+,\dot{H}^{-1/2-\varepsilon}(\mathbb{T}^3))} = 0.$$

Applying Sobolev product rules and the decreasing Sobolev chain spaces, we have

$$\|(u^i_{\Psi(n)} - u^i)g\theta\|_{L^2(\mathbb{R}_+,\dot{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))} \leq C\|u^i_{\Psi(n)}i - u^i\|_{L^\infty([0,T_p],\dot{H}^{1-\varepsilon})}\,\|g\theta\|_{L^2(\mathbb{R}_+,\dot{H}^1)}.$$

This guarantees the convergence of the second term of the right-hand side of (3.14) in $\mathcal{D}(\mathbb{R}_+{}^* \times \mathbb{T}^3)$.

As above, we go further by rewriting

$$J_n(\theta_nu^i_n) - \theta u^i = J_n(\theta_nu^i_n\theta - u^i) + (J_n - \mathrm{Id})\theta u^i$$

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and noting that

$$\|(J_n - \mathrm{Id})\theta u^i\|_{L^2(\mathbb{R}_+,\dot{H}^{-1/2-\varepsilon}(\mathbb{T}^3))} \leq n^{-\varepsilon}\|u\|_{L^\infty(\mathbb{R}_+,L^2(\mathbb{T}^3))}\,\|\theta\|_{L^2(\mathbb{R}_+,\dot{H}^1(\mathbb{T}^3))}.$$

So the limit (u,θ) satisfies system (1.2)–(1.6) in the weak sense, *i.e.*, in $\mathcal{D}'(\mathbb{R}_+{}^* \times \mathbb{T}^3)$. The assertions

$$u \in C([0,T],\dot{H}^1) \cap L^2([0,T],\dot{H}^2)$$

and

$$\theta \in C([0,T],L^2) \cap L^2([0,T],\dot{H}^1)$$

follow from the fact that each term of the sequence (u_n,θ_n) satisfies the energy estimates (3.6) and (3.8), and (u_n,θ_n) converges almost everywhere to (u,θ), since it converges strongly to this limit according to (3.9) and (3.10).

To prove the energy inequality, we again use equations (3.9) and (3.10) to infer that $\theta_n(t)$ converges weakly in $L^2(\mathbb{T}^3)$ towards $\theta(t)$ and $u_n(t)$ converges weakly in $\dot{H}^1(\mathbb{T}^3)$ towards $u(t)$. Hence

$$\|\theta(t)\|_{L^2}^2 \leq \liminf_{n\to\infty}\|\theta_n(t)\|_{L^2}^2$$

and

$$\|u(t)\|_{L^2}^2 + \alpha^2\|\nabla u(t)\|_{L^2}^2 \leq \liminf_{n\to\infty}\big(\|u_n(t)\|_{L^2}^2 + \alpha^2\|\nabla u_n(t)\|_{L^2}^2\big).$$

On the other hand (θ_n) converges weakly to θ in $L^2\big([0,T],\dot{H}^1(\mathbb{T}^3)\big)$ and (u_n) converges weakly to u in $L^2\big([0,T],\dot{H}^2(\mathbb{T}^3)\big)$, so for all non-negative time t, we have

$$\int_0^t \|\nabla\theta(\tau)\|_{L^2}^2\,d\tau \leq \liminf_{n\to\infty}\int_0^t \|\nabla\theta_n(\tau)\|_{L^2}^2\,d\tau$$

and

$$\int_0^t (\|\nabla u\|_{L^2(\mathbb{T}^3)}^2 + \alpha^2\|\Delta u\|_{L^2(\mathbb{T}^3)}^2)\,d\tau \leq \liminf_{n\to\infty}\int_0^t (\|\nabla u_n\|_{L^2(\mathbb{T}^3)}^2 + \alpha^2\|\Delta u_n\|_{L^2}^2)\,d\tau$$

Taking the lower limit as n goes to infinity in the energy inequality (3.6) yields the energy inequality (1.7).

To prove that the solution (u,θ) is continuous, we take the scalar product of equation (1.3) by u and the scalar product of equation (1.2) by θ to infer that $\frac{d}{dt}\|u(t)\|_{L^2(\mathbb{T}^3)}^2$ and $\frac{d}{dt}\|\theta(t)\|_{L^2(\mathbb{T}^3)}^2$ both belong to $L^1[0,T]$. Hence, $\|u(t)\|_{L^2(\mathbb{T}^3)}^2$ and $\|\theta(t)\|_{L^2(\mathbb{T}^3)}^2$ belong to $C[0,T]$. Taking the scalar product of equations (1.3) and (1.2) by a suitable test function, we deduce that the functions $t \rightsquigarrow u(t)$ and $t \rightsquigarrow \theta(t)$ are weakly continuous. This leads to the fact that both u and θ are strongly continuous. The proof of the existence result is finished.

4 Continuous Dependence of Weak Solutions on the Initial Data and Uniqueness

Let T be a strictly positive time, and let (u,θ) and $(\overline{u},\overline{\theta})$ be two weak solutions of the system (Bq_α) on the time interval $[0,T]$ associated respectively to the initial data (u^0,θ^0) and $(\overline{u}^0,\overline{\theta}^0)$. We let $\delta u = u-\overline{u}$, $\delta v = v-\overline{v}$, $\delta\theta = \theta-\overline{\theta}$ and $\delta p = p-\overline{p}$. Taking differences in related (Bq_α) systems, classical computations lead to the one denoted (δBq_α) and given by

$$\partial_t\delta\theta - \kappa\Delta\delta\theta + \delta u\cdot\nabla\theta + \overline{u}\cdot\nabla\delta\theta = 0 \quad \text{in } \mathbb{R}_+\times\mathbb{T}^3, \tag{4.1}$$

$$\partial_t\delta v - \nu\Delta\delta v + \delta u\cdot\nabla u + \overline{u}\cdot\nabla\delta u = -\nabla\delta p + \delta\theta e_3 \quad \text{in } \mathbb{R}_+\times\mathbb{T}^3, \tag{4.2}$$

$$\delta v = \delta u - \alpha^2\Delta\delta u \quad \text{in }, \mathbb{R}_+\times\mathbb{T}^3 \tag{4.3}$$

$$\operatorname{div}\delta v = \operatorname{div}\delta u = 0 \quad \text{in } \mathbb{R}_+\times, \mathbb{T}^3 \tag{4.4}$$

$$(\delta u,\delta\theta)(0) = (u^0-\overline{u}^0, \theta^0-\overline{\theta}^0) \quad \text{in } \mathbb{T}^3. \tag{4.5}$$

According to the proof of the existence result, the temperature time derivative $\frac{d}{dt}\delta\theta$ belongs to the space $Ł^2\big([0,T],\dot{H}^{-1}(\mathbb{T}^3)\big)$, and both temperatures θ and $\overline{\theta}$ and their difference $\delta\theta$ belong to the space $Ł^2\big([0,T],\dot{H}^1(\mathbb{T}^3)\big)$. Also, the filtered velocity field time derivative $\frac{d}{dt}\delta v$ belongs to the space $Ł^2\big([0,T],\dot{H}^{-2}(\mathbb{T}^3)\big)$ and the velocity field δu belongs to the space $Ł^2\big([0,T],\dot{H}^2(\mathbb{T}^3)\big)$. Hence, taking the appropriate duality action, we have, in the framework of divergence-free vector fields and equations (4.1)–(4.2), for almost every time t in $[0,T]$,

$$\Big\langle \frac{d}{dt}\delta\theta,\delta\theta\Big\rangle_{\dot{H}^{-1}(\mathbb{T}^3)} + \kappa\|\nabla\delta\theta\|^2_{L^2(\mathbb{T}^3)} + \langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot{H}^{-1}(\mathbb{T}^3)} = 0$$

$$\Big\langle \frac{d}{dt}\delta v,\delta u\Big\rangle_{\dot{H}^{-2}(\mathbb{T}^3)} + \nu(\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta\delta v\|^2_{L^2(\mathbb{T}^3)}) + \langle\delta u\cdot\nabla u,\delta u\rangle_{\dot{H}^{-2}(\mathbb{T}^3)} = \langle\delta\theta,\delta u\rangle_{\dot{H}^{-1}(\mathbb{T}^3)}.$$

Applying the Lions–Magenes lemma concerning the derivatives of functions with values in Banach spaces (*cf.* [22, Chap. 3, p. 169]), we obtain

$$\Big\langle \frac{d}{dt}\delta\theta,\delta\theta\Big\rangle_{\dot{H}^{-1}(\mathbb{T}^3)} = \frac{d}{dt}\|\delta\theta\|^2_{L^2(\mathbb{T}^3)}$$

and

$$\Big\langle \frac{d}{dt}\delta v,\delta u\Big\rangle_{\dot{H}^{-2}(\mathbb{T}^3)} = \frac{d}{dt}(\|\delta u\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}).$$

It follows that

$$\frac{d}{dt}\|\delta\theta\|^2_{L^2(\mathbb{T}^3)} + \kappa\|\nabla\delta\theta\|^2_{L^2(\mathbb{T}^3)} + \langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot{H}^{-1}(\mathbb{T}^3)} = 0 \tag{4.6}$$

$$\frac{d}{dt}\big(\|\delta u\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)}\big) + \nu\big(\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta\delta v\|^2_{L^2(\mathbb{T}^3)}\big) + \langle\delta u\cdot\nabla u,\delta u\rangle_{\dot{H}^{-2}(\mathbb{T}^3)} = \langle\delta\theta,\delta u\rangle_{\dot{H}^{-1}(\mathbb{T}^3)}. \tag{4.7}$$

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We sum equations (4.6) and (4.7) to obtain

$$\begin{aligned}\frac{d}{dt}(\|\delta u\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)} + \|\delta\theta\|^2_{L^2(\mathbb{T}^3)})\\ + \nu(\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta\delta v\|^2_{L^2(\mathbb{T}^3)}) + \kappa\|\nabla\delta\theta\|^2_{L^2(\mathbb{T}^3)}\\ = \underbrace{\langle\delta\theta,\delta u\rangle_{\dot H^{-1}(\mathbb{T}^3)}}_{Q_3} - \underbrace{\langle\delta u\cdot\nabla u,\delta u\rangle_{\dot H^{-2}(\mathbb{T}^3)}}_{Q_1} - \underbrace{\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)}}_{Q_2}.\end{aligned}$$

Let us estimate Q_1, Q_2 and Q_3. To do so, we recall the following Sobolev inequalities: for all $\vartheta\in\dot H^1(\mathbb{T}^3)$, it holds that

$$\|\vartheta\|_{L^3(\mathbb{T}^3)} \le c\|\vartheta\|^{1/2}_{L^2(\mathbb{T}^3)}\,\|\vartheta\|^{1/2}_{\dot H^1(\mathbb{T}^3)} \tag{4.8}$$

and

$$\|\vartheta\|_{L^6(\mathbb{T}^3)} \le \|\vartheta\|_{\dot H^1(\mathbb{T}^3)}. \tag{4.9}$$

- For Q_1 we use the Cauchy–Schwartz inequality twice to obtain

$$Q_1 \le \|\delta u\|_{L^3(\mathbb{T}^3)}\,\|\nabla u\|_{L^2(\mathbb{T}^3)}\,\|\delta u\|_{L^6(\mathbb{T}^3)}.$$

  Sobolev inequalities (4.8)–(4.9) imply that

$$Q_1 \le c\|\delta u\|^{3/2}_{\dot H^1(\mathbb{T}^3)}\,\|\delta u\|^{1/2}_{L^2(\mathbb{T}^3)}\,\|u\|_{\dot H^1(\mathbb{T}^3)}.$$

  Using the Young product inequality, we obtain

$$\langle\delta u\cdot\nabla u,\delta u\rangle_{\dot H^{-1}(\mathbb{T}^3)} \le c(\|\delta u\|^2_{L^2(\mathbb{T}^3)} + \|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)})\|u\|_{\dot H^1(\mathbb{T}^3)}. \tag{4.10}$$

- For Q_2, as for the preceding item, we use the Cauchy–Schwartz inequality twice to obtain

$$\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)} \le \|\delta u\|_{L^3(\mathbb{T}^3)}\,\|\nabla\theta\|_{L^2(\mathbb{T}^3)}\,\|\delta\theta\|_{L^6(\mathbb{T}^3)}.$$

  Then inequalities (4.8)–(4.9) and the Sobolev norm definition imply that

$$\begin{aligned}\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)} &\le c\|\delta u\|^{1/2}_{L^2(\mathbb{T}^3)}\,\|\delta u\|^{1/2}_{\dot H^1(\mathbb{T}^3)}\,\|\nabla\theta\|_{L^2(\mathbb{T}^3)}\,\|\delta\theta\|_{\dot H^1(\mathbb{T}^3)}\\ &\le c\|\delta u\|^{1/2}_{L^2(\mathbb{T}^3)}\,\|\nabla\delta u\|^{1/2}_{L^2(\mathbb{T}^3)}\,\|\nabla\theta\|_{L^2(\mathbb{T}^3)}\,\|\nabla\delta\theta\|_{L^2(\mathbb{T}^3)}.\end{aligned}$$

  Using twice the Young product inequality, we obtain

$$\begin{aligned}\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)} &\le c\|\delta u\|^{1/2}_{L^2(\mathbb{T}^3)}\,\|\nabla\delta u\|^{1/2}_{L^2(\mathbb{T}^3)}\,\|\nabla\theta\|_{L^2(\mathbb{T}^3)}\,\|\nabla\delta\theta\|_{L^2(\mathbb{T}^3)}\\ &\le c\|\delta u\|_{L^2(\mathbb{T}^3)}\,\|\nabla\delta u\|_{L^2(\mathbb{T}^3)}\,\|\nabla\theta\|^2_{L^2(\mathbb{T}^3)} + \frac{\kappa}{2}\|\nabla\delta\theta\|^2_{L^2(\mathbb{T}^3)}\\ &\le c(\|\delta u\|^2_{L^2(\mathbb{T}^3)} + \|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)})\,\|\nabla\theta\|^2_{L^2(\mathbb{T}^3)} + \frac{\kappa}{2}\|\nabla\delta\theta\|^2_{L^2(\mathbb{T}^3)}.\end{aligned}$$

  Finally,

$$\langle\delta u\cdot\nabla\theta,\delta\theta\rangle_{\dot H^{-1}(\mathbb{T}^3)} \le c(\|\delta u\|^2_{L^2(\mathbb{T}^3)}+\|\nabla\delta u\|^2_{L^2(\mathbb{T}^3)})\|\nabla\theta\|^2_{L^2(\mathbb{T}^3)}+\frac{\kappa}{2}\|\nabla\delta\theta\|^2_{L^2(\mathbb{T}^3)}. \tag{4.11}$$

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- For Q_3 we use respectively the Cauchy–Schwartz inequality and Young's inequality to obtain

$$\langle \delta\theta, \delta u\rangle_{\dot{H}^{-1}(\mathbb{T}^3)} \leq \|\delta u\|^2_{L^2(\mathbb{T}^3)} + \|\delta\theta\|^2_{L^2(\mathbb{T}^3)}. \tag{4.12}$$

Summing up estimations (4.10), (4.11), and (4.12), we infer that

$$\begin{aligned}\frac{d}{dt}(\|\delta u\|^2_{L^2} + \|\nabla\delta u\|^2_{L^2} + \|\delta\theta\|^2_{L^2}) + \nu(\|\delta u\|^2_{L^2} + \|\nabla\delta u\|^2_{L^2}) + \frac{\kappa}{2}\|\nabla\delta\theta\|^2_{L^2} \\ \leq C(1 + \|\nabla\theta\|^2_{L^2} + \|u\|_{\dot{H}^1}) \times (\|\delta u\|^2_{L^2} + \|\nabla\delta u\|^2_{L^2} + \|\delta\theta\|^2_{L^2}).\end{aligned} \tag{4.13}$$

Dropping the dissipative positive term from the left-hand side of (4.13) and putting

$$g(t) = C(\|\nabla\theta\|^2_{L^2} + \|u\|_{\dot{H}^1} + 1),$$

we deduce that

$$\frac{d}{dt}(\|\delta u\|^2_{L^2} + \|\nabla\delta u\|^2_{L^2} + \|\delta\theta\|^2_{L^2}) \leq g(t)(\|\delta u\|^2_{L^2} + \|\nabla\delta u\|^2_{L^2} + \|\delta\theta\|^2_{L^2}). \tag{4.14}$$

This differential inequality is of Gronwall type where $\theta \in L^2\big([0,T], \dot{H}^1(\mathbb{T}^3)\big)$ and $u \in L^\infty\big([0,T], \dot{H}^1(\mathbb{T}^3)\big)$. Gronwall's lemma applied to inequality (4.14) leads to

$$(\|\delta u\|^2_{L^2} + \|\nabla\delta\|^2_{L^2} + \|\delta\theta\|^2_{L^2})(t) \leq (\|\delta u^0\|^2_{L^2} + \|\nabla\delta u^0\|^2_{L^2} + \|\delta\theta^0\|^2_{L^2}) \times \exp\int_0^t g(s)\,ds.$$

Hence, the continuous dependence of the weak solution on the initial data in any bounded interval of time $[0,T]$ follows. In particular, the solution is unique.

5 Convergence Results

As a result of the preceding sections, we dispose of a family of solutions $(u_\alpha, \theta_\alpha)$ that depends on the parameter α and that is continuously dependent on the initial data (u^0, θ^0).

In this section, we are dealing with convergence results as the parameter α goes to zero ($\alpha \to 0^+$). Hence, we can suppose that there exists a fixed value of α denoted α_0, such that $0 < \alpha \leq \alpha_0$. Consequently, taking $\alpha = \alpha_0$ in the right-hand side, the energy estimates (1.7) reads, for all $t \in [0,T]$,

$$\begin{aligned}&\|u_\alpha(t)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\nabla u_\alpha(t)\|^2_{L^2(\mathbb{T}^3)} + \|\theta_\alpha(t)\|^2_{L^2(\mathbb{T}^3)} \\ &\quad + 2\int_0^t \nu\big(\|\nabla u_\alpha(t)\|^2_{L^2(\mathbb{T}^3)} + \alpha^2\|\Delta u_\alpha(t)\|^2_{L^2(\mathbb{T}^3)}\big) + \kappa\|\theta_\alpha(\tau)\|^2_{L^2(\mathbb{T}^3)}\,d\tau \\ &\qquad \leq \|u^0\|^2_{L^2(\mathbb{T}^3)} + \alpha_0^2\|\nabla u^0\|^2_{L^2(\mathbb{T}^3)} + \|\theta^0\|^2_{L^2(\mathbb{T}^3)} + 2\rho_{\alpha_0}(T),\end{aligned} \tag{5.1}$$

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where $\rho_{\alpha_0}(T)$ is defined by

$$\rho_{\alpha_0}(T) = \frac{1}{2}(e^{2T} - 1)(\|u^0\|^2_{L^2(\mathbb{T}^3)} + \alpha_0^2\|\nabla u^0\|^2_{L^2(\mathbb{T}^3)}) + \left(\left(T - \frac{1}{2}\right)e^{2T} + T + \frac{1}{2}\right)\|\theta^0\|^2_{L^2(\mathbb{T}^3)}.$$

Here, as u, θ and v are families of solution that depend on the parameter α, we changed the notation to u_α θ_α and v_α instead of u, θ and v. When $\alpha \to 0^+$, we obtain the following uniform bounds, which are independent of the parameter α. Namely, we have:

both of θ and u are uniformly bounded in the space $L^2\big([0,T], \dot{H}^1(\mathbb{T}^3)\big)$

and

v is uniformly bounded in the space $L^2\big([0,T], L^2(\mathbb{T}^3)\big)$.

Hence, applying the Banach–Alaoglu theorem (see [15, Section 3.15, p. 68]) in the framework of Hilbert spaces allows us to extract subsequences $(u_{\alpha_k})_k$, $(v_{\alpha_k})_k$, and $(\theta_{\alpha_k})_k$ of u_α, v_α, and θ_α respectively such that

$$(\theta_{\alpha_k}, u_k) \rightharpoonup (\breve{\theta}, \breve{u}) \quad \text{weakly in } L^2\big([0,T], \dot{H}^1(\mathbb{T}^3)\big)$$

as $\alpha_k \to 0^+$ (or equivalently as $k \to +\infty$) and

$$v_{\alpha_k} \rightharpoonup \breve{u} \quad \text{weakly in } L^2\big([0,T], L^2(\mathbb{T}^3)\big)$$

as $\alpha_k \to 0^+$ (or equivalently as $k \to +\infty$).

Let us now establish uniform estimates to the time derivatives of θ_{α_k} and u_{α_k} in the appropriate spaces. At this step, we note that our target is to derive estimations that are independent of the parameter α. Hence, it is clear that estimations derived in the proof of the existence result do not apply because of their dependence on α. For a fixed positive time, since θ_{α_k} is uniformly bounded independently of α in the space $L^2\big([0,T], \dot{H}^1(\mathbb{T}^3)\big)$, the diffusion term $\Delta\theta_{\alpha_k}$ belongs to $L^2\big(\dot{H}^{-1}([0,T], \mathbb{T}^3)\big)$. The advection will be dealt with as

$$\begin{aligned}\int_0^T \|\operatorname{div}\theta_{\alpha_k}u_{\alpha_k}\|^2_{\dot{H}^{-3/2}} &\leq \int_0^T \|\theta_{\alpha_k}u_{\alpha_k}\|^2_{\dot{H}^{-1/2}}\, d\tau \\ &\leq \int_0^T \|\theta_{\alpha_k}\|^2_{L^2}\,\|u_{\alpha_k}\|^2_{\dot{H}^1} \\ &\leq \|\theta_{\alpha_k}\|^2_{L^\infty_T(L^2)}\,\|u_{\alpha_k}\|^2_{L^2_T(\dot{H}^1)},\end{aligned}$$

where we used the Sobolev norm definition, product laws and classical computation. Since u_{α_k} and θ_{α_k} are subsequences of u_α and θ_α, the energy estimate (5.1) applies also for u_{α_k} and θ_{α_k}, and we have

$$\|\theta_{\alpha_k}\|^2_{L^\infty_T(L^2)}\,\|u_{\alpha_k}\|^2_{L^2_T(\dot{H}^1)} \leq \frac{1}{2\nu}\big(\|u^0\|^2_{L^2(\mathbb{T}^3)} + \alpha_0^2\|\nabla u^0\|^2_{L^2(\mathbb{T}^3)} + \|\theta^0\|^2_{L^2(\mathbb{T}^3)} + 2\rho_{\alpha_0}(T)\big).$$

So it follows that

$$\int_0^T \|\operatorname{div}\theta_{\alpha_k}u_{\alpha_k}\|^2_{\dot H^{-3/2}} \le \frac{1}{2\nu}\big(\|u^0\|^2_{L^2(\mathbb{T}^3)} + \alpha_0^2\|\nabla u^0\|^2_{L^2(\mathbb{T}^3)} + \|\theta^0\|^2_{L^2(\mathbb{T}^3)} + 2\rho_{\alpha_0}(T)\big).$$

The above temperature diffusion and convection estimations lead to

$$\Big\|\frac{d}{dt}\theta_{\alpha_k}\Big\|_{L^2_T(\dot H^{-3/2})} \le K_1,$$

where K_1 is a real positive constant. Let us now turn to the time derivative of the velocity field u_{α_k}. Applying the operator $(I-\alpha^2\Delta)^{-1}$ to the equation (1.3), we obtain

$$\text{(5.2)}\qquad \frac{d}{dt}u_{\alpha_k} = \nu\Delta u_{\alpha_k} - (I-\alpha^2\Delta)^{-1}(u_{\alpha_k}\cdot\nabla)u_{\alpha_k} - (I-\alpha^2\Delta)^{-1}\nabla p_{\alpha_k} + (I-\alpha^2\Delta)^{-1}\theta_{\alpha_k}e_3 \quad \text{in } \mathbb{R}_+\times\mathbb{T}^3.$$

For a fixed positive time, since u_{α_k} is uniformly bounded independently of α in the space $L^2\big([0,T],\dot H^1(\mathbb{T}^3)\big)$, the dissipation Δu_{α_k} belongs to $L^2\big(\dot H^{-1}([0,T],\mathbb{T}^3)\big)$. For the remaining terms, we recall that operator $(I-\alpha^2\Delta)^{-1}$ is bounded from $H^{-2}(\mathbb{T}^3)$ into $L^2(\mathbb{T}^3)$. Moreover, a direct frequency space computation implies that its norm is uniformly bounded independently of the parameter α and satisfies

$$\text{(5.3)}\qquad |||(I-\alpha^2\Delta)^{-1}||| \le 1.$$

Since θ_α is time square integrable with value in $\dot H^1$, then

$$\|(I-\alpha^2\Delta)^{-1}\theta e_3\|_{L^2(\dot H^3)} \le K_2',$$

where K' is a real positive constant. The convection will be estimated as

$$\begin{aligned}\int_0^T \|(I-\alpha^2\Delta)^{-1}\operatorname{div}(u_{\alpha_k}\otimes u_{\alpha_k})\|^2_{L^2} &\le \int_0^T \|\operatorname{div}(u_{\alpha_k}\otimes u_{\alpha_k})\|^2_{H^{-2}}\\ &\le \int_0^T \|\operatorname{div}(u_{\alpha_k}\otimes u_{\alpha_k})\|^2_{H^{-3/2}}\\ &\le \|u_{\alpha_k}\|^2_{L^\infty_T(L^2)}\|u_{\alpha_k}\|^2_{L^2_T(\dot H^1)},\end{aligned}$$

where we used respectively the inequality (5.3), the Sobolev norm definition and product laws. Finally the energy estimate (5.1) implies that

$$\int_0^T \|(I-\alpha^2\Delta)^{-1}\operatorname{div}(u_{\alpha_k}\otimes u_{\alpha_k})\|^2_{L^2} \le \frac{1}{2\nu}\big(\|u^0\|^2_{L^2(\mathbb{T}^3)} + \alpha_0^2\|\nabla u^0\|^2_{L^2(\mathbb{T}^3)} + \|\theta^0\|^2_{L^2(\mathbb{T}^3)} + 2\rho_{\alpha_0}(T)\big).$$

Using the preceding bounds of both the temperature and the convection terms, and recalling equation (1.1), we infer that

$$\|(I-\alpha^2\Delta)^{-1}\nabla p\|_{L^2(\dot{H}^{-1})} \leq K_2''.$$

So equation (5.2) implies that

$$\Big\|\frac{d}{dt}u_{\alpha_k}\Big\|_{L^2_T(\dot{H}^{-1})} \leq K_2.$$

Using the Arzelà–Ascoli theorem and the Cantor diagonal extraction process, we can extract subsequences of u_{α_k}, θ_{α_k}, which we relabel also by u_{α_k}, θ_{α_k}, such that $u_{\alpha_k} \to \breve{u}$ and $\theta_{\alpha_k} \to \breve{\theta}$ strongly in $C\big([0,T],\dot{H}^{-\varepsilon}(\mathbb{T}^3)\big)$ for all $\varepsilon > 0$ as $\alpha_k \to 0^+$. Following the lines of the proof of the existence result, we can extract further subsequences that we relabel again u_{α_k}, v_{α_k} and θ_{α_k} and show that as $\alpha_k \to 0^+$, $\breve{u}$ and $\breve{\theta}$ are solutions in the weak sense of the system (Bq). In fact, we use weak convergence to take the limit in the linear part and obtain for almost every time t of $[0,T]$, any test solenoidal vector field Λ, and any test scalar function Ξ,

$$(v_{\alpha_k}-\breve{u},\Lambda)_{L^2} \to 0$$

and

$$(\theta_{\alpha_k}-\breve{\theta},\Xi)_{L^2} \to 0$$

as $k \to +\infty$, or equivalently as $\alpha_k \to 0^+$. Mainly, $u_{\alpha_k}(t)$ converges to $\breve{u}(t)$ and $\theta_{\alpha_k}(t)$ converges to $\breve{\theta}(t)$ weakly in L^2 and uniformly in $[0,T]$. Moreover,

$$\int_0^t (v_{\alpha_k}-\breve{u},\Delta\Lambda)_{L^2}\,d\tau \to 0,$$

and

$$\int_0^t (\theta_{\alpha_k}-\breve{\theta},\Xi)_{L^2}\,d\tau \to 0.$$

Likewise for the existence part, to handle the nonlinear terms, as a first step, we prove that as $\alpha_k \to 0^+$ (or equivalently $k \to +\infty$) that

$$\lim_{k\to+\infty} u^i_{\Phi_1(k)}u^j_{\Phi_1(k)} = \breve{u}^i\breve{u}^j \quad \text{in } \mathcal{D}'(\mathbb{R}_+{}^* \times \mathbb{T}^3)$$

and

$$\lim_{k\to+\infty} u^i_{\Phi_1(k)}\theta^j_{\Psi_1(k)} = \breve{u}^i\breve{\theta}^j \quad \text{in } \mathcal{D}'(\mathbb{R}_+{}^* \times \mathbb{T}^3).$$

After that, we establish that

$$\lim_{k\to+\infty} J_k u^i_{\Phi_1(k)}u^j_{\Phi_1(k)} = \breve{u}^i\breve{u}^j \quad \text{in } \mathcal{D}'(\mathbb{R}_+{}^* \times \mathbb{T}^3)$$

and

$$\lim_{k\to+\infty} J_k u^i_{\Phi_1(k)}\theta^j_{\Psi_1(k)} = \breve{u}^i\breve{\theta}^j \quad \text{in } \mathcal{D}'(\mathbb{R}_+{}^* \times \mathbb{T}^3).$$

Finally, we infer, for any divergence-free test vector field Λ and any test scalar function Ξ belonging both of them to $C^1([0,T],C_0^\infty)$, that

$$(\breve{\theta},\Xi)+\int_0^t(\breve{\theta},\Delta\Xi)\,d\tau+\int_0^t(u\cdot\nabla\theta,\Xi)\,d\tau=0$$

$$(\breve{u},\Lambda)+\int_0^t(\breve{u},\Delta\Lambda)\,d\tau+\int_0^t(u\cdot\nabla u,\Lambda)\,d\tau+\int_0^t(\theta e_3,\Lambda)\,d\tau=0.$$

As for the energy estimate, we note that every weak solution fulfills the energy estimates (1.7), so we deduce the energy estimates (1.8) by taking the lower limit as $\alpha_k\to 0^+$.

To prove that u_{α_k} converges to $\breve{u}$ strongly in $L^2([0,T],L^2)$, we note that

$$\begin{aligned}\|u_{\alpha_k}-\breve{u}\|_{L^2([0,T],L^2)}&\le\|u_{\alpha_k}-\breve{u}\|_{L^\infty([0,T],\dot{H}^{-1})}\,\|u_{\alpha_k}-\breve{u}\|_{L^2([0,T],\dot{H}^1)}\\&\le\|u_{\alpha_k}-\breve{u}\|_{C([0,T],\dot{H}^{-1})}\,\|u_{\alpha_k}-\breve{u}\|_{L^2([0,T],\dot{H}^1)}\\&\le\|u_{\alpha_k}-\breve{u}\|_{C([0,T],\dot{H}^{-1})}(\|u_{\alpha_k}\|_{L^2([0,T],\dot{H}^1)}+\|\breve{u}\|_{L^2([0,T],\dot{H}^1)}).\end{aligned}$$

We use the energy estimate (1.8) to bound $\|\breve{u}\|_{L^2([0,T],H^1)}$ in addition to the uniform bound of u_{α_k} in $L^2([0,T],H^1)$. Hence, the fact that $u_{\alpha_k}\to\breve{u}$ strongly in $C\big([0,T],\dot{H}^{-\varepsilon}(\mathbb{T}^3)\big)$ for all $\varepsilon>0$ as $\alpha_k\to 0^+$ accomplishes the job. The same proof applies to show that θ_{α_k} converges to $\breve{\theta}$ strongly in $L^2([0,T],L^2)$.

To show that $v_{\alpha_k}\to\breve{u}$ strongly in $L^2([0,T],\dot{H}^{-1})$, we note that

$$\begin{aligned}\|v_{\alpha_k}-u_{\alpha_k}\|^2_{L^2([0,T],\dot{H}^{-1})}&=\alpha^4\int_0^T\Big(\sum_{k\in\mathbb{Z}^3}|k|^{-2}\,|\widehat{\Delta u}_{\alpha_k}|^2\Big)\\&=\alpha^4\int_0^T\Big(\sum_{k\in\mathbb{Z}^3}|k|^{2}\,|\widehat{u}_{\alpha_k}|^2\Big)\\&=\alpha^4\|u_{\alpha_k}\|^2_{L^2([0,T],\dot{H}^1)}.\end{aligned}$$

The fact that u_{α_k} belongs to $L^2([0,T],\dot{H}^1)$ implies the aim.

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