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A SINGULAR PERTURBATION PROBLEM AND A NEUTRAL DIFFERENTIAL-DIFFERENCE EQUATION

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1. **Introduction.** Vasil'eva, [2], demonstrates a close connection between the explicit formulae for solutions to the linear difference equation with constant coefficients

$$z(t) = Az(t-\tau) \tag{1.1}$$

where z is an n-vector, A an $n\times n$ constant matrix, $\tau>0$, and a corresponding differential equation with constant coefficients

$$\tau\dot{z} = Bz. \tag{1.2}$$

(1.2) is obtained from (1.1) by replacing the difference $z(t-\tau)$ by the first two terms of its Taylor Series expansion, combined with a suitable rearrangement of the terms.

We consider the vector differential-difference equation

$$\dot{x}(t-\tau) = G(t, x(t), \dot{x}(t)). \tag{1.3}$$

On replacing $\dot{x}(t-\tau)$ by the first two terms of its Taylor Series expansion, and suitably rearranging, we obtain

$$\tau\dot{x}(t) = H(t, y(t), \dot{y}(t)). \tag{1.4}$$

A comparison is made of the solutions between (1.3) and (1.4).

2. **The vector differential-difference equation.** The basic result is given in the following theorem:

THEOREM. *Suppose $y(t)$, $x(t)$ are n-vectors. Let $\dot{y}(t)=\dot{y}(t-\tau)+f(y, \dot{y}, t)$, $0\le t\le b$ with $y(t)=\phi(t)$ for $-\tau\le t\le 0$ where $\phi(t)\in C^3, f(y, \dot{y}, t)$ is differentiable and k_1 and k_2 are positive constants such that*

$$\|f(x_1, y_1, t)-f(x_2, y_2, t)\| \le k_1\,\|x_1-x_2\|+k_2\,\|y_1-y_2\|, \qquad 0\le t\le b.$$

Let $\tau\ddot{x}(t)=f(x, \ddot{x}, t)$, with $x(t)=\phi(t)$ for $-\tau\le t\le 0$ and

$$\|e(t)\| \equiv \|x(t)-y(t)\|.$$

If (*i*) $\|\dddot{}\,(t)\| \le a_1$ *for* $-\tau \le t \le b$,

(ii) $$k_2 < 1$$

(iii) $$\max\left\{\frac{1}{2(1-k_2)}, \frac{k_2}{2(1-k_2)}\right\} < 1,$$

then there exists a constant a_2 *such that* $\|e(t)\| \le a_2\tau^2$ *for* $0 \le t \le b$. *If* (*i*) *is replaced by* $\|\ddot{x}(t)\| \le a_3/\tau$, *then*

$$\|e(t)\| \le a_4\tau, \quad for \quad 0 \le t \le b.$$

We consider the vector differential-difference equation

$$\dot{y}(t) = \dot{y}(t-\tau) + f(y, \dot{y}, t), \qquad 0 \le t \le b, \tag{2.1}$$

with

$$y(t) = \phi(t) \quad \text{for} \quad -\tau \le t \le 0,$$

$\phi(t) \in C^3$; $f(y, \dot{y}, t)$ is continuous and differentiable, and k_1 and k_2 are constants such that

$$\|f(x_1, y_1, t) - f(x_2, y_2, t)\| \le k_1 \|x_1 - x_2\| + k_2 \|y_1 - y_2\|.$$

We want to compare (2.1) with the equation

$$\tau\ddot{x}(t) = f(x, \dot{x}, t), \qquad 0 \le t \le b, \tag{2.2}$$

where $x(t) = \phi(t)$ for $-\tau \le t \le 0$.

Now

$$\dot{x}(t-\tau) = \dot{x}(t) - \tau\ddot{x}(t) + \frac{\tau^2}{2!}\dddot{x}(t-\theta\tau), \tag{2.3}$$

where $0 < \theta < 1$. Substituting from (2.2) into (2.3),

$$\dot{x}(t) = \dot{x}(t-\tau) + f(x, \dot{x}, t) - \frac{\tau^2}{2}\dddot{x}(t-\theta\tau). \tag{2.4}$$

Hence, if $x(t)$ solves (2.2), then it satisfies (2.1) approximately, the error $E(t)$ being

$$E(t) = \frac{\tau^2}{2}\dddot{x}(t-\theta\tau). \tag{2.4A}$$

Define the error e between the solutions x and y by $e \equiv x - y$, so subtracting (2.1) from (2.4)

$$\dot{e}(t) = \dot{e}(t-\tau) + f(x, \dot{x}, t) - f(y, \dot{y}, t) + E(t). \tag{2.5}$$

Integrating the above equation,

$$e(t) - e(0) = e(t-\tau) - e(-\tau) + \int_0^t \{f(x, \dot{x}, s) - f(y, \dot{y}, s)\}\, ds + \int_0^t E(s)\, ds \tag{2.6}$$

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But $e(0)=e(-\tau)=0$ because of the conditions imposed on y and x in (2.1) and (2.2) respectively, and so (2.6) implies

$$\text{(2.7)}\qquad \|e(t)\| \le \|e(t-\tau)\|+k_1\int_0^\tau \|e(s)\|\,ds+k_2\int_0^t \|\dot e(s)\|\,ds+\int_0^t \|E(s)\|\,ds$$

Also, (2.5) implies

$$\|\dot e(t)\| \le \|\dot e(t-\tau)\|+k_1\|e(t)\|+k_2\|\dot e(t)\|+\|E(t)\|.$$

Assume $1>k_2$, so

$$\|\dot e(t)\| \le \frac{1}{1-k_2}\|\dot e(t-\tau)\|+\frac{k_1}{1-k_2}\|e(t)\|+\frac{1}{1-k_2}\|E(t)\|.$$

Substitute for $\|e(t)\|$, from (2.7), into the above inequality; then

$$\text{(2.9)}\qquad \|\dot e(t)\| \le \frac{1}{1-k_2}\|\dot e(t-\tau)\|+\frac{1}{1-k_2}E_m+\frac{k_1}{1-k_2}\|e(t-\tau)\|$$
$$+\frac{k_1^2}{1-k_2}\int_0^t\|e(s)\|\,ds+\frac{k_1k_2}{1-k_2}\int_0^t\|\dot e(s)\|\,ds+\frac{k_1}{1-k_2}\int_0^t\|E(s)\|\,ds,$$

where we define $E_m=\max_{0\le t\le b}\|E(t)\|$.

From (2.4A)

$$E_m \le \frac{\tau^2}{2}\|\dddot x(t-\theta\tau)\| \le \frac{\tau^2}{2}\max_{-\tau\le t\le b}\|\dddot x(t)\|.$$

If $t\in[-\tau,0]$, then $\dddot x(t)=\dddot\phi(t)$, so $E_m=\le c_2\tau^2$ if $\dddot\phi(t)$ is bounded by c_1. If $t\in[0,b]$, then $E_m\le c_5\tau^2$ if $\dddot x(t)\le c_4$, and $E_m\le c_6\tau$ if $x(t)\le c_6/\tau$.

We want to show that the error $e(t)$ between the solutions $x(t)$ and $y(t)$ is, under certain conditions, "small" and tends to zero as τ tends to zero. The resultant inequality (2.7) for $e(t)$ involves $\dot e(t)$ and the inequality (2.9) for $\dot e(t)$ involves $e(t)$. However $\|e(t)\|\le\|e(t)\|+\|\dot e(t)\|$, so this leads us to define the vector

$$z(t)=\begin{bmatrix}e(t)\\ \dot e(t)\end{bmatrix}$$

with a norm such that

$$\|z(t)\| \equiv \tfrac{1}{2}\{\|e(t)\|+\|\dot e(t)\|\}.$$

We thus have from (2.7) and (2.9),

$$\text{(2.10)}\qquad \|z(t)\| \le \left\{\tfrac{1}{2}+\frac{k_1}{2(1-k_2)}\right\}\|e(t-\tau)\|+\frac{1}{2(1-k_2)}\|\dot e(t-\tau)\|$$
$$+\int_0^t\left\{\left(\tfrac{1}{2}k_1+\frac{k_1^2}{2(1-k_2)}\right)\|e(s)\|+\left(\tfrac{1}{2}k_2+\frac{k_1k_2}{2(1-k_2)}\right)\|\dot e(s)\|\right\}ds$$
$$+\frac{1}{2(1-k_2)}E_m+\int_0^t\left\{\tfrac{1}{2}+\frac{k_1}{2(1-k_2)}\right\}\|E(s)\|\,ds$$
$$\le A+C\|z(t-\tau)\|+B\int_0^t\|z(s)\|\,ds,$$

where

$$A \equiv \frac{1}{2(1-k_2)} E_m + \int_0^b \left\{\tfrac{1}{2} + \frac{k_1}{2(1-k_2)}\right\} \|E(s)\| \, ds,$$

$$B \equiv \max\left\{\frac{k_1^2}{2(1-k_2)}, \frac{k_1 k_2}{2(1-k_2)}, \frac{k_1}{2}, \frac{k_2}{2}\right\},$$

$$C \equiv \max\left\{\frac{1}{2(1-k_2)}, \frac{k_1}{2(1-k_2)} + \tfrac{1}{2}\right\}$$

Equation (2.10) is valid for $t \geq \tau$; thus there is a unique integer $n \geq 1$, such that $n\tau \leq t \leq (n+1)\tau$, and so (2.10) holds for $t \in [n\tau, (n+1)\tau]$.

We now use the method of induction to show, with suitable conditions on $\ddot{x}(t)$ and for $t \in [n\tau, (n+1)\tau]$, that $\|z(t)\| \leq \tau C^*$ where C^* is used as the generic symbol for a constant, (independent of n and τ), i.e. C^* may vary from equation to equation.

We firstly consider the situation when $n=0$, i.e. $t \in [0, \tau]$. Then, in (2.5), $\dot{e}(t) = f(x, \dot{x}, t) - f(y, \dot{y}, t) + E(t)$ since $\dot{e}(t-\tau) = 0$ for $0 \leq t \leq \tau$; so

$$\|\dot{e}(t)\| \leq k_1 \|e(t)\| + k_2 \|\dot{e}(t)\| + E_m; \quad \text{and for} \quad 1 < k_2,$$

(2.11)
$$\|\dot{e}(t)\| \leq \frac{k_1}{1-k_2} \|e(t)\| + \frac{1}{1-k_2} E_m.$$

In (2.7), for $0 \leq t \leq \tau$,

(2.12)
$$\|e(t)\| \leq \tau E_m + k_1 \int_0^t \|e(s)\| \, ds + k_2 \int_0^t \|\dot{e}(s)\| \, ds.$$

From (2.11) and (2.12),

$$\|\dot{e}(t)\| \leq \frac{(1+k_1\tau)}{1-k_2} E_m + \frac{k_1^2}{1-k_2} \int_0^t \|e(s)\| \, ds + \frac{k_1 k_2}{1-k_2} \int_0^t \|\dot{e}(s)\| \, ds.$$

As before, we define

$$z(t) \equiv \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix};$$

also, $\|z(t)\| = \tfrac{1}{2}\{\|e(t)\| + \|\dot{e}(t)\|\}$.

Hence, for $0 \leq t \leq \tau$,

$$\|z(t)\| \leq \frac{(1+\tau(1+k_1-k_2))}{2(1-k_2)} E_m + \tfrac{1}{2} M \int_0^t \|z(s)\| \, ds,$$

where

$$M = \max\left(k_1, k_2, \frac{k_1^2}{1-k_2}, \frac{k_1 k_2}{1-k_2}\right).$$

Let

$$L \equiv \frac{1+\tau(1+k_1-k_2)}{2(1-k_2)},$$

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then by Gronwall's Inequality,

$$\|z(t)\| \leq E_m L \exp(M\tau).$$

Since $E_m \leq C_3\tau$, under appropriate conditions as given previously, then

$$\|z(t)\| \leq \tau C^* \quad \text{for} \quad t \in [0, \tau].$$

We now assume the induction hypothesis, viz. that $\|z(t)\| \leq \tau C^*$, is true for some integer $(n-1)$, that is for $t \in [(n-1)\tau, n\tau]$. We want to prove that, for $t \in [n\tau, (n+1)\tau]$, $\|z(t)\| \leq \tau C^*$.

We note, in passing, that $E_m \leq \tau C^*$ and so, therefore, is A. (2.10) may be written as

$$\|z(t)\| \leq A + C(\tau C^*) + B\int_0^t \|z(s)\|\, ds,$$
$$\leq \tau C^* + B\int_0^t \|z(s)\|\, ds.$$

On using Gronwall's Inequality,

$$\|z(t)\| \leq \tau C^* e^{Bt} \leq \tau C^* e^{Bb} \leq \tau C^*.$$

Hence, by the principle of induction,

$$\|z(t)\| \leq \tau C^*,$$

and so

$$\|e(t)\| = \|x(t) - y(t)\| \leq \tau C^* \quad \text{for} \quad 0 \leq t \leq b.$$

The theorem is now proved.

3. **Boundedness of derivative.** Condition (i) of the theorem on page 2 immediately poses the problem as to when $\ddot{x}(t)$ is bounded. This leads to a consideration of the conditions under which the solution of the singular perturbation problem (2.2) converges to the solution of the equation obtained from (2.2) by putting $\tau = 0$; this equation is called the degenerate equation.

Following Wasow [3], we summarise the results pertaining to the autonomous differential system

$$\begin{aligned} \dot{x} &= u \\ \tau\dot{u} &= g(x, u). \\ x(0) = \alpha, &\qquad y(0) = \beta \end{aligned} \tag{3.1}$$

where x is a scalar and u a two-dimensional vector. The mathematical analysis of the problem is analogous for the non-autonomous problem and also for the situation where x and u are vectors of any dimension.

We assume that g is continuous in an open region Ω of the (x, u) space and that there is a function $\phi(x)$ continuous in $\zeta_1 \leq x \leq \zeta_2$ such that the points $(x, \phi(x))$, $\zeta_1 \leq x \leq \zeta_2$, are in Ω and

$$g(x, \phi(x)) \equiv 0.$$

$\phi(x)$ is called the root of the equation $g(x, u)=0$. A further assumption is that there exists a positive η, independent of x, such that

$$\|u-\phi(x)\| < \eta, \qquad u \neq \phi(x) \quad \text{in} \quad \zeta_1 \leq x \leq \zeta_2$$

imply that $g(x, u)\neq 0$, in $\zeta_1\leq x\leq \zeta_2$. Such a root $\phi(x)$ is called isolated in $\zeta_1\leq x\leq\zeta_2$.

The boundary layer equation belonging to (2.15) is defined as

$$\frac{du}{dT} = g(x, u) \tag{3.2}$$

where x is a parameter.

Our next assumption is that the singular point $u=\phi(x)$ of (3.2) is asymptotically stable for all x in $\zeta_1\leq x\leq\zeta_2$. Such a root is called a stable root. Finally, we assume that (3.1) and the degenerate equation

$$\begin{aligned} \dot{x} &= u \\ u &= \phi(x) \\ x(0) &= \alpha \end{aligned} \tag{3.3}$$

have a unique solution in an internal $0\leq t\leq b$.

We define a point $(\alpha, \beta)\in\Omega$, $\zeta_1\leq\alpha\leq\zeta_2$, to lie in the domain of influence of the stable root $u=\phi(x)$ if the solution of the problem

$$\frac{du}{dT} = g(\alpha, u), \qquad y(0) = \beta$$

exists and remains in Ω for all $T>0$, and if it tends to $\phi(\alpha)$, as $T\to+\infty$.

We now state Tihonov's convergence theorem:

THEOREM. *Let the above assumptions be satisfied and let* (α, β) *be a point in the domain of influence of the root* $u=\phi(x)$. *Then the solution* $x(t)$, $u(t)$ *of* (3.1) *is related to the solution* $x_0(t)$, $u_0(t)=\phi(x_0(t))$ *of* (3.3) *by the fact that as* $\tau\to 0$, $x(t)\to x_0(t)$, $u(t)\to u_0(t)=\phi(x_0(t))$ *for* $0\leq t\leq T_0$.

Here T_0 *is any number such that* $u=\phi(x_0(t))$ *is an isolated stable root of* $g(x_0(t), u)=0$ *for* $\{t:0\leq t\leq T_0\}$. *The convergence is uniform in* $\{t:0\leq t\leq T_0\}$, *for* $x(t)$, *and in any interval* $I=\{t:0<t_1\leq t\leq T_0\}$ *for* $u(t)$.

Tihonov's theorem admits the following interpretation. If (α, β) lies on the curve $C:g(x, u)=0$, or is within a "tube" of width $0(\tau)$ centred on C, and if the other assumptions of the theorem are satisfied, then the trajectory described in the $x-u$ space by the solution of (3.1) is a slowly traced path which takes place near the curve $g(x, u)=0$, or $g(x, \dot{x})=0$. Since the equation $g(x, \dot{x})=0$ contains no τ, then $\ddot{x}(t)$ will be bounded for some finite time interval I, and for some sufficiently small τ.

This interpretation and conclusion carries over immediately to the non-autonomous case and in particular to (2.2).

The above reasoning suggests that the solution to $f(x_0, \dot{x}_0, t)=0$ would supply an approximation to the solution, $y(t)$, of (2.1). However, if $\|\ddot{x}(t)\|$ is bounded by some constant a_1, and if $\|y(t)-x_0(t)\| \leq a_3\tau$ then by the theorem on page 2, $\|x(t)-y(t)\| \leq a_2\tau^2$, so that the agreement between $x(t)$ and $y(t)$ is possibly better than the agreement between $x(t)$ and $x_0(t)$.

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