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ON ALGEBRAS GENERATED BY COMPOSITION OPERATORS

J. A. CIMA AND W. R. WOGEN

1. Introduction and definitions. Let Δ be the open unit disk in the complex plane and let $\mathscr{L}$ be the group of automorphisms of Δ onto Δ, define by

$$\mathscr{L} = \left\{ \phi : \Delta \rightarrow \Delta | \phi(z) = a \frac{z - \lambda}{1 - \bar{\lambda} z} \quad \text{where } |a| = 1, |\lambda| < 1 \right\}.$$

The Banach spaces $H^p = H^p(\Delta)$, $1 \leqq p < \infty$, are the Hardy spaces of functions analytic in Δ with their integral p means bounded,

$$\sup_{r<1} \left(\frac{1}{2\pi} \int |f(re^{i\theta})|^p d\theta \right) = M < \infty .$$

The Banach space $H^\infty(\Delta) = H^\infty$ consists of the bounded analytic functions on Δ. If X is a Banach space and $\mathscr{B}(X)$ is the space of all bounded linear operators on X, then a vector $x \in X$ is said to be a cyclic vector for an algebra $\mathscr{A} \subset \mathscr{B}(X)$ if the closure of the set

$$\{T(x) : T \in \mathscr{A}\}$$

is all of X. We recall that if X is a Banach space and $\mathscr{A}$ is a weakly closed algebra of operators on X then $\mathscr{A}$ is called reflexive if $T \in \mathscr{B}(X)$ and T leaves invariant the common invariant subspaces of $\mathscr{A}$ implies $T \in \mathscr{A}$.

The principal result of this paper is concerned with the set

$$L = \{C_\phi \in \mathscr{B}(H^p) : \phi \in \mathscr{L}\}$$

consisting of composition operators on the Hardy spaces H^p, $1 \leqq p < \infty$. Let $\mathscr{A}(L)$ denote the weakly closed subalgebra of $\mathscr{B}(H^p)$ generated by L. We show that every non-constant vector $f \in H^p$ is a cyclic vector for $\mathscr{A}(L)$. We also show that this result is sharp in the sense that the theorem fails if $\mathscr{L}$ is replaced by any abelian subgroup of $\mathscr{L}$. It is a straightforward consequence of this result, using a technique of S. Fisher [**1**], that the linear span of $\mathscr{L}$ is uniformly dense in the disk algebra (the Banach space of functions continuous on $\bar{\Delta}$, and analytic in Δ).

A second result shows that if $\mathscr{A}(L)$ is the weakly closed algebra of $\mathscr{B}(H^2)$ generated by the set L then $\mathscr{A}(L)$ is reflexive.

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2. The principal result. It is well-known (cf. [**5**]) that a function f is in H^p $(1 \leqq p < \infty)$ if there exists a harmonic function u, $u(z) \geqq 0$, such that

$$|f(z)|^p \leqq u(z)$$

for all $z \in \Delta$. It is then clear that for each $\phi \in \mathscr{L}$ the composition operator on H^p,

$$C_\phi(f) = f \circ \phi,$$

is linear and into H^p. An easy computation (see [**6**, p. 7]) yields the estimate

$$||C_\phi|| \leqslant \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|}\right)^{1/p}.$$

THEOREM 1. *Let $f \in H^p$, $1 \leqq p < \infty$, with f non-constant. Then*

$$f(\mathscr{L}) = \{C_\phi(f) : \phi \in \mathscr{L}\}$$

has dense span in H^p.

The proof of Theorem 1 will require two lemmas.

LEMMA 1. *If $f \in H^p$, $1 \leqq p < \infty$ and $\phi(z) = (z - \lambda)/(1 - \bar{\lambda}z)$, then*

$$D^n(C_\phi(f))_{z=0} = \sum_{j=1}^{n} A_j(1 - |\lambda|^2)^j f^j(-\lambda)(\bar{\lambda})^{n-j}$$

where $A_j > 0$, $j = 1, 2, 3, \ldots, n$.

LEMMA 2. *If $f \in H^p$, $1 \leqq p < \infty$ and f is non-constant, then given $n > 0$ there is a $\phi \in \mathscr{L}$ so that*

$$D^n(f \circ \phi)|_{z=0} \neq 0.$$

Proof of Theorem 1. Assume the validity of Lemma's 1 and 2. We consider first $1 < p < \infty$. By the Hahn-Banach Theorem it suffices to show that if $\psi \in (H^p)^*$ and $\psi(C_\phi f) = 0$ for all $\phi \in \mathscr{L}$ then $\psi = 0$. Hence, assume there is a $g \in H^q$ $(1/p + 1/q = 1)$, such that

$$\Psi(f) = \frac{1}{2\pi} \int_{|z|=1} \overline{g(z)} f(\phi(z)) \frac{dz}{z} = 0$$

for every $\phi \in \mathscr{L}$. Suppose the Fourier expansions of g and f are given by

$$g(z) \simeq \sum_{n=0}^{\infty} b_n z^n$$

$$f(z) \simeq \sum_{n=0}^{\infty} a_n z^n, \quad |z| = 1.$$

Choose $\phi \in \mathscr{L}$ to be a rotation, $\phi(z) = az$, $|a| = 1$. We have assumed

$$0 = \frac{1}{2\pi} \int_{|z|=1} \overline{g(z)} f(az) \frac{dz}{z}$$

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for all $|a| = 1$. On the other hand the Hausdorff-Young inequalities imply that

$$\frac{1}{2\pi i}\int_{|z|=1}\overline{g(z)}f(az)\frac{dz}{z} = \sum_{n=0}^{\infty} a_n\bar{b}_n a^n$$

and the series converges absolutely. Let $z = re^{i\theta}$, $r < 1$ and note that

$$p(z) = \frac{1}{2\pi i}\int_0^{2\pi} f(re^{i(\theta+t)})\overline{g(e^{it})}dt$$

$$= \sum_{n=0}^{\infty} a_n\bar{b}_n z^n$$

is in H^∞. By Abel's theorem

$$\lim_{r\to 1} p(re^{i\theta}) = 0$$

a.e. on $|z| = 1$. Hence $a_n\bar{b}_n = 0$ for $n = 0, 1, 2, 3, \ldots$. It follows that $b_n = 0$ for all n such that $a_n \neq 0$. We apply Lemma 2. Given n, there is a $\phi \in \mathscr{L}$ with $D^n(C_\phi f)|_{z=0} \neq 0$. Thus we have

$$f(\phi(z)) = \sum_{k=0}^{\infty} A_k z^k$$

with $A_n \neq 0$. Replacing f by $C_\phi f$ in the above argument we see that $b_n = 0$, $n = 0, 1, 2, \ldots$. Thus $g = 0$ (and so ψ is the zero functional).

Now for $p = 1$ and $\psi \in (H^1)^*$ we know there is a $g \in L^\infty$ such that

$$\psi(f) = \frac{1}{2\pi}\int_{|z|=1} f(t)\overline{g(t)}dt$$

all $f \in H^1$. A similar proof shows that if $g(t) \simeq \sum_{n=-\infty}^{+\infty} b_n e^{int}$ then $b_n = 0$ for $n = 0, 1, 2, 3, \ldots$. Hence $\overline{g(t)} \in H_0^\infty$ and so

$$\psi(f) = \{f(0)\overline{g(0)}\} = 0$$

for all $f \in H^1$.

We proceed now to the proofs of Lemma's 1 and 2.

Proof of Lemma 1. First note that if $\phi(z) = (z - \lambda)/(1 - \bar{\lambda}z)$ then

(1) $$D^k\phi(z) = \frac{k!(1 - |\lambda|^2)(\bar{\lambda})^{k-1}}{(1 - \bar{\lambda}z)^{k+1}}$$

if $k > 0$ and thus

(2) $$D^k\phi(z)|_{z=0} = k!\,(1 - |\lambda|^2)(\bar{\lambda})^{k-1}$$

and also

(3) $$f^k(\phi(z))|_{z=0} = f^k(-\lambda).$$

We will show that $D^n(C_\phi f)$ has the following form:

(4) $$D^n(C_\phi f) = \sum_{k=1}^{n} d_k f^k \circ \phi,$$

where d_k is a sum of terms of the form

(5) $$a(\phi^{(1)})^{l_1} (\phi^{(2)})^{l_2} \ldots (\phi^{(n)})^{l_n},$$

with $a > 0$ and

(6) $$\sum_{i=1}^{n} il_i = n,$$

(7) $$\sum_{i=1}^{n} l_i = k.$$

Clearly $D(C_\phi f) = (f' \circ \phi)(\phi')$ and $D^2(C_\phi f) = (f'' \circ \phi)(\phi')^2 + (f' \circ \phi)(\phi'')$ are of the desired form. We proceed by induction.

Suppose that $D^n(C_\phi f)$ has the desired form. Since

$$D^{n+1}(f \circ \phi) = D(D^n(f \circ \phi))$$

it suffices to consider the form of

$$D[(f^{(k)} \circ \phi)(\phi^{(1)})^{l_1} \ldots (\phi^{(n)})^{l_n}]$$

where the l_i satisfy (6) and (7). The derivative is

$$(f^{(k+1)} \circ \phi)(\phi^{(1)})^{l_1+1} (\phi^{(2)})^{l_2} \ldots (\phi^{(n)})^{l_n} +$$
$$(f^{(k)} \circ \phi) l_1 (\phi^{(1)})^{l_1-1} (\phi^{(2)})^{l_2+1} \ldots (\phi^{(n)})^{l_n} + \ldots$$
$$(f^{(k)} \circ \phi)(\phi^{(1)})^{l_1} \ldots l_n (\phi^{(n)})^{l_n-1} (\phi^{(n+1)}).$$

For the first term,

$$1(l_1 + 1) + 2l_2 + \ldots + nl_n = 1 + \sum_{i=1}^{n} il_1 = n + 1$$

and $(l_1 + 1) + l_2 + \ldots + l_n = 1 + \sum_{i=1}^{n} l_i = k + 1$ so that (6) and (7) hold. For the jth term $(2 \leqq j \leqq n)$

$$l_1 + \ldots + (j - 1)(l_{j-1} - 1) + j(l_j + 1) + \ldots + nl_n = n + 1$$

and

$$l_1 + \ldots + (l_{j-1} - 1) + (l_j + 1) + \ldots + l_n = k$$

so that (6) and (7) again hold. Thus $D^{n+1}(C_\phi f)$ has the desired form.

Now from (2),

$$a(\phi^1)^{l_1} \ldots (\phi^n)^{l_n}|_{z=0} = a_k(1 - |\lambda|^2)^k (\bar{\lambda})^s,$$

where

$$a_k = a \prod_{j=1}^{k} (j!)^{l_j} > 0$$

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and

$$s = \sum_{j=1}^{n} (j-1)l_j = n - k.$$

$d_k|_{z=0}$ is a sum of such terms, so it has the form $b_k(1 - |\lambda|^2)^k(\bar{\lambda})^{n-k}$. Thus from (4)

$$D(f \circ \phi)|_{z=0} = \sum_{k=1}^{n} b_k(1 - |\lambda|^2)^k(\bar{\lambda})^{n-k} f^k(-\lambda).$$

Proof of Lemma 2. Consider $\phi(z) = (z - \lambda)/(1 - \bar{\lambda}z)$. The lemma is true for $n = 1$. In fact

$$D(f \circ \phi)|_{z=0} = f'(-\lambda)(1 - |\lambda|^2) \neq 0$$

for some λ, since f is not constant. Suppose that the lemma holds for n. We may assume that $D^n f|_{z=0} \neq 0$. (If $D^n f|_{z=0} = 0$, we can replace f by $f \circ \phi_1$, where $\phi_1 \in \mathscr{L}$ is chosen so that $D^n(f \circ \phi_1)|_{z=0} \neq 0$. Here we need the fact that $\mathscr{L}$ is a semigroup.)

Suppose that the lemma fails for $n + 1$. Then

$$0 = D^{n+1}(f \circ \phi)|_{z=0} = \sum_{j=1}^{n} b_j(1 - |\lambda|^2)^j(\bar{\lambda})^{n+1-j} f^j(-\lambda) + b_{n+1}(1 - |\lambda|^2)^{n+1} f^{n+1}(-\lambda).$$

Hence,

$$f^{n+1}(-\lambda) = -\sum_{j=1}^{n} B_j(1 - |\lambda|^2)^{j-(n+1)}(\bar{\lambda})^{n+1-j} f^j(-\lambda)$$

identically for all λ, $|\lambda| < 1$. In particular $f^{n+1}(0) = 0$, so

$$\frac{f^{n+1}(-\lambda) - f^{n+1}(0)}{\lambda} = -\frac{\sum_{j=1}^{n-1} B_j(1 - |\lambda|^2)^{j-(n+1)}(\bar{\lambda})^{n+1-j} f^1(\lambda)}{\lambda} - \frac{B_n(1 - |\lambda|^2)^{-1}(\bar{\lambda}) f^n(-\lambda)}{\lambda}.$$

Let $\lambda \to 0$. The left side approaches $-f^{n+2}(0)$. The first term on the right side approaches zero. However,

$$\lim_{\lambda \to 0} B_n(1 - |\lambda_n|^2)^{-1} \frac{\bar{\lambda}}{\lambda} f^n(-\lambda)$$

fails to exist. This contradiction completes the proof of the lemma.

Let $\mathscr{S}$ be the linear span of the functions in $\mathscr{L}$.

LEMMA 3. *The uniform closure of $\mathscr{S}$ contains the constant functions.*

Proof of Lemma 3. If $\delta > 0$, then on $\bar{\Delta} - \{z \mid |z - e^{i\theta}| < \delta\}$,

$$\lim_{r \to 1} \frac{z - re^{i\theta}}{1 - re^{-i\theta}z} = -e^{i\theta}$$

and the convergence is uniform. Now let n be a positive integer and for $k = 0, 1, 2, \ldots, n - 1$ define $\phi_{r,k,n}(z) = \phi_{r,k}(z)$ as follows:

$$\phi_{r,k}(z) = -e^{-ak}\left(\frac{z - re^{ak}}{1 - re^{-ak_z}}\right), \quad a_k \equiv \frac{\pi k i}{n}.$$

We claim that the means

$$\frac{1}{n} \sum_{k=1}^{n} \phi_{rk}(z)$$

are uniformly close to one on $\bar{\Delta}$ if n is sufficiently large and if r is sufficiently close to one. For if $\epsilon > 0$ is given choose N so that for $n \geqq N$, $2/n < \epsilon/2$. Then choose $\delta > 0$ so small that the sets

$$B_k = \{z \mid |z - e^{a_k}| \leqq \delta\}$$

are disjoint for $k = 0, 1, \ldots, n - 1$. We can now choose r so large ($r < 1$) that

$$|\phi_{r,k}(z) - 1| < \epsilon/2$$

for $z \in \bar{\Delta} - B_k$. For $z \in \bar{\Delta} - \bigcup_{k=1}^{n} B_k$ we have

$$\left| \sum_{k=1}^{n} \frac{1}{n} \phi_{r,k}(z) - 1 \right| \leqq \sum_{k=1}^{n} \left| \frac{\phi_{r,k}(z) - 1}{n} \right| < \frac{\epsilon}{2}.$$

If $z \in B_j$ for some j, then

$$\left| \sum_{k=1}^{n} \frac{1}{n} \phi_{r,k}(z) - 1 \right| \leqq \sum_{k \neq j} \frac{|\phi_{r,k}(z) - 1|}{n} + \frac{2}{n} < \epsilon.$$

As a corollary to Theorem 1 we obtain the following result about $\mathscr{S}$ as a subset of the disk algebra (the algebra of functions continuous on $\bar{\Delta}$ and analytic in Δ).

COROLLARY. *$\mathscr{S}$ is uniformly dense in the disk algebra.*

Proof. We imitate the proof of S. Fisher [**1**]. Let f be in the disk algebra and set $f_t(z) = f(tz)$ for $0 < t < 1$, $z \in \Delta$. $\mathscr{S}$ is a dense subset of H^p by Theorem 1. Hence, there is a sequence $\{\psi_n\}$ in $\mathscr{S}$ tending to f in H^p and consequently $\{\psi_n\}$ tends to f uniformly on compacta. If $\epsilon > 0$ is given we can find a $\psi \in \mathscr{S}$ and a $0 < t < 1$ such that

$$\|f - \psi_t\| \leqq \|f - f_t\| + \|f_t - \psi_t\| < \epsilon.$$

We show $\psi_t \in \overline{\mathscr{S}}$. From the definitions of ψ and ψ_t it is sufficient to show that

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$\phi_t \in \overline{\mathscr{S}}$, where $\phi(z) = (z - \lambda)/(1 - \bar{\lambda}z)$. But if $\lambda = re^{i\theta}$, then

$$\phi_t(z) = \frac{tz - \lambda}{1 - \bar{\lambda}tz} = \frac{t(1 - r^2)}{1 - r^2 t} \left\{ \frac{z - \lambda t}{1 - \bar{\lambda}tz} \right\} - \frac{r(1 - t^2)}{1 - r^2 t^2} e^{i\theta}.$$

The first term is in $\mathscr{S}$ and the latter in $\overline{\mathscr{S}}$ by Lemma 3.

3. Composition operators. We restrict ourselves in this section to the Hilbert space H^2. Recall from Section 1 that $L = \{C_\phi : \phi \in \mathscr{L}\}$ is a subset of $B(H^2)$ and that $\mathscr{A}(L)$ is the weakly closed algebra generated by L. Theorem 1 has some consequences concerning invariant subspaces and the reflexiveness of $\mathscr{A}(L)$.

COROLLARY. *The only subspaces of* H^2 (more generally H^p, $1 \leqq p < \infty$) *which are invariant under every* C_ϕ *in* L *are* $\{0\}$, $\mathbf{C}$ *and* H^2.

THEOREM 2. $\mathscr{A}(L)$ is reflexive.

Proof. Let us recall first a theorem of Radjavi-Rosenthal [**4**]. They have shown that if $\mathscr{A}$ is a weakly closed algebra with a totally ordered invariant subspace lattice and containing a maximal abelian self-adjoint algebra, then $\mathscr{A}$ is reflexive. Our corollary shows that the lattice of $\mathscr{A}(L)$ is totally ordered. Consider then C_ψ, where $\psi(z) = az$, $|a| = 1$ and a is irrational mod 2π. It is easy to see that C_ψ is a unitary operator with cyclic vector with (simple) pure point spectrum. (Any $f(z) = \sum C_n z^n$ in H^2 with $C_n \neq 0$ for $n = 0, 1, 2, \ldots$ is a cyclic vector, and for each n, a^n is a simple eigenvalue with eigenvector z^n.) Also $C_\psi^* = C_\tau$ where $\tau(z) = \bar{a}z$. Thus $\mathscr{A}(C_\psi, C_\psi^*) \subset \mathscr{A}(L)$, and $\mathscr{A}(C_\psi, C_\psi^*)$ is maximal abelian since C_ψ is normal and cyclic (cf., e.g., [**7**, §5, Theorem 5]). The Radjavi-Rosenthal theorem now applies to complete the proof.

Let H_0^2 denote the functions in H^2 vanishing at $z = 0$, and let P denote the orthogonal projection of H^2 onto $H_0^2 = H^2 \ominus \mathbf{C}$.

COROLLARY. $P\mathscr{A}(L)|_{H_0^2} = \mathscr{B}(H_0^2)$.

Proof. Since $\mathscr{A}(L)$ is reflexive, it follows that $\mathscr{A}(L)^* = \{T^* : T \in \mathscr{A}(L)\}$ is reflexive. Further, the invariant subspaces of $\mathscr{A}(L)^*$ are $\{0\}$, H_0^2, and H^2. Thus it is easy to see that $\mathscr{A}(L)^*|_{H_0^2} = \mathscr{B}(H_0^2)$ so that

$$\mathscr{B}(H_0^2) = (\mathscr{A}(L)^*|_{H_0^2})^* = P\mathscr{A}(L)|_{H_0^2}.$$

Finally, we note that Theorem 1 fails if $\mathscr{L}$ is replaced by an abelian subgroup $\mathscr{L}'$ of $\mathscr{L}$. In fact if $\phi \in \mathscr{L}'$, Nordgren [**3**] has shown that C_ϕ has nonconstant eigenfunctions. Suppose M_λ is an eigenspace for C_ϕ and $f \in M_\lambda$, f nonconstant. If ψ commutes with ϕ then C_ψ commutes with C_ϕ and it follows that M_λ is invariant under C_ψ. Thus $f(\mathscr{L}') \subseteq M_\lambda$. Some examples of abelian subgroups are

(i) $$\left\{ \phi(z) = \frac{z - \lambda}{1 - \lambda z} \mid -1 < \lambda < 1 \right\} = \{\phi \in \mathscr{L} \mid \phi(1) = 1, \phi(-1) = -1\}.$$

More generally given μ_1, μ_2 with $|\mu_1| = |\mu_2| = 1$, $\mu_1 \neq \mu_2$

$$\{\phi \in \mathscr{L} | \phi(\mu_1) = \mu_1, \phi(\mu_2) = \mu_2\}$$

is an abelian subgroup. Also

(ii) $\{\phi \in \mathscr{L} | \phi(z) = az, |a| = 1\} = \{\phi \in \mathscr{L} | \phi(0) = 0\}$

is abelian. More generally, given μ, $|\mu| < 1$,

$$\{\phi \in \mathscr{L} | \phi(\mu) = \mu\}$$

is abelian.

We pose the following question: For which closed (nonabelian) subgroups of $\mathscr{L}$ does Theorem 1 hold?

References

1. S. Fisher, *The convex hull of finite Blaschke products*, Bull. Amer. Math. Soc. *74* (1968), 1128–1129.
2. K. Hoffman, *Banach spaces of analytic functions* (Prentice Hall, New York 1962).
3. E. Nordgren, *Composition operators*, Can. J. Math. *20* (1968), 442–449.
4. H. Radjavi and P. Rosenthal, *On invariant subspaces and reflexive algebras*, Amer. J. Math. *91* (1969), 683–692.
5. W. Rudin, *Analytic functions of class* H_p, Trans. Amer. Math. Soc. *78* (1955), 46–66.
6. H. Schwartz, *Composition operators on* H^p, Ph.D. thesis, University of Toledo, 1969.
7. D. Topping, *Lectures on von Newmann algebras* (can Nostrand Reinhold, London, New York, 1971).

University of North Carolina,
Chapel Hill, North Carolina

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