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MC2 RINGS AND WQD RINGS*

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Abstract. We introduce in this paper the concepts of rings characterized by minimal one-sided ideals and concern ourselves with rings containing an injective maximal left ideal. Some known results for idempotent reflexive rings and left HI rings can be extended to left $MC2$ rings. As applications, we are able to give some new characterizations of regular left self-injective rings with non-zero socle and extend some known results for strongly regular rings.

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1. Introduction. Throughout this paper, R denotes an associative ring with identity, and all modules are unitary. For any non-empty subset X of a ring R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the set of right annihilators of X and the set of left annihilators of X, respectively. We write $J(R)$, $P(R)$, $Z_l(R)$ ($Z_r(R)$), $N(R)$, $U(R)$, $E(R)$, $S_l(R)$ and $S_r(R)$ for the Jacobson radical, the prime radical, the left (right) singular ideal, the set of all nilpotent elements, the set of all invertible elements, the set of all idempotent elements, the left socle and the right socle of R, respectively.

An element k of R is called left minimal if Rk is a minimal left ideal. An element e of R is called left minimal idempotent if $e^2 = e$ is left minimal. Similarly, the notion of right minimal (idempotent) element is defined. We denote $M_l(R)$, $ME_l(R)$, $M_r(R)$ and $ME_r(R)$ for the set of left minimal elements, the set of left minimal idempotent elements, the set of right minimal elements and the set of right minimal idempotent elements of R, respectively. A ring R is called left $MC2$ if every minimal left ideal which is isomorphic to a summand of $_RR$ is a summand. Left $MC2$ rings were initiated by Nicholson and Yousif in [**11**], related to the left mininjective rings. In [**16**, **18**], the authors discuss their properties. In [**11**], a ring R is called left mininjective if $rl(k) = kR$ for every $k \in M_l(R)$, where $rl(k)$ denotes the set of right annihilators of $l(k)$ in R, and R is said to be left minsymmetric if $k \in M_l(R)$ always implies $k \in M_r(R)$. According to [**11**], left mininjective $\Longrightarrow$ left minsymmetric $\Longrightarrow$ left $MC2$ and the converse are not true.

A ring R is called left PS [**9**] if Rk is projective as left R-module for every $k \in M_l(R)$ and R is said to be left universally mininjective [**11**] if Rk is an idempotent left ideal of R for every $k \in M_l(R)$. [**16**] uses the term left DS for the left universally mininjective and shows that R is left DS if and only if R is left PS and left $MC2$ (see [**16**, Theorem 3.1]). According to [**11**, Lemma 5.1], left DS rings are left mininjective.

A ring R is called left min-abel if for each $e \in ME_l(R)$, e is left semi-central in R, and R is said to be strongly left min-abel if every element of $ME_l(R)$ is central

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in R. In [**17**], strongly left min-abel rings are studied for characterizing the strongly regular rings. In [**17**], we proved that R is strongly regular if and only if R is strongly left min-abel, left weakly regular and $MELT$, where a ring R is $MELT$ ($MERT$) if every essential maximal left (right) ideal of R is an ideal. A ring R is called left (right) quasi-duo if every maximal left (right) ideal of R is an ideal. Proposition 4.7 of [**14**] shows that R is strongly regular if and only if R is left weakly regular left quasi-duo, and [**5**, Theorem 6] shows that R is strongly regular if and only if R is abelian left quasi-duo whose simple singular left R-modules are YJ-injective, where a ring R is called abelian if every idempotent of R is central. In [**15**], a ring whose simple left R-modules are YJ-injective is called left GP–V-ring. R is called left $SGP-V$-ring if every simple singular left R-modules are YJ-injective. Clearly, left $GP-V$-rings are left $SGP-V$-rings, but the converse is not true in general. For example, let R be a 2×2 lower triangular matrix ring over a field. Then R is a left quasi-duo left $SGP-V$-ring but R is not left $GP-V$-ring.

Left ideal L of R is called GW-ideal if, for any $a \in L$, there exists a positive integer n such that $a^nR \subseteq L$. Similarly, the notion of $GW-$ideal for a right ideal K of R is defined. Clearly, ideal is $GW-$ideal, but the converse is not true by [**15**, Example 1.2]. A ring R is called left (right) WQD if every maximal left (right) ideal of R is a $GW-$ideal, and R is called $WMELT$ ($WMERT$) if every essential maximal left (right) ideal of R is a $GW-$ideal. Clearly, left quasi-duo $\Longrightarrow MELT \Longrightarrow WMELT$ and left quasi-duo $\Longrightarrow$ left $WQD \Longrightarrow WMELT$. [**15**, Theorem 2.2] shows that R is strongly regular if and only if R is left $GP-V-$ring and left WQD if and only if R is left $GP-V-$ring and right WQD.

A ring R is called reflexive [**7**] if $aRb = 0$ implies $bRa = 0$ for all $a, b \in R$, and R is said to be left idempotent reflexive [**3**] if $aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in E(R)$. Clearly, semi-prime $\Longrightarrow$ reflexive $\Longrightarrow$ left idempotent reflexive $\Longrightarrow$ left $MC2$.

R is called reduced if it contains no non-zero nilpotent elements. Clearly, reduced $\Longrightarrow$ semi-prime $\Longrightarrow$ left DS.

A left R-module M is called YJ-injective (see [**2**], [**8**]) if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any left R-homomorphism from Ra^n to M extends to one from R to M.

In Section 2, we discuss some classes of rings characterized by minimal one-sided ideals, give some interesting and valuable characterizations of these rings.

In Section 3, we introduce the notions of left $SGP-V$-rings and left WQD rings. And then, we give some characterizations of strongly regular rings, which generalize some known results that appeared in [**5**, **14**, **15**].

In Section 4, we prove that a left $MC2$ ring containing an injective maximal left ideal is left self-injective. This result generalizes [**4**, Proposition 5], and as byproducts of the result, we obtain new characterizations of regular left self-injective rings with non-zero left socle. These characterizations are then used to prove that a left $MC2$ left HI ring is semi-simple Artinian.

2. Rings characterized by minimal one-sided ideals. We start with the following theorem.

THEOREM 2.1. *The following conditions are equivalent for a ring R.*

(1) *R is left $MC2$.*

(2) *Every left minimal idempotent element of R is right minimal.*

(3) *$aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in ME_l(R)$.*

(4) *For any $k \in M_l(R)$ and $g \in ME_l(R)$ with $k^2 = 0$, $Rk \cong Rg$ as left R-modules always implies $Rk = Re$, $e^2 = e \in R$.*

Proof. (1) $\Longrightarrow$ (4) is trivial.

(4) $\Longrightarrow$ (3) Assume that $aRe = 0$, where $a \in R$ and $e \in ME_l(R)$. If $eRa \neq 0$, then there exists a $b \in R$ such that $eba \neq 0$. Clearly, $eba \in M_l(R)$, $(eba)^2 = 0$ and $Reba \cong Re$ as left R-modules, so, by (4), we have $Reba = Rg$ for some $g \in ME_l(R)$. Hence $Rg = RgRg = RebaReba = 0$, which is a contradiction. This shows that $eRa = 0$.

(3) $\Longrightarrow$ (2) Let $e \in ME_l(R)$ and $a \in R$ with $ea \neq 0$. By (3), $eaRe \neq 0$. So, we have $Re = ReaRe$, consequently, $Re = ReRe = ReaReReaRe \subseteq ReaReaRe$. This implies $(Rea)^2 \neq 0$. Since Rea is a minimal left ideal of R, $Rea = Rg, g \in ME_l(R)$. Let $g = cea$, $c \in R$ and $h = eac$. Then $h \in ME_l(R)$ and $eaR = hR$. Since $l(h) = l(ea) = l(e)$, $eaR = hR = rl(h) = rl(e) = eR$. This shows that eR is a minimal right ideal of R.

(2) $\Longrightarrow$ (1) Suppose that $k \in M_l(R)$ and $e \in ME_l(R)$ such that ${}_RRe \cong {}_RRk$. It is easy to compute that there exists a $g \in ME_l(R)$ such that $gk = k$ and $l(g) = l(k)$. By (2), gR is a minimal right ideal of R. Hence $kR = gkR = gR$. Let $g = kc$, $c \in R$ and $h = ck$, then, Clearly, $Rk = Rh, h \in ME_l(R)$. This shows that R is left $MC2$ ring. □

It is easy to see that R is left minsymmetric if and only if $M_l(R) \subseteq M_r(R)$. So, by Theorem 2.1, left minsymmetric rings are left $MC2$. In fact, we have the following corollary.

COROLLARY 2.2. *The following conditions are equivalent for a ring R.*

(1) *R is left minsymmetric.*

(2) *R is left $MC2$ and $Z_l(R) \cap M_l(R) \subseteq M_r(R)$.*

(3) *$S_l(R) \subseteq S_r(R)$ and for any $a, b \in M_l(R)$, if $aR \subseteq bR$, then $aR = bR$.*

Proof. (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are trivial.

(2) $\Longrightarrow$ (1) Let $k \in M_l(R)$. If ${}_RRk$ is projective, then $Rk = Re$, $e \in ME_l(R)$ because R is left $MC2$. Let $e = ck$, $c \in R$ and $g = kc$, then $g \in ME_l(R)$ and $kR = gR$. By Theorem 2.1, $g \in ME_r(R)$, so $k \in M_r(R)$. If ${}_RRk$ is not projective, then $k \in Z_l(R)$, so, we have $k \in Z_l(R) \cap M_l(R)$. By hypothesis, $k \in M_r(R)$. This implies R is left minsymmetric.

(3) $\Longrightarrow$ (1) Let $k \in M_l(R)$. Then by (3), $k \in S_l(R) \subseteq S_r(R)$. Hence there exists a $b \in M_r(R)$ such that $bR \subseteq kR$. Since $l(k) \subseteq l(b) \neq R$ and $l(k)$ is a maximal left ideal of R, $l(b) = l(k)$. So, we have $b \in M_l(R)$. By hypothesis, $bR = kR$, so, we have $k \in M_r(R)$. Hence R is left minsymmetric. □

THEOREM 2.3. *The following conditions are equivalent for a ring R.*

(1) *R is strongly left min-abel.*

(2) *R is left min-abel and left $MC2$.*

(3) *R is left min-abel and $ME_l(R) \subseteq ME_r(R)$.*

(4) *Every left minimal idempotent element of R is right semi-central.*

(5) *R is left min-abel ring and every simple projective left R-module is injective.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) is an immediate corollary of Theorem 2.1.

(3) $\Longrightarrow$ (4) Let $e \in ME_l(R)$. Then, by (3), e is left semi-central and right minimal in R. For any $a \in R$, set $h = ea - eae$, then $eh = h$, $he = 0$. If $h \neq 0$, then $hR = ehR = eR$. Therefore $eR = eReR = hReR = heReR = 0$, which is a contradiction. So, $h = 0$ and then $ea = eae$ for any $a \in R$. This shows that e is right semi-central.

(4) $\Longrightarrow$ (5) First, we claim R is left min-abel. Let $e \in ME_l(R)$. Then by (4), e is right semi-central. For any $a \in R$, write $h = ae - eae$. If $h \neq 0$, then $he = h$, $eh = 0$ and $Rh = Re$. Consequently, $Re = ReRe = ReRh = ReReh = 0$, which is a contradiction. Hence we have shown that e is central, and so R is strongly left min-abel. Next, we show that simple projective left $R-$module W is injective. Since $W \cong R/K$, where K is a maximal left ideal of R and since R/K is projective, $R = K \oplus U$, where $U = Re$, $e^2 = e \in R$, is a minimal left ideal of R. We claim that ${}_RU$ is injective. Let L be any proper essential left ideal of R, and $f : L \longrightarrow U$ any non-zero left R-homomorphism. Then $L/N \cong U$, where $N = kerf$ is a maximal sub-module of L. Now $L = N \oplus V$, where $V(\cong U)$ is a minimal left ideal of R. Since R is strongly left min-abel ring, $V = Rg$, where $g^2 = g$ is central in R. For any $y \in L$, let $y = d + ag$, where $d \in N$, $a \in R$. Then $dg = gd \in N \cap V = 0$, so $f(y) = f(d + ag) = f(ag) = f(dg) + f(ag) = f((d + ag)g) = (d + ag)f(g) = yf(g)$. Hence ${}_RU$ is injective, and so is ${}_RW$.

(5) $\Longrightarrow$ (1) Assume that $e \in ME_l(R)$. By hypothesis, e is left semi-central. For any $a \in R$, set $h = ea - eae$, then $eh = h$, $he = 0$. If $h \neq 0$, then Rh is simple projective left $R-$module, so, Rh is injective. Hence $Rh = Rt$, $t \in ME_l(R)$, so, we have $h = ht$. Since t is left semi-central by hypothesis, $h = ht = tht \in Rhht = Rheht = 0$, which is a contradiction. Hence $h = 0$ and so e is right semi-central. Therefore R is strongly left min-abel. □

Let F be a division ring and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Clearly, R is left quasi-duo. Then, by the following Corollary 2.5, we know that R is left min-abel. But R is not left $MC2$ because the left minimal idempotent element $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not right semi-central. Hence R is not strongly left min-abel by Theorem 2.3.

THEOREM 2.4. *The following conditions are equivalent for a ring R.*

(1) *R is left min-abel.*

(2) *Every non-essential maximal left ideal of R is an ideal.*

(3) *For every left minimal regular element $k \in R$ (that is $k \in M_l(R)$ and $k = kck$ for some $c \in R$), $Rk + R(kc - 1) = R$.*

(4) *For every minimal projective left ideal Rk of R, $l(k)$ is an ideal of R.*

Proof. (1) $\Longrightarrow$ (2) Suppose that M is a maximal left ideal of R and M is not essential in ${}_RR$. Hence there exists a non-zero left ideal L of R such that $L \cap M = 0$. Since M is maximal, $L \oplus M = R$. Consequently, $M = Rf$, $f^2 = f \in R$. Clearly, $1 - f$ is left minimal idempotent. Since R is left min-abel ring, $1 - f$ is left semi-central. Hence, clearly, Rf is an ideal of R.

(2) $\Longrightarrow$ (4) Since Rk is projective left R-module, $l(k)$ is a summand maximal left ideal of R. By (2), $l(k)$ is an ideal.

(4) $\Longrightarrow$ (1) Let e is any left minimal idempotent of R. By (4), $l(e) = R(1 - e)$ is an ideal of R. Hence e is left semi-central in R and so R is left min-abel.

(1) $\Longrightarrow$ (3) Let $k \in M_l(R)$ and $k = kck$ for some $c \in R$. Set $e = kc$, $g = ck$. Then $k = ek = kg$ and $e, g \in ME_l(R)$. By (1), e, g are all left semi-central. If $Rk + R(kc - 1) \neq R$, then $Rk \subseteq R(kc - 1)$, because $R(kc - 1) = R(e - 1) = R(1 - e)$ is maximal left ideal of R. Hence $k = k(1 - e)$, so, we have $ke = 0$. Consequently, $k = kg = gkg = ckkck = ckek = 0$, which is a contradiction. Hence $Rk + R(kc - 1) = R$

(3) $\Longrightarrow$ (1) Let $e \in ME_l(R)$. For any $a \in R$, write $h = ae - eae$. If $h \neq 0$, then $h = he$, $eh = 0$ and $Rh = Re$, consequently, $h^2 = 0$ and $h \in M_l(R)$. Write $e = ch$ for some

$c \in R$. Hence $h = he = hch$ and by (3), $Rh + R(hc - 1) = R$. Let $1 = dh + b(hc - 1)$, where $d, b \in R$. Thus $h = 1h = dh^2 + b(hc - 1)h = b(hch - h) = 0$, which is a contradiction. This shows that e is left semi-central in R, and so R is left min-abel. □

According to [**6**, Theorem 3.2], R is a left quasi-duo ring if and only if $R(xy - 1) + Rx = R$ for any $x, y \in R$. Hence, by Theorem 2.4, we have the following corollary.

COROLLARY 2.5. *R is left quasi-duo if and only if R is $MELT$ and left min-abel.*

Recall that a ring R is semi-abelian [**1**] if every idempotent of R is either right semi-central or left semi-central. By the proof of (4) $\Longrightarrow$ (5) of Theorem 2.3, we have the following corollary.

COROLLARY 2.6. *Every semi-abelian ring R is left min-abel.*

It is easy to show that $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ is semi-abelian but not left $MC2$ for any division ring F. Hence semi-abelian rings are not left idempotent reflexive in general. But, in contrast with Theorem 2.3, we can easy obtain the following result: R is abelian if and only if R is semi-abelian and left idempotent reflexive.

By simply computing, we know that the ring $S = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$ is not semi-abelian, so, by [**1**, Corollary 2.4], $S[x]$ is not semi-abelian but $S[x]$ is strongly left min-abel. On the other hand, by [**20**, Proposition 2.1], $S[x]$ is left quasi-duo. Hence left quasi-duo rings are not semi-abelian in general.

THEOREM 2.7. *The following conditions are equivalent for a ring R.*

(1) *R is left WQD.*

(2) *R is left min-abel and $WMELT$.*

(3) *$R/J(R)$ is left WQD.*

In this case, R is directly finite (that is, $ab = 1$ implies $ba = 1$ for any $a, b \in R$) and $R/J(R)$ is reduced.

Proof. By Theorem 2.4, (2) $\Longrightarrow$ (1) is trivial. (1) $\Longleftrightarrow$ (3) is evident.

(1) $\Longrightarrow$ (2) It is only to show that R is left min-abel. Assume that $e \in ME_l(R)$. Then $R(1 - e) = l(e)$ is a maximal left ideal of R, by (1), $R(1 - e)$ is a $GW-$ideal, so there exists a positive integer n such that $(1 - e)^n R \subseteq R(1 - e)$, this implies $(1 - e)R \subseteq R(1 - e)$. It follows that e is left semi-central. Therefore R is left min-abel.

Now let $a \in R$ with $a^2 \in J(R)$. If $a \notin J(R)$, then there exists a maximal left ideal M such that $a \notin M$. Hence $R = M + Ra$. Let $1 = m + ba$ for some $m \in M$ and $b \in R$. Therefore $a = ma + ba^2$. Since M is $GW-$ideal and $m \in M$, there exists a positive integer n such that $m^n a \in M$. Hence $m^{n-1}a = m^n a + m^{n-1}ba^2 \in M$ because $a^2 \in M$. Further, we have $m^{n-2}a = m^{n-1}a + m^{n-2}ba^2 \in M$. Continuing in this process, we have $ma \in M$ and so $a = ma + ba^2 \in M$, which is a contradiction. This shows that $a \in J(R)$ and therefore $R/J(R)$ is reduced. Since reduced rings are always directly finite, $R/J(R)$ is directly finite. Consequently, R is directly finite. □

THEOREM 2.8. *The following conditions are equivalent for a ring R.*

(1) *R is left DS.*

(2) *R is left mininjective and for every minimal right ideal A and minimal left ideal B, we have $A \cap B = AB$.*

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(3) *R is left minsymmetric and for every minimal right ideal A and minimal left ideal B, we have $A \cap B = AB$.*

(4) *$S_l \subseteq S_r$ and for every minimal right ideal A and minimal left ideal B, we have $A \cap B = AB$.*

(5) *R is left PS and $S_l(R) \subseteq S_r(R)$.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (1) $\Longrightarrow$ (5) always hold.

(1) $\Rightarrow$ (2) Let A be a minimal right ideal and B a minimal left ideal of R. If $A \cap B = 0$, then $A \cap B = AB$ always holds. If $A \cap B \neq 0$, then for every $0 \neq k \in A \cap B$, we have $B = Rk$. Since R is a left DS ring, $Rk = RkRk$. Let $k = kbk$ for some $b \in R$. Then $k \in AB$, because $kb \in A$ and $k \in B$. Hence $A \cap B \subseteq AB$.

(4) $\Rightarrow$ (1) Let Rk be a minimal left ideal of R. Then, by (4), $k \in S_l \subseteq S_r$, hence there exists a minimal right ideal cR of R such that $cR \subseteq kR$. Let $c = kb$ for some $b \in R$. Since $Rk \cong Rkb$ as left R-module, Rc is a minimal left ideal of R. Hence $c \in cR \cap Rc = cRc$ by hypothesis, so we have $Rc = RcRc$. Therefore, we have $cR = cRcR$. Since $0 \neq cR = cRcR \subseteq kRkR$, $RkRk \neq 0$, so $Rk = RkRk$.

(5) $\Longrightarrow$ (1) Assume that $k \in M_l(R)$. If $(Rk)^2 = 0$, then $RkR \subseteq l(k)$. Let L be the complement right ideal of RkR in R. Then $S_r(R) \subseteq RkR \oplus L$. Clearly $L \subseteq l(k)$, so, we have $S_l(R) \subseteq S_r(R) \subseteq RkR \oplus L \subseteq l(k)$. Since R is left PS, $l(k) = Re$, $e^2 = e \in R$. Evidently, $1 - e \in ME_l(R)$, so, we have $1 - e \in S_l(R) \subseteq l(k) = Re$, which is a contradiction. Therefore, we have $(Rk)^2 \neq 0$, and so R is left DS. $\square$

THEOREM 2.9. *The following conditions are equivalent for a ring R.*

(1) *$k \in M_l(R)$ implies $k^2 \neq 0$.*

(2) *R is left DS and left min-abel.*

(3) *For every $k \in M_l(R)$, there exists a central left minimal idempotent $e \in R$ such that $k = ke$.*

(4) *For every $k \in M_l(R)$, there exists a $e \in ME_l(R)$ such that $k = ke = ek$.*

(5) *R is left DS and for any $k, l \in M_l(R)$, $RkRl = RlRk$.*

(6) *R is left DS and for any $k, l \in M_l(R)$, $kl = 0$ always implies $lk = 0$.*

(7) *$N(R) \cap S_l(R) = 0$.*

(8) *For any $k, l \in M_l(R)$, $Rk \cap Rl = Rkl$.*

Proof. (1) $\Longrightarrow$ (2) Assume that $k \in M_l(R)$. By (1), $k^2 \neq 0$, so, we have $(Rk)^2 \neq 0$, which implies that R is a left DS. Now let $e \in ME_l(R)$. We claim that e is left semi-central. Otherwise there exists a $b \in R$ such that $be \neq ebe$. Let $h = be - ebe$, then $h \in M_l(R)$, $he = h$ and $eh = 0$. Hence $h^2 = hh = heh = 0$, which is a contradiction by (1). This shows that R is left min-abel.

(2) $\Longrightarrow$ (3) Let $k \in M_l(R)$. By (2), $Rk = Re$ where $e^2 = e \in R$ is left semi-central. Hence $k = ke$. We claim that e is right semi-central. Otherwise $t = ec - ece \neq 0$ for some $c \in R$. Clearly, $et = t \in M_l(R)$, $te = 0$, $t^2 = 0$, so we have $Rt = Rg$, $g^2 = g \in R$ by (2). Since g is left semi-central, $t = tg = gtg \in Rt^2g = 0$, which is a contradiction. Therefore, e is right semi-central and so e is central.

(3) $\Longrightarrow$ (4) is trivial.

(4) $\Longrightarrow$ (5) Evidently, R is left DS. Now let $k, l \in M_l(R)$. If $RkRl = 0$, then $RlRk = 0$. Otherwise $RlRk = Rk$. Since R is left DS, $Rk = RkRk = RkRlRk = 0$, which is a contradiction. Hence $RlRk = 0 = RkRl$. If $RkRl \neq 0$, then the proof above implies that $RlRk \neq 0$. By hypothesis, $k = ek = ke$ for some $e \in ME_l(R)$. We claim that e is right semi-central. In fact, if there exists a $c \in R$ such that $h = ec - ece \neq 0$, then $eh = h \in M_l(R)$, $he = 0$, $h^2 = 0$, so, there exists $g \in ME_l(R)$ such that $h = hg = gh$

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by hypothesis. Therefore, $Rh = Rhg = Rg$, Write $g = dh, d \in R$, then $h = gh = dhh = dh^2 = 0$, which is a contradiction. Hence e is right semi-central. Consequently, $RkRl = ReRl = ReRle = Re = Rk = RlRk$, we are done.

(5) $\Longrightarrow$ (6) Assume that $k, l \in M_l(R)$ with $kl = 0$, but $lk \neq 0$. Then $Rlk = Rk = Re, e^2 = e \in R$. Let $c \in R$ and $h = ec - ece$, then $eh = h, he = 0$. If $h \neq 0$, then $Rh = Reh = ReRh = RhRe = Re$ by hypothesis. Thus $h = he = 0$, which is a contradiction. This implies that e is right semi-central. Clearly, $Re = Rk = Rlk = RlRk = RkRl = Rl$. Write $e = ck, c \in R$ and $e = dl, d \in R$. Then $el = ckl = 0$ and $e = ee = edl = edel = 0$, which is a contradiction. Hence $lk = 0$.

(6) $\Longrightarrow$ (7) If there exists a left minimal element $k \in N(R)$, then $k^2 = 0$. By (6), $Rk = Re, e^2 = e \in R$ and so $k = ke \neq 0$. By (6), $ek \neq 0$. Write $e = ck, c \in R$, then $ek = ckk = ck^2 = 0$, which is a contradiction. Hence $N(R) \cap S_l(R) = 0$.

(7) $\Longrightarrow$ (8) First, we note that every left minimal idempotent e of R is central (in fact, for any $c, d \in R$, $h = ec - ece$, $t = de - ede$ are contained in $N(R)$. If $h \neq 0$ and $t \neq 0$, then $h, t \in S_l(R)$. This is impossible because $N(R) \cap S_l(R) = 0$). Next, We observe that R is left DS (in fact, if $k \in M_l(R)$, then $k^2 \neq 0$ by (7), hence $(Rk)^2 \neq 0$). Hence every minimal left ideal of R is an ideal. Finally, we assume that $k, l \in M_l(R)$. If $Rk \cap Rl = 0$, then $Rkl \subseteq Rl \cap Rk = 0$ and so $Rk \cap Rl = 0 = Rkl$. If $Rk \cap Rl \neq 0$, then $Rk = Rk \cap Rl = Rl = Re$, where $e \in R$ is a central idempotent element. Hence $Rkl = Rel = Rle = Re = Rl = Rk \cap Rl$.

(8) $\Longrightarrow$ (1) Assume that $k \in M_l(R)$, then $Rk = Rk \cap Rk = Rkk = Rk^2$ by hypothesis. Hence $k^2 \neq 0$. □

A ring with the properties of the Theorem 2.9 is denoted in the literature as a strongly left DS ring. Clearly, strongly left DS ring is left DS left min-abel and every left minimal idempotent is central. So, reduced rings and strongly regular rings are all strongly left DS.

It is well known that R is strongly regular if and only if R is von Neumann regular left quasi-duo and there exists a $MELT$ von Neumann regular ring which is not strongly regular, so there exists a $MELT$ von Neumann regular ring which is not left min-abel by Corollary 2.5. Therefore there exists a $MELT$ left DS ring which is not left min-abel. Consequently, there exists a $MELT$ left DS ring which is not strongly left DS by Theorem 2.9. Hence there exists a left DS ring which is not strongly left DS.

3. Left $SGP - V$-rings and strongly regular rings.

THEOREM 3.1. *If R be a left $SGP - V$-ring, then the following conditions are equivalent.*

(1) $J(R) = 0$.
(2) *R is semi-prime.*
(3) *R is left DS.*
(4) *R is left mininjective.*
(5) *R is left minsymmetric.*
(6) *R is left $MC2$.*
(7) $S_l(R) \subseteq S_r(R)$.
(8) *R is left idempotent reflexive.*
(9) *R is reflexive.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6), (2) $\Longrightarrow$ (9) $\Longrightarrow$ (8) $\Longrightarrow$ (6) and (5) $\Longrightarrow$ (7) are trivial.

(7) $\Longrightarrow$ (6) Let $e \in ME_l(R)$ and $a \in R$ such that $aRe = 0$. If $eRa \neq 0$, then there exists a $b \in R$ such that $eba \neq 0$. Clearly, $RebaR \subseteq l(eba) = l(e)$. Let L be the complement right ideal of $RebaR$ in R. Then, evidently, $L \subseteq l(eba) = l(e)$. Since $Reba \oplus L$ is an essential right ideal of R, $e \in S_l(R) \subseteq S_r(R) \subseteq RebaR \oplus L \subseteq l(eba) = l(e) = R(1-e)$, which is a contradiction. Hence $eRa = 0$. By Theorem 2.1, R is left $MC2$.

(6) $\Longrightarrow$ (1) If $J(R) \neq 0$, then there exists $0 \neq a \in J(R)$. We claim that $RaR + \cup_{i=1}^{\infty} l(a^i) = R$. Otherwise, there exists a maximal left ideal M of R containing $RaR + \cup_{i=1}^{\infty} l(a^i)$. If M is not essential in ${}_RR$, then $M = l(e)$ for some $e \in ME_l(R)$. So, we have $aRe = 0$. Since R is left $MC2$, $eRa = 0$, and then $e \in l(a) \subseteq M = l(e)$, which is a contradiction. This shows that M is essential in ${}_RR$ and so R/M is simple singular left R-module. Since R is a left $SGP-V$-ring, R/M is $YJ-$injective, hence there exists a positive integer n such that $a^n \neq 0$, and any left R-homomorphism from Ra^n to R/M can be extended to an R-homomorphism from R to R/M. Now we define a map $f : Ra^n \longrightarrow R/M$ by $f(ra^n) = r + M$ for any $r \in R$. Clearly, f is well defined. Thus, there exists $b \in R$ such that $1 - a^n b \in M$ and so $1 \in M$ because $a^n b \in RaR \subseteq M$, which is a contradiction. Therefore we have $RaR + \cup_{i=1}^{\infty} l(a^i) = R$. Since $a \in J(R)$, $RaR \subseteq J(R)$, so, we have $\cup_{i=1}^{\infty} l(a^i) = R$. This implies a is nilpotent. We can assume that $a^2 = 0$. By the proof above, we have $l(a) = R$, which also is a contradiction because $a \neq 0$. Therefore $J(R) = 0$. □

COROLLARY 3.2. *Let R be a left $GP-V$-ring. Then the following conditions are equivalent.*

(1) *R is strongly left min-abel.*
(2) *R is left min-abel.*
(3) *R is strongly left DS.*

Proof. (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2) are trivial.

(2) $\Longrightarrow$ (3) Since R is a left $GP-V-$ring, $J(R) = 0$, hence R is left DS. By Theorem 2.9, R is strongly left DS because R is left min-abel. □

COROLLARY 3.3. *Let R be a left $SGP-V$-ring. Then the following conditions are equivalent.*

(1) *R is strongly left DS.*
(2) *R is left min-abel and $J(R) = 0$.*
(3) *R is left min-abel and semi-prime.*

A ring R is called NI if $N(R)$ forms an ideal of R, and R is said to be 2-*primal* if $N(R) = P(R)$. Clearly, 2-*primal* rings are NI.

COROLLARY 3.4. *Let R be a left $SGP-V$-ring. Then the following conditions are equivalent.*

(1) *R is reduced.*
(2) *R is left $MC2$ and $N(R)$ forms left ideal of R.*
(3) *R is left $MC2$ and $N(R)$ forms right ideal of R.*
(4) *R is NI and left $MC2$.*

Proof. (1) $\Longrightarrow$ (4) $\Longrightarrow$ (i), $i = 2, 3$ are trivial.

(2) $\Longrightarrow$ (1) Since R is a left $SGP-V$-ring and a left $MC2$ ring, by Theorem 3.1, $J(R) = 0$. Since $N(R)$ forms a left ideal of R, $N(R) \subseteq J(R)$. So, we have $N(R) = 0$.

Similarly, we have (3) $\Longrightarrow$ (1). □

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THEOREM 3.5. *The following conditions are equivalent for a ring R.*
(1) *R is strongly regular.*
(2) *R is strongly left min-abel, $WMELT$ and a left $SGP-V$-ring.*
(3) *R is left min-abel, $WMELT$ and a left $GP-V$-ring.*
(4) *R is left $MC2$, left WQD and a left $SGP-V$-ring.*

Proof. (1) $\Longrightarrow$ (i), $i=3,4$ are trivial.
(4) $\Longrightarrow$ (2) is an immediate consequence of Theorem 2.3 and 2.7.
(3) $\Longrightarrow$ (2) follows from Corollary 3.2.
(2) $\Longrightarrow$ (1) First, we show that R is reduced. If it is not the case, then there exists $0\neq a\in R$ such that $a^2=0$. Hence $l(a)$ is contained in a maximal left ideal M of R. If M is not an essential left ideal of R, then $M=l(e)$ for some $e\in ME_l(R)$. Since R is strongly left min-abel, e is central. So, we have $ea=ae=0$ because $a\in l(a)\subseteq M=l(e)$. Therefore $e\in l(a)\subseteq l(e)$, which is a contradiction. Hence M is essential and then R/M is a simple singular left R-module. Since R is a left $SGP-V$-ring, R/M is YJ-injective, hence the left R-homomorphism $f:Ra\longrightarrow R/M$ defined by $f(ra)=r+M$ can be extended to one from R to R/M. In other words, there exists a $b\in R$ such that $1-ab\in M$, so, we have $b-bab=b(1-ab)\in M$. Since R is $WMELT$ and $ba\in M$, there exists a positive integer n such that $(ba)^nb\in M$. Hence $(ba)^{n-1}b=(ba)^{n-1}(b-bab)+(ba)^nb\in M$. Continuing this process, we have $bab\in M$. Thus $b=b-bab+bab\in M$ and $ab\in M$. Therefore, $1\in M$, which is a contradiction. This proves that R is reduced.

Now we prove that R is strongly regular. If not, there exists $0\neq a\in R$ such that $Ra+l(a)\subseteq M$ for some maximal left ideal M of R. Clearly, M is an essential left ideal of R. Thus R/M is $YJ-$injective, hence there exists a positive integer n such that $a^n\neq 0$, and any left R-homomorphism from Ra^n to R/M can be extended to an R-homomorphism from R to R/M. Now we define a map $f:Ra^n\longrightarrow R/M$ by $f(ra^n)=r+M$ for any $r\in R$. Since R is reduced, f is well defined. Thus, there exists $b\in R$ such that $1-a^nb\in M$. Since R is $WMELT$, M is GW-ideal. As the proof in the first part, we obtain $ba^nb\in M$. Furthermore, $b=b(1-a^nb)+ba^nb\in M$, and then $a^nb\in M$, whence $1\in M$. This contradiction shows that R is strongly regular. □

In [**14**], Rege proved that R is strongly regular if and only if R is left weakly regular left quasi-duo. We can generalize the result as follows.

THEOREM 3.6. *The following conditions are equivalent for a ring R.*
(1) *R is strongly regular.*
(2) *R is left weakly regular and left WQD.*
(3) *R is left weakly regular and right WQD.*
(4) *R is left weakly regular and right quasi-duo.*

Proof. (1) $\Longrightarrow$ (4) $\Longrightarrow$ (3) and (1) $\Longrightarrow$ (2) are trivial.
(2) $\Longrightarrow$ (1) If there exists $a\in R$ such that $Ra+l(a)\neq R$, then there exists a maximal left ideal M containing $Ra+l(a)$. Since R is left WQD and $a\in R$, there exists a positive integer n such that $a^nR\subseteq M$. Since R is left weakly regular, $a^n=xa^n$ for some $x\in Ra^nR$. Therefore $x\in M$. Since R is left WQD, $R/J(R)$ is reduced by Theorem 2.7. But R is left weakly regular, $J(R)=0$, so, we have that R is reduced. Hence $1-x\in l(a^n)=l(a)\subseteq M$ and $1\in M$, which is a contradiction. This shows that R is strongly regular.
(3) $\Longrightarrow$ (1) Similar to the proof of (2) $\Longrightarrow$ (1), we have that R is reduced. For any $a\in R$, we claim that $aR+r(a)=R$. Otherwise there exists $0\neq b\in R$ satisfying

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$bR + r(b) \subseteq K$ for some maximal right ideal K of R. So, by hypothesis, there exists a positive integer m such that $Rb^m \in K$. Since $b^m = yb^m$ for some $y \in Rb^m R \subseteq K$, $1 - y \in l(b^m) = l(b) = r(b) \subseteq K$, we have $1 \in K$, which is a contradiction. Therefore, R is strongly regular. □

THEOREM 3.7. *The following conditions are equivalent for a ring R.*
(1) *R is strongly regular.*
(2) *R is strongly left min-abel, $WMERT$ and left $SGP - V$-ring.*
(3) *R is left $SGP - V$-ring, left $MC2$ and right WQD.*
(4) *R is left weakly regular, left min-abel and $WMERT$.*

Proof. (1) $\Longrightarrow$ (2) evidently.

(2) $\Longrightarrow$ (3) By Theorem 2.3, R is left min-abel left $MC2$. Now let $e \in ME_r(R)$. Since R is a left $MC2$ left $SGP - V$-ring, R is semi-prime by Theorem 3.1, hence R is right $MC2$. Consequently, $e \in ME_l(R)$, and so e is central in R because R is strongly left min-abel. This proves R is right min-abel. By Theorem 2.7, R is right WQD because R is $WMERT$.

(3) $\Longrightarrow$ (4) First, we show that R is reduced. Since R is left $MC2$ left $SGP - V$-ring, $J(R) = 0$ by Theorem 3.1. Since R is right WQD, R is $WMERT$ and $R/J(R)$ is reduced by Theorem 2.7. Therefore R is reduced and so R is left min-abel. Next, we claim that R is left weakly regular. Otherwise, there exists $0 \neq a \in R$ such that $RaR + l(a) \subseteq M$. Clearly, M is essential in $_RR$, hence R/M is YJ-injective by hypothesis. It is easy to show that there exists a positive integer n and $b \in R$ such that $1 - a^n b \in M$. This implies $1 \in M$ because $a^n b \in M$. This contradiction shows that R is left weakly regular.

(4) $\Longrightarrow$ (1) Since R is left weakly regular, R is semi-prime, hence R is right $MC2$. Now let $e \in ME_r(R)$. By Theorem 2.1, $e \in ME_l(R)$. Since R is left min-abel, e is left semi-central. By Theorem 2.3, R is strongly right min-abel. By Theorem 2.7, R is right WQD. By Theorem 3.6, R is strongly regular. □

4. Injective maximal left ideals. In general, the existence of an injective maximal left ideal in a ring R can not guarantee the left self-injectivity of R. [**4**, Proposition 5] proves that if R is idempotent reflexive and R contains an injective maximal left ideal, then R is left self-injective. We can generalize the result as follows.

THEOREM 4.1. *Let R be left $MC2$. If R contains an injective maximal left ideal, then R is left self-injective.*

Proof. Let M be an injective maximal left ideal of R. Then $R = M \oplus N$ for some minimal left ideal N of R. Hence we have $M = Re$ and $N = R(1 - e)$ for some $e^2 = e \in R$. If $MN = 0$, then we have $eR(1 - e) = 0$. Since R is left $MC2$ and $1 - e \in ME_l(R)$, $(1 - e)Re = 0$. So e is central. Now let L be any proper essential left ideal of R and $f : L \longrightarrow N$ any non-zero left $R-$homomorphism. Then $L/U \cong N$, where $U = kerf$ is a maximal sub-module of L. Now $L = U \oplus V$, where $V \cong N = R(1 - e)$ is a minimal left ideal of R. Since e is central, $V = R(1 - e)$. For any $z \in L$, let $z = x + y$, where $x \in U, y \in V$. Then $f(z) = f(x) + f(y) = f(y)$. Since $y = y(1 - e) = (1 - e)y$, $f(z) = f(y) = f(y(1 - e)) = yf(1 - e)$. Since $x(1 - e) = (1 - e)x \in V \cap U = 0$, $xf(1 - e) = f(x(1 - e) = f(0) = 0$. Thus $f(z) = yf(1 - e) = y(f(1 - e) + xf(1 - e) = (y + x)f(1 - e) = zf(1 - e)$. Hence $_RN$ is injective. If $MN \neq 0$, by the proof of [**4**, Proposition 5], we have $_RN$ is injective. Hence $R = M \oplus N$ is left self-injective. □

Since strongly left $DS \Longrightarrow$ left $DS \Longrightarrow$ left mininjective $\Longrightarrow$left minsymmetric $\Longrightarrow$ left $MC2$ and strongly left min-abel $\Longrightarrow$left $MC2$, we have the following corollary.

COROLLARY 4.2. *Let R contain an injective maximal left ideal. If R satisfies one of the following conditions, then R is left self-injective.*
(1) *R is strongly left DS.*
(2) *R is left DS.*
(3) *R is left mininjective.*
(4) *R is left minsymmetric.*
(5) *R is strongly min-abel.*

It is well known that if R is left self-injective, then $J(R) = Z_l(R)$. Therefore by [**16**, Theorem 5.1] and Theorem 4.1, we have the following corollary.

COROLLARY 4.3. *Let R contain an injective maximal left ideal. Then R is left self-injective if and only if* $J(R) = Z_l(R)$.

A ring R is right *Kasch* if every simple right R-module can be embedded in R_R and R is said to be left $C2$ [**12**] if every left ideal that is isomorphic to a direct summand of ${}_RR$ is itself a direct summand. Clearly, left $C2$ rings are left $MC2$ and by [**19**, Lemma 1.15], right *Kasch* ring are left $C2$. Hence, we have the following corollary.

COROLLARY 4.4. *Let R contain an injective maximal left ideal. If R satisfies one of the following conditions, then R is left self-injective.*
(1) *R is right Kasch.*
(2) *R is left C2.*

Recall that a ring R is left *pp* if every principal left ideal of R is projective. As an application of Theorem 4.1, we have the following result.

THEOREM 4.5. *The following conditions are equivalent for a ring R.*
(1) *R is a von Neumann regular left self-injective ring with* $S_l(R) \neq 0$.
(2) *R is a left* $MC2$ *left pp ring containing an injective maximal left ideal.*
(3) *R is a left minsymmetric left pp ring containing an injective maximal left ideal.*
(4) *R is a left mininjective left pp ring containing an injective maximal left ideal.*
(5) *R is a left DS left pp ring containing an injective maximal left ideal.*

Proof. (1) $\Longrightarrow$ (5) $\Longrightarrow$ (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) are trivial.
(2) $\Longrightarrow$ (1) By Theorem 4.1, R is left self-injective. Hence, by [**10**, Theorem 1.2], R is left $C2$, so, R is von Neumann regular because R is left *pp*. Also we have $S_l(R) \neq 0$ since there is an injective maximal left ideal. □

By [**21**], a ring R is said to be left HI if R is left hereditary containing an injective maximal left ideal. Osofsky [**13**] proves that left self-injective left hereditary ring is semi-simple Artinian. We can generalize the result as follows.

COROLLARY 4.6. *The following conditions are equivalent for a ring R.*
(1) *R is semi-simple Artinian.*
(2) *R is left* $MC2$ *left* HI.
(3) *R is left minsymmetric left* HI.
(4) *R is left mininjective left* HI.
(5) *R is left DS left* HI.

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