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ON UCHIMURA'S CONNECTION BETWEEN PARTITIONS AND THE NUMBER OF DIVISORS

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ABSTRACT. A combinatorial proof is given for Uchimura's identity

$$\sum_{n\geq 1}\frac{x^n}{1-x^n}=\sum_{n>1} nx^n \prod_{j\geq n+1}(1-x^j).$$

As a corollary to this proof we derive a formula for the sum of the nth powers of the divisors of m in terms of partitions of m.

Uchimura has proved [1] that

$$\sum_{n\geq 1}\frac{x^n}{1-x^n}=\sum_{n\geq 1} nx^n \prod_{j\geq n+1}(1-x^j). \tag{1}$$

If both sides are expanded as power series in x and the coefficients of x^m are compared, equation (1) is seen to be equivalent to

$$d(m)=-\sum_{\pi\vdash m}{}' (-1)^{\#(\pi)}\lambda(\pi) \tag{2}$$

where $d(m)$ is the number of divisors of m, $\pi\vdash m$ means that π is a partition of m, the prime on the summation restricts the sum to those partitions which have distinct parts, $\#(\pi)$ is the number of parts in π and $\lambda(\pi)$ is the smallest part in π. The purpose of this paper is to give a direct combinatorial proof of equation (2). As a corollary to this proof, we shall derive the more general identity

$$\sigma_n(m)=-\sum_{\pi\vdash m}{}' (-1)^{\#(\pi)}\sum_{j=1}^{\lambda(\pi)}(L(\pi)-\lambda(\pi)+j)^n \tag{3}$$

where $\sigma_n(m)$ is the sum of the nth powers of the divisors of m and $L(\pi)$ is the largest part in π.

For each positive integer N, let $C(N)$ denote the set of partitions π into

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distinct parts satisfying the following inequalities:

$$L(\pi) \geq N > L(\pi) - \lambda(\pi).$$

Thus, for example, $C(2) = \{(2), (3), (4), \ldots, (1+2), (2+3), (3+4), \ldots\}$.

For each partition π into distinct parts, there are exactly $\lambda(\pi)$ integers N such that $\pi \in C(N)$, namely $N = L(\pi) - \lambda(\pi) + j$, $1 \leq j \leq \lambda(\pi)$. We thus have that

$$(4) \qquad -\sum_{\pi \vdash m}{}' (-1)^{\#(\pi)} \lambda(\pi) = -\sum_{N} \sum_{\substack{\pi \vdash m \\ \pi \in C(N)}} (-1)^{\#(\pi)}.$$

To prove equation (2), it is therefore sufficient to show that

$$(5) \qquad -\sum_{\substack{\pi \vdash m \\ \pi \in C(N)}} (-1)^{\#(\pi)} = \begin{cases} 1, & \text{if } N \mid m \\ 0, & \text{otherwise.} \end{cases}$$

We shall prove this equation by exhibiting an algorithm which pairs partitions of m in $C(N)$ which have oppositive parity in the number of parts. If $N \nmid m$, then all of the partitions will be paired. If $N \mid m$, then the only partition which will remain unpaired is the partition consisting of a single part divisible by N.

If π contains a part which is a multiple of N and if π has at least one other part, then we remove the multiple of N and add N to the smallest remaining part. We continue to create new partitions by adding N to the smallest part in the previous partition until we again have a partition of m. As an example, if $N = 7$ then the parition $11+13+14+16$ is sucessively transformed into $11+13+16$, $13+16+18$, $16+18+20$.

If π does not contain a part which is a multiple of N, we reverse the procedure given above, which is to say that we subtract N from the largest part in the previous partition until we reach that unique partition for which the total amount subtracted is less than the smallest part plus N and more than the largest part minus N. This total amount subtracted is then inserted as a new part. As an example, if $N = 7$ then the partition $12+13+16+18$ is successively transformed into $11+12+13+16$ (7 subtracted), $9+11+12+13$ (14 subtracted), $9+11+12+13+14$.

This concludes the proof of equation (5). Equation (3) is a simple corollary of (5), for we have that

$$\sigma_n(m) = -\sum_N N^n \sum_{\substack{\pi \vdash m \\ \pi \in C(N)}} (-1)^{\#(\pi)}$$

$$= -\sum_{\pi \vdash m}{}' (-1)^{\#(\pi)} \sum N^n,$$

the inner sum being over all N such that $\pi \in C(N)$. As was shown above, these are given by $N = L(\pi) - \lambda(\pi) + j$, $1 \leq j \leq \lambda(\pi)$.

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If we let $D(N)$ denote the set of partitions π satisfying

$$L(\pi) \geq N \geq L(\pi) - \lambda(\pi)$$

and not necessarily having distinct parts but with at most one part which is a multiple of N, then the same algorithm as before pairs partitions in $D(N)$ with opposite parity and leaves unpaired only the partition consisting of a single part which is a multiple of N, if it exists. We thus also have

$$-\sum_{\substack{\pi \vdash m \\ \pi \in D(N)}} (-1)^{\#(\pi)} = \begin{cases} 1, & \text{if } N \mid m \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Summing over N as before yields

$$\sigma_n(m) = -\sum_{\pi \vdash m} (-1)^{\#(\pi)} \sum N^n, \tag{7}$$

where the inner sum is over all N such that $\pi \in D(N)$: that is to day,

$$\sum N^n = \sum_{i=1}^{\lambda(i)} \delta^n(L(\pi) - l(\pi) + i, \pi)$$

where

$$\delta(a, \pi) = \begin{cases} a, & \text{if } a \text{ divides at most one part of } \pi \\ 0, & \text{otherwise.} \end{cases}$$

It is worth noting that Uchimura [2] has generalized equation (1) in a different direction, namely that

$$\sum_{n \geq 1} n^r x^n \prod_{j \geq n+1} (1 - x^j) = Y_r(K_1, \ldots, K_r), \qquad r \geq 1, \tag{8}$$

where Y_r is the rth Bell polynomial

$$Y_r(K_1, \ldots, K_r) = \sum_{\pi \vdash r} \frac{r!}{f_1! \, f_2! \cdots f_r!} \left(\frac{K_1}{1!}\right)^{f_1} \cdots \left(\frac{K_r}{r!}\right)^{f_r},$$

f_i being the frequency of the part i in the partition π, and K_{j+1} being the generating function for the sum of the jth powers of the divisors.

$$K_{j+1} = K_{j+1}(x) = \sum_{n \geq 1} \sigma_j(n) x^n.$$

References

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