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# INTEGRAL MEANS OF HOLOMORPHIC FUNCTIONS AS GENERIC LOG-CONVEX WEIGHTS

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**Abstract**

Let $\mathcal{H}ol(B_d)$ denote the space of holomorphic functions on the unit ball $B_d$ of $\mathbb{C}^d$, $d \ge 1$. Given a log-convex strictly positive weight $w(r)$ on $[0, 1)$, we construct a function $f \in \mathcal{H}ol(B_d)$ such that the standard integral means $M_p(f, r)$ and $w(r)$ are equivalent for any $p$ with $0 < p \le \infty$. We also obtain similar results related to volume integral means.

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## 1. Introduction

Let $\mathcal{H}ol(B_d)$ denote the space of holomorphic functions on the unit ball $B_d$ of $\mathbb{C}^d$, $d \ge 1$. For $0 < p < \infty$ and $f \in \mathcal{H}ol(B_d)$, the standard integral means $M_p(f, r)$ are

$$M_p(f, r) = \left( \int_{\partial B_d} |f(r\zeta)|^p \, d\sigma_d(\zeta) \right)^{1/p}, \quad 0 \le r < 1,$$

where $\sigma_d$ denotes the normalised Lebesgue measure on the unit sphere $\partial B_d$. For $p = \infty$,

$$M_\infty(f, r) = \sup\{|f(z)| : |z| = r\}, \quad 0 \le r < 1.$$

A function $w : [0, 1) \to (0, +\infty)$ is called a weight if $w$ is continuous and nondecreasing. A weight $w$ is said to be *log-convex* if $\log w(r)$ is a convex function of $\log r$ for $0 < r < 1$. It is known that $M_p(f, r)$ is a log-convex weight for any $f \in \mathcal{H}ol(B_d)$ with $f(0) \ne 0$, $d \ge 1$ and $0 < p \le \infty$. In fact, for $d = 1$, this result constitutes the classical Hardy convexity theorem (see [2]). The proof can be extended to all dimensions $d \ge 2$ (see for example [7, Lemma 1]).

Let $u, v : X \to (0, +\infty)$. We say that $u$ and $v$ are equivalent ($u \asymp v$, in brief) if there exist constants $C_1, C_2 > 0$ such that

$$C_1 u(x) \le v(x) \le C_2 u(x), \quad x \in X.$$

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In the present paper, for each $p$ with $0 < p \le \infty$, we show that the functions $M_p(f, r)$ with $f \in \mathcal{H}ol(B_d)$ and $f(0) \neq 0$ are generic log-convex weights up to equivalence.

THEOREM 1.1. *Let $d \ge 1$ and let $w : [0, 1) \to (0, +\infty)$ be a log-convex weight. There exists $f \in \mathcal{H}ol(B_d)$ such that*

$$M_p(f, r) \asymp w(r), \quad 0 \le r < 1,$$

*for each $p$ with $0 < p \le \infty$.*

We also consider volume integral means for $0 < q < \infty$. The logarithmic convexity properties for such integral means have been investigated recently (see, for example, [5–7]). Applying Theorem 1.1, we obtain, in particular, the following result.

COROLLARY 1.2. *Let $d \ge 1$ and $0 < q < \infty$ and let $w : [0, 1) \to (0, +\infty)$ be a weight. The following properties are equivalent:*

(i) *$w(r)$ is equivalent to a log-convex weight on $[0, 1)$;*
(ii) *there exists $f \in \mathcal{H}ol(B_d)$ such that*

$$\left(\frac{1}{\nu_d(rB_d)} \int_{rB_d} |f(z)|^q \, d\nu_d(z)\right)^{1/q} \asymp w(r), \quad 0 < r < 1,$$

*where $\nu_d$ denotes the normalised volume measure on $B_d$.*

Section 2 is devoted to the proof of Theorem 1.1. Corollary 1.2 and other results related to volume integral means are obtained in Section 3.

## 2. Proof of Theorem 1.1

Let $\mathbb{D} = B_1$ and $\mathbb{T} = \partial\mathbb{D}$. For a log-convex weight $w$ on $[0, 1)$, [1, Theorem 1.2] provides functions $f_1, f_2 \in \mathcal{H}ol(\mathbb{D})$ such that $|f_1(z)| + |f_2(z)| \asymp w(|z|)$ for $z \in \mathbb{D}$. These functions are almost sufficient for a proof of Theorem 1.1 with $d = 1$. However, we will need additional technical information contained in [1]. Namely, applying [1, Lemma 2.2] and arguing as in the proof of [1, Theorem 1.2], we obtain the following lemma.

LEMMA 2.1. *Let $w$ be a log-convex weight on $[0, 1)$. There exist $a_k > 0$, $n_k \in \mathbb{N}$, $k = 1, 2, \dots$, and constants $r_0 \in (\frac{9}{10}, 1)$, $C_1, C_2 > 0$ with the following properties:*

$$n_k < n_{k+1}, \quad k = 1, 2, \dots; \tag{2.1}$$

$$\sum_{k=1}^{\infty} a_k r^{n_k} \le C_1 w(r), \quad r_0 \le r < 1; \tag{2.2}$$

$$|g_1(r\zeta)| + |g_2(r\zeta)| \ge C_2 w(r), \quad r_0 \le r < 1, \ \zeta \in \mathbb{T}; \tag{2.3}$$

*where*

$$g_1(z) = \sum_{j=1}^{\infty} a_{2j-1} z^{n_{2j-1}}, \quad g_2(z) = \sum_{j=1}^{\infty} a_{2j} z^{n_{2j}}, \quad z \in \mathbb{D}.$$

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PROOF OF THEOREM 1.1. We are given a log-convex weight $w$ on $[0,1)$. First, assume that $d=1$. Let $a_k$ and $n_k$ ($k=1,2,\dots$), $g_1$ and $g_2$ be as provided by Lemma 2.1. By (2.3),

$$|g_1(r\zeta)|^2 + |g_2(r\zeta)|^2 \ge C_3 w^2(r), \quad r_0 \le r < 1,\ \zeta \in \mathbb{T}.$$

Using (2.1) and integrating the above inequality with respect to Lebesgue measure $\sigma_1$ on $\mathbb{T}$,

$$\sum_{k=1}^{\infty} a_k^2 r^{2n_k} \ge C_3 w^2(r), \quad r_0 \le r < 1.$$

Therefore,

$$1 + \sum_{k=1}^{\infty} a_k^2 r^{2n_k} \ge C_4 w^2(r), \quad 0 \le r < 1.$$

So, by (2.1),

$$M_2(f,r) \ge w(r), \quad 0 \le r < 1, \tag{2.4}$$

where

$$\sqrt{C_4} f(z) = 1 + \sum_{k=1}^{\infty} a_k z^{n_k}, \quad z \in \mathbb{D}.$$

Also, (2.2) guarantees that

$$|f(r\zeta)| \le C_0 w(r), \quad 0 \le r < 1,\ \zeta \in \mathbb{T}. \tag{2.5}$$

Hence, $M_2(f,r) \le M_\infty(f,r) \le Cw(r)$ for $0 \le r < 1$. Combining these estimates and (2.4), we conclude that $M_2(f,r) \asymp M_\infty(f,r) \asymp w(r)$. Thus, $M_p(f,r) \asymp w(r)$ for $0 \le r < 1$ and any $p$ with $2 \le p \le \infty$.

Also, we claim that $M_p(f,r) \asymp w(r)$ for any $p$ with $0 < p < 2$. Indeed, (2.4) and (2.5) guarantee that

$$\sigma_1 \left\{ \zeta \in \mathbb{T} : |f(r\zeta)| \ge \frac{w(r)}{2} \right\} \ge \frac{1}{2C_0^2}.$$

Therefore, $M_\infty(f,r) \ge M_p(f,r) \ge C_p w(r)$ for $0 \le r < 1$. This completes the proof of the theorem for $d=1$.

Now, assume that $d \ge 2$. Let $W_k$, $k=1,2,\dots$, be a Ryll–Wojtaszczyk sequence (see [3]). By definition, $W_k$ is a holomorphic homogeneous polynomial of degree $k$, $\|W_k\|_{L^\infty(\partial B_d)} = 1$ and $\|W_k\|_{L^2(\partial B_d)} \ge \delta$ for a constant $\delta > 0$ which does not depend on $k$. Let

$$F(z) = 1 + \sum_{k=1}^{\infty} a_k W_k(z), \quad z \in B_d.$$

Clearly, (2.2) guarantees that $|F(r\zeta)| \le Cw(r)$ for $0 \le r < 1$ and $\zeta \in \partial B_d$. Also, the polynomials $W_k$, $k=1,2,\dots$, are mutually orthogonal in $L^2(\partial B_d)$; hence, we have $M_2(F,r) \ge C(\delta) w(r)$ for $0 \le r < 1$. So, arguing as in the case $d=1$, we conclude that $M_p(F,r) \asymp w(r)$ for any $p$ with $0 < p \le \infty$, as required. □

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As indicated in the introduction, for any $f \in \mathcal{H}ol(B_d)$, the function $M_p(f,r)$ is log-convex; hence, Theorem 1.1 implies the following analogue of Corollary 1.2.

Corollary 2.2. *Let $d \geq 1$ and $0 < p \leq \infty$ and let $w : [0,1) \to (0,+\infty)$ be a weight. The following properties are equivalent:*

(i) *$w(r)$ is equivalent to a log-convex weight on $[0,1)$;*
(ii) *there exists $f \in \mathcal{H}ol(B_d)$ such that*

$$M_p(f,r) \asymp w(r), \quad 0 \leq r < 1.$$

## 3. Volume integral means

In this section, we consider integral means based on volume integrals. Recall that $\nu_d$ denotes the normalised volume measure on the unit ball $B_d$. For $f \in \mathcal{H}ol(B_d)$, $0 < q < \infty$ and a continuous function $u : [0,1) \to (0,+\infty)$, define

$$M_{q,u}(f,r) = \left(\frac{1}{r^{2d}} \int_{rB_d} |f(z)|^q u(|z|)\, d\nu_d(z)\right)^{1/q}, \quad 0 < r < 1;$$
$$M_{q,u}(f,0) = |f(0)| u^{1/q}(0).$$

Proposition 3.1. *Let $0 < q < \infty$ and let $u, w : [0,1) \to (0,+\infty)$ be log-convex weights. There exists $f \in \mathcal{H}ol(B_d)$ such that*

$$M_{q,1/u}(f,r) \asymp w(r), \quad 0 \leq r < 1.$$

Proof. By Theorem 1.1 with $p = 2$, there exist $a_k \geq 0$, $k = 0, 1, \ldots$, such that

$$w^q(t) \asymp \sum_{k=0}^{\infty} a_k t^k, \quad 0 \leq t < 1.$$

Let

$$\varphi^q(t) = \sum_{k=0}^{\infty} (k+2d) a_k t^k, \quad 0 \leq t < 1.$$

The functions $\varphi^q(t)$ and $\varphi(t)$ are correctly defined log-convex weights on $[0,1)$. Hence, $\varphi(t) u^{1/q}(t)$ is a log-convex weight, being the product of two log-convex weights. By Theorem 1.1, there exists $f \in \mathcal{H}ol(B_d)$ such that

$$\int_{\partial B_d} |f(t\zeta)|^q \, d\sigma_d(\zeta) \asymp \varphi^q(t) u(t), \quad 0 \leq t < 1,$$

or, equivalently,

$$\frac{t^{2d-1}}{u(t)} \int_{\partial B_d} |f(t\zeta)|^q \, d\sigma_d(\zeta) \asymp \sum_{k=0}^{\infty} (k+2d) a_k t^{k+2d-1}, \quad 0 \leq t < 1.$$

Representing $M^q_{q,1/u}(f,r)$ in polar coordinates and integrating the above estimates with respect to $t$,

$$\begin{aligned} M^q_{q,1/u}(f,r) &= \frac{2d}{r^{2d}} \int_0^r \int_{\partial B_d} |f(t\zeta)|^q \, d\sigma_d(\zeta) \, \frac{t^{2d-1}}{u(t)} \, dt \\ &\asymp \sum_{k=0}^{\infty} a_k r^k, \\ &\asymp w^q(r), \quad 0 \le r < 1, \end{aligned}$$

as required. □

Clearly, Proposition 3.1 is of special interest if $M_{q,1/u}(f,r)$ is log-convex or equivalent to a log-convex function for any $f \in \mathcal{H}ol(B_d)$. Also, we have to prove Corollary 1.2. So, assume that $u \equiv 1$ and define

$$\begin{aligned} \mathfrak{M}_q(f,r) &= \left( \frac{1}{\nu_d(rB_d)} \int_{rB_d} |f(z)|^q \, d\nu_d(z) \right)^{1/q}, \quad 0 < r < 1, \\ \mathfrak{M}_q(f,0) &= |f(0)|, \end{aligned}$$

where $0 < q < \infty$.

Proof of Corollary 1.2. By Proposition 3.1, (i) implies (ii). To prove the reverse implication, assume that $w(t)$ is a weight on $[0,1)$ and $w(r) \asymp \mathfrak{M}_q(f,r)$ for some $f \in \mathcal{H}ol(B_d)$ with $f(0) \neq 0$.

If $d = 1$ and $0 < q < \infty$, then $\mathfrak{M}_q(f,r)$ is log-convex by Theorem 1 from [5]. So, (ii) implies (i) for $d = 1$. The function $\mathfrak{M}_q(f,r)$ is also log-convex if $1 \le q < \infty$ and $d \ge 2$. Indeed,

$$\mathfrak{M}_q(f,r) = \left( \int_{B_d} |f(rz)|^q \, d\nu_d(z) \right)^{1/q}, \quad 0 \le r < 1.$$

Thus, Taylor's Banach space method applies (see [4, Theorem 3.3]).

Now, assume that $d \ge 2$ and $0 < q < 1$. The function $M^q_q(f,t)$ is a log-convex weight. Hence, by Theorem 1.1 with $p = 2$, there exist $a_k \ge 0$, $k = 0, 1, \ldots$, such that

$$M^q_q(f,t) \asymp \sum_{k=0}^{\infty} a_k t^k, \quad 0 \le t < 1.$$

Thus,

$$\begin{aligned} \mathfrak{M}^q_q(f,r) &= \frac{2d}{r^{2d}} \int_0^r M^q_q(f,t) t^{2d-1} \, dt \\ &\asymp \sum_{k=0}^{\infty} \frac{a_k}{k+2d} r^k, \quad 0 \le r < 1. \end{aligned}$$

In other words, $\mathfrak{M}_q(f,r)$ is equivalent to a log-convex weight on $[0,1)$. So, (ii) implies (i) for all $d \ge 1$ and $0 < q < \infty$. The proof of the corollary is finished. □

For $\alpha > 0$, Proposition 3.1 also applies to the integral means:

$$\frac{1}{r^{2d}} \int_{rB_d} |f(z)|^p (1 - |z|^2)^\alpha \, dv_d(z), \quad 0 \le r < 1.$$

However, in general, these integral means are not log-convex.

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