Can. J. Math., Vol. XXVI, No. 1, 1974, pp. 115–120

OPERATORS WITH COMPACT SELF-COMMUTATOR

CARL PEARCY AND NORBERTO SALINAS

1. Introduction. Let $\mathscr{H}$ be a fixed separable, infinite dimensional complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all (bounded, linear) operators on $\mathscr{H}$. The ideal of all compact operators on $\mathscr{H}$ will be denoted by $\mathscr{C}$ and the canonical quotient map from $\mathscr{L}(\mathscr{H})$ onto the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{C}$ will be denoted by π.

Some open problems in the theory of extensions of C^*-algebras (cf. [**1**]) have recently motivated an increasing interest in the class of all operators in $\mathscr{L}(\mathscr{H})$ whose self-commutator is compact. (Observe that the self-commutator of an operator $T \in \mathscr{L}(\mathscr{H})$ is compact if and only if $\pi(T)$ is normal in $\mathscr{L}(\mathscr{H})/\mathscr{C}$.) A general question of particular interest in this context is the following:

Under what conditions can an operator whose self-commutator is compact be expressed as the sum of a normal operator and a compact operator?

For a given operator T on $\mathscr{H}$, let $E(T)$ denote the essential spectrum of T, i.e., the spectrum of $\pi(T)$ in $\mathscr{L}(\mathscr{H})/\mathscr{C}$, or equivalently, $\{\lambda \in \mathbf{C}: T - \lambda$ is not Fredholm$\}$. Also, let $\Omega(T)$ be the Weyl spectrum of T, i.e.,

$$\Omega(T) = \{\lambda \in \mathbf{C}: T - \lambda \text{ is not a Fredholm operator of index zero}\}.$$

It appears that a reasonable conjecture in regard to the above question is the following:

($*$) *Let T be an operator on $\mathscr{H}$ whose self-commutator is compact. Then T can be written as the sum of a normal operator and a compact operator if and only if* $\Omega(T) = E(T)$.

In the present note we make some remarks concerning this conjecture. In particular, in Theorem 2.3 we prove that conjecture ($*$) has an affirmative answer if and only if the following statement holds: If T and S are any two operators on $\mathscr{H}$ such that $\pi(T)$ and $\pi(S)$ are normal elements in $\mathscr{L}(\mathscr{H})/\mathscr{C}$ and $\Omega(T) = E(T) = \Omega(S) = E(S)$, then $\pi(T)$ and $\pi(S)$ are unitarily equivalent in the C^*-algebra $\mathscr{L}(\mathscr{H})/\mathscr{C}$ (i.e., there exists a unitary W in $\mathscr{L}(\mathscr{H})/\mathscr{C}$ satisfying $W^*\pi(S)W = \pi(T)$). For the sake of brevity, we shall denote by (NC) the set of all operators that can be expressed as the sum of a normal operator and a compact operator. A good operator to examine to test the possible validity of the conjecture ($*$) is the operator given by the direct sum of a unilateral shift U of multiplicity one and a normal operator M whose spectrum is the closed unit disk. In § 3 we prove that $U \oplus M$ is in the uniform

Received August 15, 1972 and in revised form, December 11, 1972. This research was supported in part by the National Science Foundation.

closure of (NC). Since every operator in $\overline{(NC)}$ is quasidiagonal, the result just mentioned also proves that the direct sum of a non-quasitriangular operator and a diagonal operator can be quasidiagonal, and hence Problem 4 of [**5**] is answered in the negative. (Recall that an operator T on $\mathscr{H}$ is said to be quasitriangular if $\lim \inf_{P\in\mathscr{P}}\|TP - PTP\| = 0$, and T is called quasidiagonal if $\lim \inf_{P\in\mathscr{P}}\|TP - PT\| = 0$, where $\mathscr{P}$ is the directed set of all finite rank (orthogonal) projections in $\mathscr{L}(\mathscr{H})$ [**3**, Problem 4].). Let (QD) be the set of all quasidiagonal operators in $\mathscr{L}(\mathscr{H})$ whose self-commutator is compact, and let (QT) be the set of all operators $T \in \mathscr{L}(\mathscr{H})$ with compact self-commutator such that both T and T^* are quasitriangular. Furthermore, let

$$(QNC) = \{T \in \mathscr{L}(\mathscr{H}): T^*T - TT^* \in \mathscr{C}, \Omega(T) = E(T)\}.$$

It can easily be proved that the sets (QD), (QT) and (QNC) are uniformly closed in $\mathscr{L}(\mathscr{H})$. Thus the following inclusion formulas hold:

$$(NC) \subset \overline{(NC)} \subset (QD) \subset (QT) \subset (QNC).$$

The conjecture $(*)$ mentioned previously is equivalent to the statement $(NC) = (QNC)$. Therefore, the validity of the conjecture $(*)$ would answer in the affirmative the following interesting open question: Is the set (NC) uniformly closed in $\mathscr{L}(\mathscr{H})$? Not only is this question unanswered, but to our knowledge, no one has thus far succeeded in proving that any of the inclusions in the above chain of inequalities is actually an equality.

2. An equivalent formulation of conjecture $(*)$. In what follows the spectrum of an operator T will be denoted by $\Lambda(T)$ and the numerical range of T by $W(T)$. The following two theorems are central to our purposes.

THEOREM 2.1. *Let N be a normal operator on $\mathscr{H}$, and let $\mathscr{M}$ be a subspace of $\mathscr{H}$ of finite codimension. Then the compression of N to $\mathscr{M}$ can be expressed as the sum of a normal operator and a compact operator.*

Proof. We know from [**4**] that there exist a Hermitian operator $H \in \mathscr{L}(\mathscr{H})$ and a continuous function $f: \Lambda(H) \rightarrow \Lambda(N)$ such that $N = f(H)$. Let P be the projection onto $\mathscr{M}$ and let $H' = (PH)|\mathscr{M}$. Since $\mathscr{M}$ has finite codimension, it can be easily proved that for every polynomial p with complex coefficients the operator $[Pp(H)]|\mathscr{M} - p(H')$ is compact. Let $\tilde{f}$ be a continuous complex-valued function on $\overline{W(H)}$ such that $\tilde{f}$ coincides with f on $\Lambda(H)$. Since $\overline{W(H)}$ is a closed interval of real numbers, there exists a sequence of polynomials $\{p_n\}$ that converges uniformly to $\tilde{f}$ on $\overline{W(H)}$. Since $W(H') \subset W(H)$ (and hence $\Lambda(H') \subset \overline{W(H)}$), it follows from the spectral mapping theorem that $\{p_n(H)\}$ converges uniformly to N and $\{p_n(H')\}$ converges uniformly to a normal operator N' acting on $\mathscr{M}$ (recall that H' is Hermitian). Since $[Pp_n(H)]|\mathscr{M}$ converges uniformly to $(PN)|\mathscr{M}$ and $[Pp_n(H)]|\mathscr{M} - p_n(H')$ is a compact operator K_n for each $n = 1, 2, \ldots$, we conclude that the sequence $\{K_n\}$ converges uniformly to a compact operator K'. Therefore $(PN)|\mathscr{M} = N' + K'$ and the proof of the theorem is complete.

THEOREM 2.2. *Let N be a normal operator on $\mathscr{H}$. Then the set of all operators T in $\mathscr{L}(\mathscr{H})$ such that $\pi(T)$ is unitarily equivalent to $\pi(N)$ in the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{C}$ is contained in (NC).*

Proof. Let $W \in \mathscr{L}(\mathscr{H})/\mathscr{C}$ be unitary, let $T \in \mathscr{L}(\mathscr{H})$ be such that $\pi(T) = W^*\pi(N)W$, and let $S \in \mathscr{L}(\mathscr{H})$ be such that $\pi(S) = W$. Note that S is a Fredholm operator. Also, if the index of S is negative, it is easy to show that there exists an isometry $V \in \mathscr{L}(\mathscr{H})$ such that $\pi(V) = W$, while if the index of S is non-negative, it can be proved that there exists a coisometry Z in $\mathscr{L}(\mathscr{H})$ such that $\pi(Z) = W$ (cf. [**1**]). Suppose first that there exists a coisometry Z in $\mathscr{L}(\mathscr{H})$ such that $\pi(Z) = W$. Since Z^*NZ is easily seen to be normal, and $\pi(T) = \pi(Z)^*\pi(N)\pi(Z) = \pi(Z^*NZ)$, we conclude that $T \in (NC)$. Now suppose that there exists an isometry V in $\mathscr{L}(\mathscr{H})$ such that $\pi(V) = W$. Since $T - V^*NV$ is clearly compact, in order to complete the proof of the theorem it suffices to prove that $V^*NV \in (NC)$. To this end we observe that V^*NV is unitarily equivalent to the compression of N to the range of VV^*. Since $1 - VV^*$ is a finite rank projection, the desired conclusion follows from Theorem 2.1.

The following theorem provides an equivalent formulation of the conjecture $(*)$.

THEOREM 2.3. *The following statements are equivalent.*

(a) *Let T be an operator in $\mathscr{L}(\mathscr{H})$ such that $\pi(T)$ is normal in $\mathscr{L}(\mathscr{H})/\mathscr{C}$ and $\Omega(T) = E(T)$. Then $T \in (NC)$.*

(b) *Let S and T be operators in $\mathscr{L}(\mathscr{H})$ such that $\pi(S)$ and $\pi(T)$ are normal in $\mathscr{L}(\mathscr{H})/\mathscr{C}$ and $\Omega(S) = E(S) = \Omega(T) = E(T)$. Then there exists a unitary W in $\mathscr{L}(\mathscr{H})/\mathscr{C}$ such that $\pi(T) = W^*\pi(S)W$.*

Proof. In order to prove that (a) implies (b), suppose that S and T are two operators in $\mathscr{L}(\mathscr{H})$ satisfying the hypotheses of statement (b). If (a) holds, then there exist two normal operators N_1 and N_2 in $\mathscr{L}(\mathscr{H})$ such that $\pi(N_1) = \pi(T)$ and $\pi(N_2) = \pi(S)$, and hence $E(N_1) = E(N_2)$. From the von Neumann converse to Weyl's theorem for normal operators [**6**] it follows that there exists a unitary operator $V \in \mathscr{L}(\mathscr{H})$ such that $N_1 - V^*N_2V \in \mathscr{C}$. Setting $W = \pi(V)$ we have $\pi(T) = W^*\pi(S)W$ and the validity of (b) follows. Conversely, suppose now that (b) is true, and let T be an operator satisfying the hypotheses of (a). Let N be a normal operator in $\mathscr{L}(\mathscr{H})$ such that $E(N) = E(T)$. Since $\Omega(N) = E(N)$, from (b) we deduce that there exists a unitary W in $\mathscr{L}(\mathscr{H})/\mathscr{C}$ such that $\pi(T) = W^*\pi(N)W$. Now Theorem 2.2 implies that $T \in (NC)$, and hence the proof that (b) implies (a) is complete.

3. An interesting example of a quasidiagonal operator. In [**2**], it is proved that if T is a non-quasitriangular operator, then so is the operator $T \oplus 0$. An easy corollary of this result is the following fact: if T is a non-quasi-triangular operator and N is a normal operator whose essential spectrum

https://doi.org/10.4153/CJM-1974-012-2 Published online by Cambridge University Press

consists of finitely many points, then $T \oplus N$ is also a non-quasitriangular operator. Thus it is natural to inquire whether the direct sum of a non-quasi-triangular operator and a normal operator is always non-quasitriangular. This question is answered in the negative by the following theorem.

THEOREM 3.1. *Let $\mathcal{K}$ be a separable infinite dimensional complex Hilbert space such that $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ and let U and M be operators in $\mathcal{L}(\mathcal{K})$ such that U is a unilateral shift of multiplicity one and M is a normal operator whose spectrum is the closed unit disk. Then $T = U \oplus M$ is in the uniform closure of (NC).*

Proof. Let B be the operator in $\mathcal{L}(\mathcal{H})$ whose representing matrix on $\mathcal{K} \oplus \mathcal{K}$ is given by

$$B = \begin{bmatrix} U^* & 0 \\ 1 - UU^* & U \end{bmatrix}.$$

It readily follows that B is a unitary operator (actually, B is a bilateral shift of multiplicity one). Let n be any positive integer greater than 1. By the von Neumann converse to Weyl's theorem for normal operators, there exists a Hilbert space isomorphism ψ_n of $\mathcal{K}$ onto the direct sum of $2n + 1$ copies of $\mathcal{K}$ and a compact operator L_n in $\mathcal{L}(\mathcal{K})$ such that

$$\psi_n(M - L_n)\psi_n^{-1} = \left[\sum_{j=1}^{n-1} \oplus \frac{n-j}{n} B\right] \oplus M.$$

Let φ_n be the Hilbert space isomorphism mapping $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ onto the direct sum of $2n + 2$ copies of $\mathcal{K}$ defined by $\varphi_n = 1_{\mathcal{K}} \oplus \psi_n$, and let K_n' be the compact operator on $\mathcal{H}$ such that $\varphi_n K_n' \varphi_n^{-1} = 0 \oplus \psi_n L_n \psi_n^{-1}$. Then

$$\varphi_n(T - K_n')\varphi_n^{-1} = U \oplus \left[\sum_{j=1}^{n-1} \oplus \frac{n-j}{n} B\right] \oplus M.$$

Since $1 - UU^*$ has rank one, there exists K_n'' in $\mathcal{C}$ such that

$$\begin{aligned}\varphi_n(T - K_n' - K_n'')\varphi_n^{-1} &= U \oplus \left[\sum_{j=1}^{n-1} \oplus \frac{n-j}{n} (U^* \oplus U)\right] \oplus M \\ &= \left[\sum_{j=1}^{n} \oplus \frac{1}{n} (U \oplus 0)\right] \\ &\quad + \left\{\left[\sum_{j=1}^{n-1} \oplus \frac{n-j}{n} (U \oplus U^*)\right] \oplus 0 \oplus M\right\}.\end{aligned}$$

Now, it follows as above that there exists a compact operator K_n''' on $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ such that

$$\begin{aligned}\varphi_n(T - K_n' - K_n'' - K_n''')\varphi_n^{-1} &= \left[\sum_{j=1}^{n} \oplus \frac{1}{n} (U \oplus 0)\right] \\ &\quad + \left\{\left[\sum_{j=1}^{n-1} \oplus \frac{n-j}{n} B^*\right] \oplus 0 \oplus M\right\}.\end{aligned}$$

Let K_n be the compact operator in $\mathscr{L}(\mathscr{H})$ given by $K_n = K_n' + K_n'' + K_n'''$, and let N_n be the normal operator in $\mathscr{L}(\mathscr{H})$ defined by

$$N_n = \varphi_n^{-1}\left\{\left[\sum_{j=1}^{n-1} \oplus \frac{n-j}{n} B^*\right] \oplus 0 \oplus M\right\}\varphi_n.$$

From the previous calculation it follows that $||T - (N_n + K_n)|| \leqq 1/n$, and hence, since n was arbitrary, we conclude that $T = U \oplus M \in \overline{(NC)}$.

Remark 3.2. It is easy to see that the ideas employed in the proof of the above theorem can be used to prove a more general result. Let $T \in (NC)$,and let P be a projection in $\mathscr{L}(\mathscr{H})$ such that $(PT)|(1 - P)\mathscr{H}$ and $[(1 - P)T]|P\mathscr{H}$ are compact. With arguments similar to those employed in the proof of Theorem 3.1 it can be shown that if N is any normal operator whose essential spectrum $E(N)$ contains the convex hull of $E(T)$, then $(PT)|P\mathscr{H} \oplus N$ and $[(1 - P)T]|(1 - P)\mathscr{H} \oplus N$ are in $\overline{(NC)}$. In particular if M is a normal operator whose essential spectrum coincides with the closed unit disk and S is an operator satisfying $E(S) \subset E(M)$ and $S \oplus S^* \in (NC)$, then $S \oplus M \in \overline{(NC)}$. In a different direction, the techniques provided in the proof of Theorem 3.1 can also be used to prove that if U is a unilateral shift of multiplicity one in $\mathscr{L}(\mathscr{H})$, r is a positive number less than one and N_r is a normal operator in $\mathscr{L}(\mathscr{H})$ whose spectrum coincides with the closed annulus $\{\lambda : r \leqq |\lambda| \leqq 1\}$, then $U \oplus rU^* \oplus N_r$ is the uniform limit of compact perturbations of normal operators.

The following corollary is an immediate consequence of Theorem 3.1, the von Neumann converse to Weyl's theorem for normal operators, and the fact that a countable direct sum of quasidiagonal operators is quasidiagonal.

COROLLARY 3.3. *Let U be a unilateral shift in $\mathscr{L}(\mathscr{H})$ of multiplicity m, $1 \leqq m \leqq \aleph_0$, and let M be a normal operator on $\mathscr{H}$ whose spectrum is the closed unit disk. Then $U \oplus M$ is a quasidiagonal operator.*

It is worth noting that Corollary 3.3 provides one with an example of a quasidiagonal subnormal operator which is not normal. To our knowledge this is the first time that an example of an operator with such a property is exhibited.

Added in proof. In the recent remarkable paper, *Unitary equivalence modulo the compact operators and extensions of C*-algebras* by L. G. Brown, R. G. Douglas, and P. A. Fillmore, in Proceedings of a Conference on Operator Theory, (Lecture Notes in Mathematics, vol. *345*, Springer-Verlag, New York, 1973), conjecture $(*)$ is established employing rather indirect arguments which are not standard in operator theory. It may be also worth noting that the techniques used in the proof of Theorem 3.1 of the present paper were the basis for one of the main ideas of the proof of Theorem 13.1 in the paper *Invariant subspaces of non-quasitriangular operators* by R. G. Douglas and Carl Pearcy which appeared in the same Springer volume.

References

1. R. G. Douglas, *Banach algebras techniques in the theory of Toeplitz operators*, lectures given in the CBMS Regional Conference at the University of Georgia, June 12-16, 1972.
2. R. G. Douglas and C. Pearcy, *A note on quasitriangular operators*, Duke Math. J. *37* (1970), 177–188.
3. P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. *76* (1970), 887–933.
4. ——— *Continuous functions of Hermitian operators*, Proc. Amer. Math. Soc. *31* (1972), 130–132.
5. John S. Lancaster, *Lifting from the Calkin Algebra*, Ph.D. thesis, Indiana University, 1972.
6. W. Sikonia, *The von Neumann converse of Weyl's Theorem*, Indiana Univ. Math. J. *21* (1971), 121–123.

University of Michigan,
Ann Arbor, Michigan;
The Institute for Advanced Study,
Princeton, New Jersey;
University of Kansas,
Lawrence, Kansas