Canad. Math. Bull. Vol. **44** (1), 2001 pp. 105–114

Convolution Equation in $\mathcal{S}'^*$—Propagation of Singularities

Stevan Pilipović

Abstract. The singular spectrum of u in a convolution equation $\mu * u = f$, where μ and f are tempered ultradistributions of Beurling or Roumieau type is estimated by

$$SSu \subset (\mathbf{R}^n \times \operatorname{Char} \mu) \cup SSf.$$

The same is done for $SS_* u$.

0 Introduction

In this paper we consider a class of convolution equations in spaces of tempered ultradistributions and study the propagation of Gevrey and analytic singularities.

Various spaces of generalized functions and hyperfunctions are introduced and used in the microlocal analysis of various classes of equations ([4], [6], [11], [13]). Hörmander gives in [4], Chapter 9 an elementary approach to the theory of hyperfunctions (*cf.* [13]) by using Poisson's kernel as well as Komatsu who develops in [6] the theory of sheaves C^* and C_* of microfunctions which correspond to spaces of ultradistributions and ultradifferentiable functions, respectively. In [10] we follow [4], Chapter 8, and analyze the microsupport of an ultradistribution in $\mathcal{S}'^*$ by the mean of a kernel introduced in [4], Section 8.4. Note that ultradistribution spaces $\mathcal{S}'^*$ are are invariant under Fourier transformation.

In this paper we investigate the singular spectrum of a solution u of $\mu * u = f$, where $\mu, f \in \mathcal{S}'^*$ and prove

$$SSu \subset (\mathbf{R}^n \times \operatorname{Char} \mu) \cup SSf.$$

The same is proved for $SS_* u$. For the corresponding assertion in distribution spaces we refer to [4], Section 8.6.

Generally, for the references related to the propagation of singularities we refer to a wast literature given in the references of [4], [6], [11] and [13].

Since ultradifferential operators with constant coefficients of $*$-class, $P(\partial)$, are convolution operators, Theorem 1 imply the appropriate assertion for $P(\partial)$ (see [6] for a detail analysis of such operators). It is known that $SS_* P(\partial)u \subset SS_* u$, $u \in \mathcal{D}'$ and $SSP(\partial)u \subset SSu$, $u \in \mathcal{D}'^*$ ([6]). Thus, if an ultradifferential operator $P(\partial)$ of $*$-class has a property $\operatorname{Char} P(\partial) = \varnothing$, then our theorem directly implies the analytic-hypoellipticity of this operator in $\mathcal{S}'^*(\mathbf{R}^n)$. An example is the analytic-hypoellipticity of $\Delta u = f$ in $\mathcal{S}'^*(\mathbf{R}^n)$ ([1], second part of Theorem 4.1).

Received by the editors January 15, 2000; revised June 9, 2000.
AMS subject classification: Primary: 32A40, 46F15; secondary: 58G07.

1 Notation and Notions

As usual, by M_p, $p \in \mathbf{N}_0$ is denoted a sequence of positive numbers with $M_0 = 1$. We refer to [5], [8] and [12] for the meaning of conditions (M.1), (M.2)$'$, (M.2), (M.3)$'$ and (M.3). Also we use the following one ([8]):

(M.1)* $M^*_{p-1} M^*_{p+1} \geq M^{*2}_p$, $p \in \mathbf{N}$, where $M^*_0 = 1$, $M^*_p = M_p/p!$, $p \in \mathbf{N}$.

Let M_p satisfy (M.1) and (M.3)$'$. The associated function $M(\rho)$ and the growth function $\tilde{M}(\rho)$ related to M_p are defined by

$$M(\rho) = \sup_{p \in \mathbf{N}_0} \ln \frac{\rho^p}{M_p},\ \tilde{M}(\rho) = \sup_{p \in \mathbf{N}_0} \ln \frac{\rho^p}{M^*_p}, \quad \rho > 0.$$

Note, for given $L > 0$ there is $L_1 > 0$ such that

$$(L|\xi|) - |\eta|\,|\xi| \leq \tilde{M}(L_1/|\eta|), \quad \xi, \eta \in \mathbf{R}^n \tag{1}$$

([12], Section 1).

We denote by Ω an open set in $\mathbf{R}^n$; $K \subset\subset \Omega$ denotes that K is a compact subset of Ω. Recall, for $\varphi \in C^\infty(\Omega)$,

$$\|\varphi\|_{K,h,M_p} = \sup_{x \in K, \alpha \in \mathbf{N}^n_0} \frac{|\varphi^{(\alpha)}(x)|}{h^{|\alpha|} M_{|\alpha|}}.$$

We use a symbol $*$ for both (M_p) and $\{M_p\}$. For the definitions of $\mathcal{D}'^*(\Omega)$, $\mathcal{D}^*_K(\Omega)$, $\mathcal{D}'^*_K(\Omega)$ and the ultradifferential operators of $*$-class we refer to [5], [8], [11] and [12]. We always assume that M_p satisfies (M.1), (M.2)$'$ and (M.3)$'$.

Komatsu [6] (see also [2]) has defined SS_*—and SS^*—singular spectrum of a hyperfunction f. We recall the definition of $SS_* f$, $f \in \mathcal{D}'^*$. $(x, \omega) \in S^*\Omega = \Omega \times S^{n-1}$ (S^{n-1} is the unit sphere in $\mathbf{R}^n$) is not in $SS_* f$ iff there exist a neighbourhood $U \subset \Omega$ of x and a conic neighbourhood Γ of ω of the form $\Gamma = \{\xi \neq 0\,;\, \big|\xi/|\xi| - \omega\big| < \eta\}$ such that for every $\phi \in \mathcal{D}^*(U)$ in (M_p) case, for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$|\widehat{\phi f}(\xi)| \leq C_\epsilon e^{-M(\epsilon|\xi|)}, \quad \xi \in \Gamma,$$

in $\{M_p\}$ case, there exist $k > 0$ and $C > 0$ such that

$$|\widehat{\phi f}(\xi)| \leq C e^{-M(k|\xi|)}, \quad \xi \in \Gamma.$$

Note, $SS_{\{M_p\}} f = WF_L f$ (see Section 8.4 in [4] and [6]).

The definition of the singular spectrum SSf, where $f \in \mathcal{B}(\Omega)$, is given by Sato (*cf.* [13]). For an $f \in \mathcal{D}'^*(\Omega)$, $(x, \omega) \in S^*\Omega$ is not in SSf if this point is not in $SS\{f\}$, where $\{f\}$ denotes the corresponding hyperfunction. Note, $SSf = WF_A f$—the analytic wave front set of f ([4], Definition 9.3.2 and Theorem 9.6.3).

The definitions of corresponding singular supports are given by

$$\operatorname{singsupp}_* f = p_1(SS_* f), \quad \operatorname{singsupp}_A f = p_1(SSf),$$

where p_1 is the first projection.

The following result ([10]) will be used in this paper.

If $u \in \mathcal{D}'^*$ and $v \in \mathcal{E}'^*$, then

$$SS_*(u * v) \subset \{(x + y, \xi)(x, \xi) \in SS_* u, (y, \xi) \in SS_* v\},$$

$$SS(u * v) \subset \{(x + y, \xi) \; ; (x, \xi) \in SSu, (y, \xi) \in SSv\}. \tag{2}$$

We recall ([7], [9]) the definitions of tempered ultradistribution spaces.

Let $m > 0$. A space of smooth functions φ on $\mathbf{R}^n$ which satisfy

$$\sigma_{m,2}(\varphi) = \left(\sum_{\alpha,\beta \in \mathbf{N}_0^n} \int_{\mathbf{R}^n} \left| \frac{m^{|\alpha+\beta|}}{M_{|\alpha|} M_{|\beta|}} (1 + |x|^2)^{|\beta|/2} \varphi^{(\alpha)}(x) \right|^2 dx \right)^{1/2} < \infty,$$

equipped with the topology induced by the norm $\sigma_{m,2}$, is denoted by $\mathcal{S}_2^{M_p,m}$.

Strong duals of

$$\mathcal{S}^{(M_p)} = \operatorname{proj} \lim_{m \to \infty} \mathcal{S}_2^{M_p,m} \quad \text{and} \quad \mathcal{S}^{\{M_p\}} = \operatorname{ind} \lim_{m \to 0} \mathcal{S}_2^{M_p,m}$$

are called spaces of tempered ultradistributions of Beurling and Roumieau type and denoted by $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$, respectively.

For every fixed $p \in [1, \infty]$, the family of norms $\{\sigma_{m,2} \; ; \; m > 0\}$ is equivalent to the family of norms $\{\sigma_{m,p} \; ; \; m > 0\}$ where instead of L^2 norm we use L^p norm. In fact, in the sequel we use the family of norms

$$s_h(\phi) = \sup \left\{ \frac{h^{|\alpha+\beta|}}{M_{|\alpha|} M_{|\alpha|}} |x^\beta \partial^\alpha \phi(x)| \; ; \alpha, \beta \in \mathbf{N}_0^n, x \in \mathbf{R}^n \right\}, \quad h > 0, \tag{3}$$

which is equivalent to $\{\sigma_{h,2} \; ; \; h > 0\}$. $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ are (FS)—and (LS)—spaces respectively. If (M.2) holds, they are (FN)—and (LN)—spaces, respectively (for these types of spaces we refer to [3]) and

$$\mathcal{D}^* \hookrightarrow \mathcal{S}^* \hookrightarrow \mathcal{E}^*, \quad \mathcal{S}^* \hookrightarrow \mathcal{S},$$

where "$A \hookrightarrow B$" means that A is dense in B and the inclusion mapping is continuous. The Fourier transformation is an isomorphism of $\mathcal{S}^*$ onto itself.

Let us recall that an $f \in \mathcal{D}'^*$ is in $\mathcal{S}'^*$ if and only if there exists a family $F_{\alpha,\beta}$, $\alpha, \beta \in \mathbf{N}_0^n$, in $L^2(\mathbf{R}^n)$ such that

$$f = \sum_{\alpha,\beta \in \mathbf{N}_0^n} \left((1 + |x|^2)^{\beta/2} F_{\alpha,\beta} \right)^{(\alpha)} \quad \text{in } \mathcal{S}'^*,$$

and in (M_p) case, there exists $k > 0$, in $\{M_p\}$ case, for every $k > 0$,

$$\left(\sum_{\alpha,\beta \in \mathbf{N}_0^n} \int_{\mathbf{R}^n} \left| \frac{M_{|\alpha|} M_{|\beta|}}{k^{|\alpha+\beta|}} F_{\alpha,\beta}(x) \right|^2 \right)^{1/2} < \infty.$$

If (M.2) and (M.3) are assumed, then $f \in \mathcal{S}'^*$ iff $f = P(\partial)F$, where F is a continuous function which satisfies $|F(x)| \le C_k e^{M(k|x|)}$, $x \in \mathbf{R}^n$, and in (M_p) case, P is an ultradifferential operator of (M_p)-class and the estimate for F holds for some $k > 0$ and some $C_k > 0$, in $\{M_p\}$ case, P is an ultradifferential operator of $\{M_p\}$-class and the estimate holds for every $k > 0$ and the corresponding $C_k > 0$.

Let $D\mathbf{R}^n = \{z \in \mathbf{C}^n \ ; \ |\operatorname{Im} z| < 1\}$ and $S^*\mathbf{R}^n = \partial D\mathbf{R}^n$. Recall ([6]), $O_*|_{D\mathbf{R}^n}$ is a sheaf over $\mathbf{C}^n$ of holomorphic functions in $D\mathbf{R}^n$ which satisfy the following growth condition near $S^*\mathbf{R}^n$.

Let U be an open set in $\mathbf{C}^n$. Then a function $F(z)$ is in $O_*|_{D\mathbf{R}^n}(U)$ if F is holomorphic in $D\mathbf{R}^n \cap U$ such that for every compact set $K \subset\subset U$, in (M_p) case, for every ultradifferential operator $P(\partial)$ of class (M_p), in $\{M_p\}$ case, for every ultradifferential operator $P(\partial)$ of class $\{M_p\}$

$$P(\partial)F(z) \quad \text{is bounded in } K \cap D\mathbf{R}^n.$$

As in [4], put

$$I(\xi) = \int_{|\omega|=1} e^{-\langle \omega, \xi \rangle}\, d\omega, \quad \xi \in \mathbf{R}^n,\ K(z) = (2\pi)^{-n} \int \frac{e^{\sqrt{-1}\langle z, \xi \rangle}}{I(\xi)}\, d\xi, \quad z \in D\mathbf{R}^n.$$

The properties of K are analyzed in [4], Chapter 4, and [10]. Note that $K(\cdot + \sqrt{-1}y) \in \mathcal{S}^*$ for every fixed y, $|y| < 1$.

Let $u \in \mathcal{S}'^*$ and

$$U(z) = (u * K)(z) = \langle u(t), K(x - t + \sqrt{-1}y) \rangle, \quad z \in D\mathbf{R}^n.$$

Then U is analytic in $D\mathbf{R}^n$ and it is proved in [10] that $q \notin SS_* u$ if and only if U is $\mathcal{O}_*$ in a neighbourhood of $x_0 - \sqrt{-1}\omega_0$ and $q \notin SSu$ if and only if U is analytic at $x_0 - \sqrt{-1}\omega_0$ (*i.e.* in a neighbourhood of this point).

2 On the Convolution Equation

First we give the definition of $\operatorname{Char} \mu$, $\mu \in \mathcal{S}'^*$; for distributions this definition is given in [4], p. 315.

Let $\mu \in \mathcal{S}'^*$ and Γ be a set of all $\xi \in \mathbf{R}^n \setminus \{0\}$ such that there is a complex conic neighbourhood V of ξ and an analytic function ϕ in $V_c = \{\zeta \in V \ ; \ |\zeta| > c\}$, for some $c > 0$, such that

$$\phi\hat{\mu} = 1 \text{ in } V \cap \mathbf{R}^n \text{ and } |\phi(\zeta)| \le C e^{M(k|\zeta|)}, \quad \zeta \in V_c, \tag{4}$$

for some $k > 0$ and $C > 0$. Then, $\operatorname{Char} \mu = (\mathbf{R}^n \setminus \{0\}) \setminus \Gamma$.

Theorem 1 *Let $u, \mu \in \mathcal{S}'^*(\mathbf{R}^n)$. Then:*

i) $SSu \subset (\mathbf{R}^n \times \operatorname{Char} \mu) \cup SS(u * \mu)$,
ii) $SS_* u \subset (\mathbf{R}^n \times \operatorname{Char} \mu) \cup SS_*(u * \mu)$.

https://doi.org/10.4153/CMB-2001-013-3 Published online by Cambridge University Press

Proof The idea of the proof is the same as for distributions ([4]) but uniform estimates of all derivatives of some functions which are needed make the proof more difficult.

We prove the first assertion. The proof of the second one is similar. Only a final conclusion has to be changed and this is done in the proof of ii).

i) Put $f = u * \mu$. We will use the notation from above. Let

$$(x_0, \omega_0) \notin SSf\,; \quad |\omega_0| = 1, \quad \omega_0 \notin \operatorname{Char}\mu.$$

We have to prove that $K * u$ is analytic at $x_0 - \sqrt{-1}\omega_0$. Let W' and W'' be closed conic neighbourhoods of ω_0 in $\mathbf{R}^n \setminus \{0\}$ such that

$$W'' \subset \operatorname{int} W', \quad W' \subset V.$$

Further, let $\kappa \in \mathcal{E}^{(M_p)}$ such that $0 \le \kappa \le 1$, $\kappa \equiv 1$ in a neighbourhood of $W''_{3c} = \{\xi \in W''\,;\, |\xi| \ge 3c\}$, $\operatorname{supp}\kappa \subset W'_{2c}$ and let κ be homogeneous of degree 0 when $|\xi| \ge 3c$. Decompose the Fourier transformation of $u * K(\cdot + y)$, $|y| < 1$ as follows:

$$\frac{\hat{u}(\xi)e^{-\langle y,\xi\rangle}}{I(\xi)} = \frac{\hat{u}(\xi)\big(1 - \kappa(\xi)\big)\, e^{-\langle y,\xi\rangle}}{I(\xi)} + \frac{\hat{f}(\xi)\phi(\xi)\kappa(\xi)e^{-\langle y,\xi\rangle}}{I(\xi)}, \quad \xi \in \mathbf{R}^n,$$

where ϕ is given in (4) and put

$$K_1(z) = (2\pi)^{-n} \int_{\mathbf{R}^n} \frac{\big(1 - \kappa(\xi)\big)\, e^{\sqrt{-1}\langle z,\xi\rangle}}{I(\xi)}\, d\xi,$$
$$K_2(z) = (2\pi)^{-n} \int_{\mathbf{R}^n} \frac{\kappa(\xi)\phi(\xi)e^{\sqrt{-1}\langle z,\xi\rangle}}{I(\xi)}\, d\xi, \quad z \in D\mathbf{R}^n.$$

Thus, $K * u = K_1 * u + K_2 * f$. Note that K_1 and K_2 are holomorphic in $D\mathbf{R}^n$. Moreover, one can easily prove that there is an $\epsilon > 0$ such that K_1 is analytic if $|\operatorname{Im} z + \omega_0| < \epsilon$. We are going to prove:

(a) For every fixed y, $|y| < 1$, $K_1(\cdot + \sqrt{-1}y) \in \mathcal{S}^*$.
This implies that $K_1 * u(\cdot + \sqrt{-1}y)$, $|y| < 1$ is a C^∞-function. Note that $K_1 * u$ is analytic in a neighbourhood of $x_0 - \sqrt{-1}\omega_0$.

(b) K_2 has an analytic extension in

$$D_\delta = \{z\,;\, |\operatorname{Im} z| < 1 - \delta + \delta(1 + |\operatorname{Re} z|^2)^{1/2}, |\operatorname{Im} z + \omega_0| < \delta\},$$

for some $\delta > 0$.

(c) $K_2(\cdot + iy) \in \mathcal{S}^*(|x| \ge d)$, $d > 0$, $y \in D_\delta$, where $\mathcal{S}^*(|x| \ge d)$ is defined in the same way as $\mathcal{S}^*$ but with the supremum in (3) taken over $\{x\,;\, |x| \ge d\}$.
This and the partition of unity imply that $K_2 * f \in \mathcal{S}'^*$.

(d) Since $K * u = K_1 * u + K_2 * f$, $|\operatorname{Im} z| < 1$, and $K_1 * u$ is analytic at $x_0 - \sqrt{-1}\omega_0$, we will finish the proof by proving that $K_2 * f$ is analytic at this point.

https://doi.org/10.4153/CMB-2001-013-3 Published online by Cambridge University Press

(a) We will prove that for some $\epsilon > 0$,

$$K_1(\cdot + \sqrt{-1}y) \in \mathcal{S}^* \quad \text{if } |y + \omega_0| < \epsilon. \tag{5}$$

Since $\operatorname{supp}\big(1 - \kappa(\xi)\big) \subset \mathbf{R}^n \setminus W''_{3c}$, there exists $\epsilon > 0$ such that

$$\langle \omega_0, \xi \rangle \leq (1 - 2\epsilon)|\xi|, \quad \xi \in \mathbf{R}^n \setminus W''_{3c}$$

and thus,

$$-\langle y, \xi \rangle - |\xi| < -\epsilon|\xi|, \quad \xi \notin W''_{3c}, \ |y + \omega_0| < \epsilon. \tag{6}$$

Assume that $|y + \omega_0| < \epsilon$. Let $x = \operatorname{Re} z \in \mathbf{R}$, $\alpha, \beta \in \mathbf{N}_0^n$. Then

$$\begin{aligned} R_{\alpha,\beta}(x) &= \frac{1}{M_{|\alpha|}M_{|\beta|}} \left| x^\beta \int_{\mathbf{R}^n} \frac{\big(1 - \kappa(\xi)\big)(\sqrt{-1}\xi)^\alpha e^{\sqrt{-1}\langle x,\xi\rangle - \langle y,\xi\rangle}}{I(\xi)}\, d\xi \right| \\ &\leq \frac{1}{M_{|\beta-p-r|}M_{|r|}M_{|p|}M_{|\alpha-s|}M_{|s|}} \\ &\qquad \left| \int_{\mathbf{R}^n} e^{\sqrt{-1}\langle x,\xi\rangle} \sum_{p \leq \beta} \binom{\beta}{p} \sum_{r \leq \beta - p} \binom{\beta - p}{r} \big(1 - \kappa(\xi)\big)^{(\beta-p-r)} \left(\frac{1}{I(\xi)}\right)^{(r)} \right. \\ &\qquad \left. \sum_{s \leq p} \binom{p}{s} \frac{\alpha!}{(\alpha - s)!} \xi^{\alpha - s} (-y)^{p-s} e^{-\langle y,\xi\rangle}\, d\xi \right|. \end{aligned}$$

Note,

$$\left| \left(\frac{1}{I(\xi)}\right)^{(r)} \right| \leq \frac{2^r r!}{I(\xi)}, \quad \frac{\alpha!}{(\alpha - s)!}\frac{1}{M_{|s|}} \leq 2^{|\alpha|}, \quad \frac{|y|^{p-s}}{M_{|p|}} < \infty\,;$$

for every a_1 there is $C_{a_1} > 0$ such that

$$\frac{\left|\big(1 - \kappa(\xi)\big)^{(\beta-p-r)}\right|}{a_1^{|\beta-p-r|} M_{|\beta-p-r|}} < C_{a_1}\,;$$

for every $a_2 > 0$

$$\frac{|\xi|^{\alpha-s}}{a_2^{|\alpha-s|} M_{|\alpha-s|}} \leq e^{M(a_2|\xi|)} \quad (\xi \in \mathbf{R}^n).$$

This implies that for every $h > 0$ and $a > 0$ there is $C > 0$ such that for every $x \in \mathbf{R}^n$ and $\alpha, \beta \in \mathbf{N}_0^n$

$$\sup\{h^{|\alpha+\beta|} R_{\alpha,\beta}(x)\,;\, \alpha, \beta \in \mathbf{R}^n\} \leq C \int_{\mathbf{R}^n \setminus W''_{3c}} e^{-\langle y,\xi\rangle + M(a|\xi|) - |\xi|}\, d\xi.$$

Now, (6) implies (5).

https://doi.org/10.4153/CMB-2001-013-3 Published online by Cambridge University Press

(b) Let us prove that K_2 has an analytic extension in

$$D_\delta = \{z\,;\,|\operatorname{Im} z| < (1-\delta+\delta(1+|\operatorname{Re} z|^2)^{1/2}, |\operatorname{Im} z+\omega_0| < \delta\}$$

for some $\delta > 0$ which will be chosen later.

Let $x+\sqrt{-1}y \in D_\delta$ and $\alpha, \beta \in \mathbf{N}_0$. We have

$$\begin{aligned} K_2(x+\sqrt{-1}y) &= \Big(\int_{W'_{2c}\setminus W''_{3c}} + \int_{W''_{3c}}\Big) \frac{\kappa(\xi)\phi(\xi)}{I(\xi)} e^{\sqrt{-1}\langle z,\xi\rangle}\, d\xi \\ &= K_{21}(x+\sqrt{-1}y) + K_{22}(x+\sqrt{-1}y). \end{aligned}$$

We will use [4], Lemma 8.4.9, which asserts that for every $\epsilon \in (0,\pi/2)$

$$I(\xi+\sqrt{-1}\eta) = (2\pi)^{(n-1)/2} e^{\langle \xi+\sqrt{-1}\eta, \xi+\sqrt{-1}\eta\rangle^{1/2}}.$$

$$(7) \qquad \langle \xi+\sqrt{-1}\eta, \xi+\sqrt{-1}\eta\rangle^{-(n-1)/4}\left(1+O\Big(\frac{1}{\langle \xi+\sqrt{-1}\eta, \xi+\sqrt{-1}\eta\rangle^{1/2}}\Big)\right)$$

if $\langle \xi+\sqrt{-1}\eta, \xi+\sqrt{-1}\eta\rangle^{1/2} \to \infty$ and

$$\big|\arg \langle \xi+\sqrt{-1}\eta, \xi+\sqrt{-1}\eta\rangle^{1/2}\big| < \frac{\pi}{2} - \epsilon.$$

By (7) for $\eta = 0$, and (6) we obtain that K_{21} has an analytic extension in a neighbourhood of $x - i\omega_0$, $x \in \mathbf{R}^n$. Let us prove assertion (b) for

$$(8) \qquad K_{22}(x+\sqrt{-1}y) = \int_{W''_{3c}} \frac{e^{\sqrt{-1}\langle x+\sqrt{-1}y,\xi\rangle}\phi(\xi)}{I(\xi)}\, d\xi.$$

Let $\kappa_1 \in \mathcal{E}^{(M_p)}$, supp $\kappa_1 \subset W''_{3c}$, $\kappa_1 \equiv 1$ in W'''_{4c} where W''' is a conic neighbourhood of ω_0 and let κ_1 be homogeneous of degree 0 for $|\xi| \geq 4c$.

We choose δ such that $0 < \delta \leq 1$,

$$\xi + \sqrt{-1}\delta\kappa_1(\xi)|\xi|x(1+x^2)^{1/2} \in V_c, \quad \xi \in \operatorname{supp}\kappa_1$$

and W' is so narrow that for some $r_0 > 0$,

$$\xi \in W'_{2c} \Rightarrow L(\xi, r_0) \subset V_c.$$

Let $|y| < 1$, $\alpha, \beta \in \mathbf{N}_0^n$. We will move the integration in (8) to the cycle

$$W''_{3c} \ni \xi \to \xi + \sqrt{-1}\delta\kappa_1(\xi)|\xi|x(1+|x|^2)^{-1/2}.$$

By Stokes' formula we have

$$K_{22}(x+\sqrt{-1}y) = \int_{W''_{3c}} \frac{e^{\sqrt{-1}\langle x+\sqrt{-1}y, \xi+\sqrt{-1}\eta\rangle}\phi(\xi+\sqrt{-1}\eta)}{I(\xi+\sqrt{-1}\eta)}\, d\xi$$

https://doi.org/10.4153/CMB-2001-013-3 Published online by Cambridge University Press

where $\eta = \delta\kappa_1(\xi)|\xi|x(1+|x|^2)^{-1/2}$. Since

$$\begin{aligned}&\mathrm{Re}\big(\sqrt{-1}\langle x+\sqrt{-1}y, \xi+\sqrt{-1}\eta\rangle - \langle \xi+\sqrt{-1}\eta, \xi+\sqrt{-1}\eta\rangle^{1/2}\big)\\ &\quad\le -|\xi|\big(1-\delta+\delta(1+|x|^2)^{1/2}\big) - \langle y,\xi\rangle,\end{aligned}$$

there is an analytic continuation of the integral to the domain D_δ.

(c) We will prove that $x \mapsto K_2(x+\sqrt{-1}y)$ is in $\mathcal{S}^{(M_p)}(|x|\ge d)$ when

$$|y| < 1-\delta+\delta(1+|d|^2)^{1/2}, \quad |y+\omega_0| < \delta.$$

Let $h>0$, $|x|\ge d$ and $\alpha,\beta\in\mathbf{N}_0$. Then we have

$$\begin{aligned}&\big|x^\beta K_2^{(\alpha)}(x+\sqrt{-1}y)\big|\\ &\le \sum_{p\le\beta}\sum_{r\le\beta-p}\sum_{j\le p}\sum_{s\le p-j}\binom{\beta}{p}\binom{\beta-p}{r}\binom{p}{j}\binom{p-j}{s}|y|^{|s|}\\ &\qquad \frac{\alpha!}{(\alpha-j)!}\int_{W'_{2c}} e^{\sqrt{-1}\langle x,\xi\rangle-\langle y,\xi\rangle}|\phi^{(\beta-p-r)}(\xi)|\left|\left(\frac{1}{I(\xi)}\right)^{(r)}\right| |\xi|^{|\alpha-j|}\\ &\qquad\qquad |\kappa^{(p-j-s)}(\xi)|\,d\xi\\ &\le \sum_{p\le\beta}\sum_{r\le\beta-p}\sum_{j\le p}\sum_{s\le p-j}\binom{\beta}{p}\binom{\beta-p}{r}\binom{p}{j}\binom{p-j}{s}|y|^{|s|}\\ &\qquad \frac{\alpha!}{(\alpha-j)!}\int_{W'_{2c}\setminus W''_{3c}} e^{\sqrt{-1}\langle x,\xi\rangle-\langle y,\xi\rangle}|\phi^{(\beta-p-r)}(\xi)|\left|\left(\frac{1}{I(\xi)}\right)^{(r)}\right| |\xi|^{|\alpha-j|}\\ &\qquad\qquad |\kappa^{(p-j-s)}(\xi)|\,d\xi\\ &\quad+\sum_{p\le\beta}\sum_{r\le\beta-p}\sum_{j\le p}\binom{\beta}{p}\binom{\beta-p}{r}\binom{p}{j}\\ &\qquad \frac{\alpha!}{(\alpha-j)!}|y|^{|p-j|}\int_{W'''_{3c}} e^{\sqrt{-1}\langle x,\xi\rangle-\langle y,\xi\rangle}|\phi^{(\beta-p-r)}(\xi)|\left|\left(\frac{1}{I(\xi)}\right)^{(r)}\right| |\xi|^{|\alpha-j|}\,d\xi\\ &= T_{21}(x)+T_{22}(x).\end{aligned}$$

By using (6), one can show (as for K_1) that for suitable $\epsilon>0$,

$$\sup_{\substack{\alpha,\beta\in\mathbf{N}_0\\ |x|\ge d}} \frac{h^{|\alpha+\beta|}T_{21}(x)}{M_{|\alpha|}M_{|\beta|}} < \infty \quad \text{if } |y+\omega_0|<\epsilon.$$

For $T_{22}(x)$, $|x| \geq d$, we have

$$\sup_{\substack{\alpha,\beta\in\mathbf{N}_0^n\\|x|\geq d}} \frac{h^{|\alpha+\beta|}T_{22}(x)}{M_{|\alpha|}M_{|\beta|}}$$

$$\leq h^{|\alpha+\beta|}\sum_{p\leq\beta}\sum_{r\leq\beta-p}\sum_{j\leq p}\binom{\beta}{p}\binom{\beta-p}{r}\binom{p}{j}$$

$$\frac{\left(1+\delta(1+d^2)^{1/2}-\delta\right)^{p-j}\alpha!}{(\alpha-j)!}\frac{2^r r!}{M_{|\beta-p-r|}M_{|p|}M_{|r|}M_{|\alpha-j|}M_{|j|}}$$

$$\int_{W'_{2c}}|e^{\sqrt{-1}\langle x+\sqrt{-1}y,\xi+\sqrt{-1}\eta\rangle}|\,|\phi^{(\beta-p-r)}(\xi+\sqrt{-1}\eta)|$$

$$\left|\frac{1}{I(\xi+\sqrt{-1}\eta)}\right|\,\left|\xi+\sqrt{-1}\eta\right|^{|\alpha-j|}d\xi$$

$$\leq C\int_{W'_{2c}}\exp\Big(-|\xi|\big(1-\delta+\delta(1+d^2)^{1/2}\big)-M(a|\xi|)-\langle y,\xi\rangle\Big)\,d\xi,$$

which is clearly a finite integral.

(d) Recall, $K*u(z)=K_1*u(z)+K_2*f(z)$, $|\operatorname{Im} z|<1$ and K_1*u is analytic at $x_0-\sqrt{-1}\omega_0$. We have to prove the same for K_2*f.

The family of norms $\sigma_{m,2}$ is equivalent to the family

$$\tilde{\sigma}_{m,2}(\phi)=\sup_{\alpha,\beta\in\mathbf{N}_0^n}\left\{\frac{m^{|\alpha+\beta|}}{M_{|\alpha|}M_{|\beta|}}\left\|\left(x^\beta\phi(x)\right)^{(\alpha)}\right\|_{L^2}\right\},\quad m>0\ ([9]).$$

This, Parseval's identity and (1) imply that in (M_p)-case, for every $m_1>0$ there is $C_1>0$ (resp. in $\{M_p\}$-case, there is $m_1>0$ and $C_1>0$) such that

$$\begin{aligned}|\langle f(t),K_2(z-t)\rangle| &\leq C_1\tilde{\sigma}_{m_1,2}\big(K_2(z-t)\big)\\ &\leq C_1\sup\frac{m_1^{|\alpha+\beta|}}{M_{|\alpha|}M_{|\beta|}}\left\|\xi^\alpha\left(\frac{\phi(\xi)\kappa(\xi)e^{\sqrt{-1}\langle x,\xi\rangle}e^{-\langle y,\xi\rangle}}{I(\xi)}\right)^{(\beta)}\right\|_{L^2}\\ &\leq Ce^{M(m|x|)}e^{\tilde{M}(\frac{m}{1-|y|})},\quad x+\sqrt{-1}y\in D\mathbf{R}^n,\end{aligned}$$

where $m>0$ and $C>0$ are suitable constants.

One can simply prove that for every $m>0$ there is $C>0$ such that

$$|K_2(z-t)|\leq Ce^{M(m|x-t|)}e^{\tilde{M}(\frac{m}{1-|y|})},\quad t\in\mathbf{R}^n,z\in D\mathbf{R}^n.$$

This implies that the boundary value $(K_2*f)(\cdot-\sqrt{-1}\omega_0)$ is equal to the convolution of f and the boundary values $K_2(\cdot-\sqrt{-1}\omega_0)$ which are analytic except at 0. Let $f=f_1+f_2$ where $f_1\in\mathcal{E}'^*$ and $f_2=0$ when $|x-x_0|<r$, $r>0$. Then,

$$SSK_2(\cdot-\sqrt{-1}w_0)\subset\{(0,tw_0)\,;\,t>0\}$$

https://doi.org/10.4153/CMB-2001-013-3 Published online by Cambridge University Press

and $x_0 \notin \mathrm{singsupp}_A(f_1 * K_2)(\cdot - \sqrt{-1}\omega_0)$, which follows from (2).

Thus, $K * u$ is analytic at $x_0 - \sqrt{-1}\omega_0$.

ii) For the estimation of $SS_* u$ we have to repeat all the arguments of the part i) and to note that

$$x_0 \notin \mathrm{singsupp}_*(f_1 * K_2)(\cdot - \sqrt{-1}\omega_0),$$

which also follows from (2). This implies that $K * u$ is O_* at $x_0 - \sqrt{-1}\omega_0$.

This completes the proof.

References

[1] C. Dong and T. Matsuzawa, *S-spaces of Gel'fand-Shilov and differential equations*. Japan. J. Math. (2) **19**(1994), 227–239.

[2] A. Eida and Pilipović, *On the microlocal decomposition of some classes of hyperfunctions*. Math. Proc. Cambridge Philos. Soc. **125**(1999), 455–461.

[3] K. Floret and J. Wloka, *Einfürung in die Theorie der lokalkonvexen Räume*. Springer, Lecture Notes in Math. **59**, Berlin, Heidelberg, New York, 1968.

[4] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*. Springer, Berlin, Heidelberg, New York, Tokyo, 1983.

[5] H. Komatsu, *Ultradistributions, I–III*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20**(1973), 25–105; **24**(1977), 607–0628; **29**(1982), 653–717.

[6] H. Komatsu, *Microlocal Analytic in Gevrey Classes and in Convex Domains*. Springer, Lecture Notes in Math. **1726**, 1989, 426–493.

[7] D. Kovačević and S. Pilipović, *Structural properties of the space of tempered ultradistributions*. Proc. Conf., Comp. Analysis and Generalized Functions, Varna, 1991, 1993, 169–184.

[8] H. J. Petzsche, *Generalized functions and the boundary values of holomorphic functions*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31**(1984), 391–431.

[9] S. Pilipović, *Characterization of bounded sets in spaces of ultradistributions*. Proc. Amer. Math. Soc. **120**(1994), 1191–1206.

[10] ———, *Microlocal Analysis of ultradistributions*. Proc. Amer. Math. Soc. **126**(1998), 105–113.

[11] L. Rodino, *Linear partial differential operators in Gevrey spaces*. World Scientific, Singapore, 1993.

[12] J. W. de Roewer, *Hyperfunctional singular support of distributions*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31**(1985), 585–631.

[13] M. Sato, T. Kawai and M. Kashiwara, *Microfunctions and Pseudo-differential Equations*. Springer, Lecture Notes in Math. **287**, 1973, 265–529.

Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad
Yugoslavia
e-mail: pilipovic@unsim.nc.ac.yu